# Integral Transforms and Their Applications

## Second Edition

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**Lokenath Debnath**
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Chapman & Hall/CRC
Taylor & Francis Group
Boca Raton London New York

Chapman & Hall/CRC is an imprint of the
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Chapman & Hall/CRC
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742


Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number-10: 1-58488-575-0 (Hardcover)
International Standard Book Number-13: 978-1-58488-575-7 (Hardcover)



**Library of Congress Cataloging-in-Publication Data**

Debnath, Lokenath.
Integral transforms and their applications. -- 2nd ed. / Lokenath Debnath and Dambaru Bhatta.
p. cm.
Includes bibliographical references and index.
ISBN 1-58488-575-0 (acid-free paper)
1. Integral transforms. I. Bhatta, Dambaru. II. Title.

QA432.D36 2006
515'.723--dc22 2006045638

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To my wife **Sadhana** and granddaughter **Princess Maya**

Lokenath Debnath

To my wife **Bisruti** and sons **Rohit** and **Amit**

Dambaru Bhatta

# *Preface to the Second Edition*

> "A teacher can never truly teach unless he is still learning himself. A lamp can never light another lamp unless it continues to burn its own flame. The teacher who has come to the end of his subject, who has no living traffic with his knowledge but merely repeats his lessons to his students, can only load their minds; he cannot quicken them."
>
> Rabindranath Tagore

When the first edition of this book was published in 1995 under the sole authorship of Lokenath Debnath, it was well received, and has been used as a senior undergraduate or graduate level text and research reference in the United States and abroad for the last ten years. We received many comments and suggestions from many students and faculty around the world. These comments and criticisms have been very helpful, beneficial, and encouraging. This second edition is the result of that input.

Another reason for adding this second edition to the literature is the fact that there have been major discoveries of several integral transforms including the Radon transform, the Gabor transform, the inverse scattering transform, and wavelet transforms in the twentieth century. It is becoming even more desirable for mathematicians, scientists and engineers to pursue study and research on these and related topics. So what has changed, and will continue to change, is the nature of the topics that are of interest in mathematics, science and engineering, the evolution of books such as this one is a history of these shifting concerns.

This new and revised edition preserves the basic content and style of the first edition. As with the previous edition, this book has been revised primarily as a comprehensive text for senior undergraduates or beginning graduate students and a research reference for professionals in mathematics, science, and engineering, and other applied sciences. The main goal of this book is on the development of the required analytical skills on the part of the reader, rather than the importance of more abstract formulation with full mathematical rigor. Indeed, our major emphasis is to provide an accessible working knowledge of the analytical methods with proofs required in pure and applied mathematics, physics, and engineering.

We have made many additions and changes in order to modernize the contents and to improve the clarity of the previous edition. We have also taken advantage of this new edition to update the bibliography and correct typographical errors, to include additional topics, examples of applications, exercises, comments, and observations, and in some cases, to entirely rewrite whole section. This edition contains a collection of over 600 challenging worked examples and exercises with answers and hints to selected exercises. There is plenty of material in the book for a year-long course. Some of the material need not be covered in a course work and can be left for the readers to study on their own in order to prepare them for further study and research. Some of the major changes, additions, and highlights in this edition and the most significant difference from the first edition include the following:

1. Chapter 1 on Integral Transforms has been completely revised and some new material on brief historical introduction was added to provide new information about the historical developments of the subject. These changes have been made to provide the reader to see the direction in which the subject has developed and find those contributed to its developments.

2. Chapter 2 on Fourier Transforms has been completely revised and new material added, including new sections on Fourier transforms of generalized functions, the Poisson summation formula, the Gibbs phenomenon, and the Heisenberg uncertainty principle. Many sections have been completely rewritten with new examples of applications.

3. Four entirely new chapters on Radon Transforms, and Wavelets and Wavelet Transforms, Fractional Calculus and its applications to ordinary and partial differential equations have been added to modernize the contents of the book. A new section on the transfer function and the impulse response function with examples of applications was included in Chapters 2 and 4.

4. The book offers a detailed and clear explanation of every concept and method that is introduced, accompanied by carefully selected worked examples, with special emphasis being given to those topics in which students experience difficulty.

5. A wide variety of modern examples of applications has been selected from areas of ordinary and partial differential equations, quantum mechanics, integral equations, fluid mechanics and elasticity, mathematical statistics, fractional ordinary and partial differential equations, and special functions.

6. The book is organized with sufficient flexibility to enable instructors to select chapters appropriate to courses of differing lengths, emphases, and levels of difficulty.

7. A wide spectrum of exercises has been carefully chosen and included at the end of each chapter so the reader may further develop both analytical skills in the theory and applications of transform methods and a deeper insight into the subject.

8. Answers and hints to selected exercises are provided at the end of the book to provide additional help to students. All figures have been redrawn and many new figures have been added for a clear understanding of physical explanations.

9. All appendices, tables of integral transforms, and the bibliography have been completely revised and updated. Many new research papers and standard books have been added to the bibliography to stimulate new interest in future study and research. Index of the book has also been completely revised in order to include a wide variety of topics.

10. The book provides information that puts the reader at the forefront of current research.

With the improvements and many challenging worked problems and exercises, we hope this edition will continue to be a useful textbook for students as well as a research reference for professionals in mathematics, science and engineering.

It is our pleasure to express our grateful thanks to many friends, colleagues, and students around the world who offered their suggestions and help at various stages of the preparation of the book. We express our sincere thanks to Veronica Martinez and Maria Lisa Cisneros for typing the final manuscript with constant changes. In spite of the best efforts of everyone involved, some typographical errors doubtless remain. Finally, we wish to express our special thanks to Bob Stern, Executive Editor, and the staff of CRC/Chapman Hall for their help and cooperation.

**Lokenath Debnath**
**Dambaru Bhatta**
The University of Texas-Pan American

# *Preface to the First Edition*

Historically, the concept of an integral transform originated from the celebrated Fourier integral formula. The importance of integral transforms is that they provide powerful operational methods for solving initial value problems and initial-boundary value problems for linear differential and integral equations. In fact, one of the main impulses for the development of the operational calculus of integral transforms was the study of differential and integral equations arising in applied mathematics, mathematical physics, and engineering science; it was in this setting that integral transforms arose and achieved their early successes. With ever greater demand for mathematical methods to provide both theory and applications for science and engineering, the utility and interest of integral transforms seems more clearly established than ever. In spite of the fact that integral transforms have many mathematical and physical applications, their use is still predominant in advanced study and research. Keeping these features in mind, our main goal in this book is to provide a systematic exposition of the basic properties of various integral transforms and their applications to the solution of boundary and initial value problems in applied mathematics, mathematical physics, and engineering. In addition, the operational calculus of integral transforms is applied to integral equations, difference equations, fractional integrals and fractional derivatives, summation of infinite series, evaluation of definite integrals, and problems of probability and statistics.

There appear to be many books available for students studying integral transforms with applications. Some are excellent but too advanced for the beginner. Some are too elementary or have limited scope. Some are out of print. While teaching transform methods, operational mathematics, and/or mathematical physics with applications, the author has had difficulty choosing textbooks to accompany the lectures. This book, which was developed as a result of many years of experience teaching advanced undergraduates and first-year graduate students in mathematics, physics, and engineering, is an attempt to meet that need. It is based essentially on a set of mimeographed lecture notes developed for courses given by the author at the University of Central Florida, East Carolina University, and the University of Calcutta.

This book is designed as an introduction to theory and applications of integral transforms to problems in linear differential equations, and to boundary and initial value problems in partial differential equations. It is appropriate for a one-semester course. There are two basic prerequisites for the course: a standard calculus sequence and ordinary differential equations. The book assumes only a limited knowledge of complex variables and contour integration, partial differential equations, and continuum mechanics. Many new examples of applications dealing with problems in applied mathematics, physics, chemistry, biology, and engineering are included. It is *not* essential for the reader to know everything about these topics, but limited knowledge of at least some of them would be useful. Besides, the book is intended to serve as a reference work for those seriously interested in advanced study and research in the subject, whether for its own sake or for its applications to other fields of applied mathematics, mathematical physics, and engineering.

The first chapter gives a brief historical introduction and the basic ideas of integral transforms. The second chapter deals with the theory and applications of Fourier transforms, and of Fourier cosine and sine transforms. Important examples of applications of interest in applied mathematics, physics statistics, and engineering are included. The theory and applications of Laplace transforms are discussed in Chapters 3 and 4 in considerable detail. The fifth chapter is concerned with the operational calculus of Hankel transforms with applications. Chapter 6 gives a detailed treatment of Mellin transforms and its various applications. Included are Mellin transforms of the Weyl fractional integral, Weyl fractional derivatives, and generalized Mellin transforms. Hilbert and Stieltjes transforms and their applications are discussed in Chapter 7.

Chapter 8 provides a short introduction to finite Fourier cosine and sine transforms and their basic operational properties. Applications of these transforms are also presented. The finite Laplace transform and its applications to boundary value problems are included in Chapter 9. Chapter 10 deals with a detailed theory and applications of Z transforms.

Chapter 12 is devoted to the operational calculus of Legendre transforms and their applications to boundary value problems in potential theory. Jacobi and Gegenbauer transforms and their applications are included in Chapter 13. Chapter 14 deals with the theory and applications of Laguerre transforms. The final chapter is concerned with the Hermite transform and its basic operational properties including the Convolution Theorem. Most of the material of these chapters has been developed since the early sixties and appears here in book form for the first time.

The book includes two important appendices. The first one deals with several special functions and their basic properties. The second appendix includes *thirteen* short tables of integral transforms. Many standard texts and reference books and a set of selected classic and recent research papers are included in the Bibliography that will be very useful for the reader interested in learning more about the subject.

The book contains 750 worked examples, applications, and exercises which include some that have been chosen from many standard books as well as recent papers. It is hoped that they will serve as helpful self-tests for understanding of the theory and mastery of the transform methods. These examples of applications and exercises were chosen from the areas of differential and difference equations, electric circuits and networks, vibration and wave propagation, heat conduction in solids, quantum mechanics, fractional calculus and fractional differential equations, dynamical systems, signal processing, integral equations, physical chemistry, mathematical biology, probability and statistics, and solid and fluid mechanics. This varied number of examples and exercises should provide something of interest for everyone. The exercises truly complement the text and range from the elementary to the challenging. Answers and hints to many selected exercises are provided at the end of the book.

This is a *text* and a *reference* book designed for use by the student and the reader of mathematics, science, and engineering. A serious attempt has been made to present almost all the standard material, and some new material as well. Those interested in more advanced rigorous treatment of the topics covered may consult standard books and treatises by Churchill, Doetsch, Sneddon, Titchmarsh, and Widder listed in the Bibliography. Many ideas, results, theorems, methods, problems, and exercises presented in this book are either motivated by or borrowed from the works cited in the Bibliography. The author wishes to acknowledge his gratitude to the authors of these works.

This book is designed as a new source for both classical and modern topics dealing with integral transforms and their applications for the future development of this useful subject. Its main features are:

1. A systematic mathematical treatment of the theory and method of integral transforms that gives the reader a clear understanding of the subject and its varied applications.

2. A detailed and clear explanation of every concept and method that is introduced, accompanied by carefully selected worked examples, with special emphasis being given to those topics in which students experience difficulty.

3. A wide variety of diverse examples of applications carefully selected from areas of applied mathematics, mathematical physics, and engineering science to provide motivation, and to illustrate how operational methods can be applied effectively to solve them.

4. A broad coverage of the essential standard material on integral transforms and their applications together with some new material that is *not* usually covered in familiar texts or reference books.

5. Most of the recent developments in the subject since the early sixties appear here in book form for the first time.

6. A wide spectrum of exercises has been carefully selected and included at the end of each chapter so that the reader may further develop both manipulative skills in the applications of integral transforms and a deeper insight into the subject.

7. Two appendices have been included in order to make the book self-contained.

8. Answers and hints to selected exercises are provided at the end of the book for additional help to students.

9. An updated Bibliography is included to stimulate new interest in future study and research.

In preparing the book, the author has been encouraged by and has benefited from the helpful comments and criticism of a number of graduate students and faculty of several universities in the United States, Canada, and India. The author expresses his grateful thanks to all these individuals for their interest in the book. My special thanks to Jackie Callahan and Ronee Trantham who typed the manuscript and cheerfully put up with constant changes and revisions. In spite of the best efforts of everyone involved, some typographical errors doubtlessly remain. I do hope that these are both few and obvious, and will cause minimal confusion. The author also wishes to thank his friends and colleagues including Drs. Sudipto Roy Choudhury and Carroll A. Webber for their interest and help during the preparation of the book. Finally, the author wishes to express his special thanks to Dr. Wayne Yuhasz, Executive Editor, and the staff of CRC Press for their help and cooperation. I am also deeply indebted to my wife, Sadhana, for all her understanding and tolerance while the book was being written.

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# About the Authors

**Lokenath Debnath** is Professor and Chair of the Department of Mathematics at the University of Texas-Pan American, Edinburg, Texas.

Professor Debnath received his M.Sc. and Ph.D. degrees in pure mathematics from the University of Calcutta, and obtained his D.I.C. and Ph.D. degrees in applied mathematics from the Imperial College of Science and Technology, London. He was a Senior Research Fellow at the Department of Applied Mathematics and Theoretical Physics at the University of Cambridge and has had several visiting appointments at the University of Oxford, Florida State University, University of Maryland, and the University of Calcutta. He served the University of Central Florida as Professor and Chair of Mathematics and as Professor of Mechanical and Aerospace Engineering from 1983 to 2001. He was Acting Chair of the Department of Statistics at the University of Central Florida and has served as Professor of Mathematics and Professor of Physics at East Carolina University for a period of fifteen years.

Among many other honors and awards, he has received a Senior Fulbright Fellowship and an NSF Scientist Award to visit India for lectures and research. He was a University Grants Commission Research Professor at the University of Calcutta and was elected President of the Calcutta Mathematical Society for a period of three years. He has served as a Lecturer of the SIAM Visiting Lecturer Program and as a Visiting Speaker of the Mathematical Association of America (MAA) from 1990. He also has served as organizer of several professional meetings and conferences at regional, national, and international levels; and as Director of six NSF-CBMS research conferences at the University of Central Florida, and East Carolina University and the University of Texas-Pan American. He has received many grants from NSF and state agencies of North Carolina, Florida, and Texas. He has also received many university awards for teaching, research, services and leadership.

Dr. Debnath is author or co-author of ten graduate level books and research monographs, including the third edition of *Introduction to Hilbert Spaces with Applications, Nonlinear Water Waves, Continuum Mechanics* published by Academic Press, the fourth edition of *Linear Partial Differential Equations for Scientists and Engineers* published by Birkhauser Verlag, the second edition of *Nonlinear Partial Differential Equations for Scientists and Engineers,* and *Wavelet Transforms and Their Applications* published by Birkhauser Verlag. He has also edited eleven research monographs including *Nonlinear Waves* published by Cambridge University Press. He is an author or co-author of over 300 research papers in pure and applied mathematics, including applied partial differential equations, integral transforms and special functions, history of mathematics, mathematical inequalities, wavelet transforms, solid and fluid mechanics, linear and nonlinear waves, solitons, mathematical physics, magnetohydrodynamics, unsteady boundary layers, dynamics of oceans, and stability theory.

Professor Debnath is a member of many scientific organizations at both national and international levels. He has been an Associate Editor and a member of the editorial boards of many refereed journals and he currently serves on the Editorial Board of many refereed journals including *Journal of Mathematical Analysis and Applications, Indian Journal of Pure and Applied Mathematics, Fractional Calculus and Applied Analysis, Bulletin of the Calcutta Mathematical Society, Integral Transforms and Special Functions, International Journal of Engineering Science,* and *International Journal of Mathematical Education in Science and Technology.* He is the current and founding Managing Editor of the *International Journal of Mathematics and Mathematical Sciences.*

Dr. Debnath delivered twenty five invited lectures at national and international conferences, presented over 200 research papers at national and international professional meetings, and given over 250 seminar and colloquium lectures at universities and institutes in the United States and abroad.

**Dambaru Bhatta** is an Assistant Professor of Mathematics at the University of Texas-Pan American, Edinburg, Texas.

Dr. Bhatta received his Ph.D. degree in applied mathematics from Dalhousie University, Halifax, Canada and M.Sc. degree in mathematics from the University of Delhi. He had worked with various companies in Montreal, Ottawa, and Atlanta. His research interests include wave-structure interaction, computational mathematics, finite element method, nonlinear partial differential equations, fractional calculus, and fractional differential equations.

# Contents

# 1

# *Integral Transforms*

"The thorough study of nature is the most fertile ground for mathematical discoveries."

Joseph Fourier

"If you wish to foresee the future of mathematics our proper course is to study the history and present condition of the science."

Henri Poincaré

"The tool which serves as intermediary between theory and practice, between thought and observation, is mathematics, it is mathematics which builds the linking bridges and gives the ever more reliable forms. From this it has come about that our entire contemporary culture, in as much as it is based the intellectual penetration and the exploitation of nature, has its foundations in mathematics."

David Hilbert

## 1.1 Brief Historical Introduction

Integral transformations have been successfully used for almost two centuries in solving many problems in applied mathematics, mathematical physics, and engineering science. Historically, the origin of the integral transforms including the Laplace and Fourier transforms can be traced back to celebrated work of P. S. Laplace (1749–1827) on probability theory in the 1780s and to monumental treatise of Joseph Fourier (1768–1830) on *La Théorie Analytique de la Chaleur* published in 1822. In fact, Laplace's classic book on *La Théorie Analytique des Probabilities* includes some basic results of the Laplace transform which is one of the oldest and most commonly used integral transforms available in the mathematical literature. This has effectively been used in finding the solution of linear differential equations and integral equations. On the other hand, Fourier's treatise provided the modern mathematical theory of heat conduction, Fourier series, and Fourier integrals with applications. In his treatise, Fourier stated a remarkable result that is universally known as the *Fourier Integral Theorem*. He gave a series of examples before stating that an arbitrary function defined on a finite interval can be expanded in terms of trigonometric series which is now universally known as the *Fourier series*. In an attempt to extend his new ideas to functions defined on an infinite interval, Fourier discovered an integral transform and its inversion formula which are now well known as the *Fourier transform* and the *inverse Fourier transform*. However, this celebrated idea of Fourier was known to Laplace and A. L. Cauchy (1789–1857) as some of their earlier work involved this transformation. On the other hand, S. D. Poisson (1781–1840) also independently used the method of transform in his research on the propagation of water waves.

However, it was G. W. Leibniz (1646–1716) who first introduced the idea of a symbolic method in calculus. Subsequently, both J. L. Lagrange (1736–1813) and Laplace made considerable contributions to symbolic methods which became known as operational calculus. Although both the Laplace and the Fourier transforms have been discovered in the nineteenth century, it was the British electrical engineer Oliver Heaviside (1850–1925) who made the Laplace transform very popular by using it to solve ordinary differential equations of electrical circuits and systems, and then to develop modern operational calculus. It may be relevant to point out that the Laplace transform is essentially a special case of the Fourier transform for a class of functions defined on the positive real axis, but it is more simple than the Fourier transform for the following reasons. First, the question of convergence of the Laplace transform is much less delicate because of its exponentially decaying kernel $\exp(-st)$, where $\operatorname{Re} s > 0$ and $t > 0$. Second, the Laplace transform is an analytic function of the complex variable and its properties can easily be studied with the knowledge of the theory of complex variable. Third, the Fourier integral formula provided the definitions of the Laplace transform and the inverse Laplace transform in terms of a complex contour integral that can be evaluated with the help the Cauchy residue theory and deformation of contour in the complex plane.

It was the work of Cauchy that contained the exponential form of the Fourier Integral Theorem as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(x-y)} f(y)\,dy\,dk. \tag{1.1.1}$$

Cauchy's work also contained the following formula for functions of the operator $D$:

$$\phi(D)f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(ik)e^{ik(x-y)} f(y)\,dy\,dk. \tag{1.1.2}$$

This essentially led to the modern form of the operational calculus. His famous treatise entitled *Memoire sur l'Emploi des Equations Symboliques* provided a fairly rigorous description of symbolic methods. The deep significance of the Fourier Integral Theorem was recognized by mathematicians and mathematical physicists of the nineteenth and twentieth centuries. Indeed, this theorem is regarded as one of the most fundamental results of modern mathematical analysis and has widespread physical and engineering applications. The generality and importance of the theorem is well expressed by Kelvin and Tait who said: "...Fourier's Theorem, which is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics. To mention only sonorous vibrations, the propagation of electric signals along a telegraph wire, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance."

During the late nineteenth century, it was Oliver Heaviside (1850–1925) who recognized the power and success of operational calculus and first used the operational method as a powerful and effective tool for the solutions of telegraph equation and the second order hyperbolic partial differential equations with constant coefficients. In his two papers entitled "On Operational Methods in Physical Mathematics," Parts I and II, published in *The Proceedings of the Royal Society*, London, in 1892 and 1893, Heaviside developed operational methods. His 1899 book on *Electromagnetic Theory* also contained the use and application of the operational methods to the analysis of electrical circuits or networks. Heaviside replaced the differential operator $D \equiv \frac{d}{dt}$ by $p$ and treated the latter as an element of the ordinary laws of algebra. The development of his operational methods paid little attention to questions of mathematical rigor. The widespread use of the Heaviside method prior to its vindication by the theory of the Fourier or Laplace transform created a lot of controversy. This was similar to the controversy put forward against the widespread use of the delta function as one of the most useful mathematical devices in Dirac's logical formulation of quantum mechanics during the 1920s. In fact, P. A. M. Dirac (1902–1984) said: "All electrical engineers are familiar with the idea of a pulse, and the $\delta$-function is just a way of expressing a pulse mathematically." Dirac's study of Heaviside's operator calculus in electromagnetic theory, his training as an electrical engineer, and his deep knowledge of the modern theory of electrical pulses seemed to have a tremendous impact on his ingenious development of modern quantum mechanics.

Apparently, the ideas of operational methods originated from the classic work of Laplace, Fourier, and Cauchy. Inspired by this remarkable work, Heaviside developed his new but less rigorous operational mathematics. In spite of the striking success of Heaviside's calculus as one of the most useful mathematical methods, contemporary mathematicians hardly recognized Heaviside's work in his lifetime, primarily due to lack of mathematical rigor. In his lecture on Heaviside and Operational Calculus at the Birth Centenary of Oliver Heaviside, J. L. B. Cooper (1952) revealed some of the controversial issues surrounding Heaviside's celebrated work, and declared: "As a mathematician he was gifted with manipulative skill and with a genius for finding convenient methods of calculation. He simplified Maxwell's theory enormously; according to Hertz, the four equations known as Maxwell's were first given by Heaviside. He is one of the founders of vector analysis...." Reviewing the history of Heaviside's calculus, Cooper gave a fairly complete account of early history of the subject along with mathematicians' varying opinions about Heaviside's contributions to operational calculus. According to Cooper, a widely publicized story that operational calculus was discovered by Heaviside remained controversial. In spite of the controversies, it is generally believed that Heaviside's real achievement was to develop operational calculus, which is one of the most useful mathematical devices in applied mathematics, mathematical physics, and engineering science. In this context Lord Rayleigh's following quotation seems to be most appropriate from a physical point of view: "In the mathematical investigation I have usually employed such methods as present themselves naturally to a physicist. The pure mathematician will complain, and (it must be confessed) sometimes with justice, of deficient rigor. But to this question there are two sides. For, however important it may be to maintain a uniformly high standard in pure mathematics, the physicist may occasionally do well to rest content with arguments which are fairly satisfactory and conclusive from his point of view. To his mind, exercised in a different order of ideas, the more severe procedure of the pure mathematician may appear not more but less demonstrative. And further, in many cases of difficulty to insist upon highest standard would mean the exclusion of the subject altogether in view of the space that would be required."

With the exception of a group of pure mathematicians, everyone has found Heaviside's work a remarkable achievement even though he did not provide a rigorous demonstration of his operational calculus. In defense of Heaviside, Richard P. Feynman's thought seems to be worth quoting. "However, the emphasis should be somewhat more on how to do the mathematics quickly and easily, and what formulas are true, rather than the mathematicians' interest in methods of rigorous proof." The development of operational calculus was somewhat similar to that of calculus of the seventeenth century. Mathematicians who invented the calculus did not provide a rigorous formulation of it. The rigorous formulation came only in the nineteenth century, even though in the transition the non-rigorous demonstration of the calculus that is still admired. It is well known that twentieth-century mathematicians have provided a rigorous foundation of the Heaviside operational calculus. So, by any standard, Heaviside deserves a lot of credit for his remarkable work.

The next phase of the development of operational calculus is characterized by the effort to provide justifications of the heuristic methods by rigorous proofs. In this phase, T. J. Bromwich (1875-1930) first successfully introduced the theory of complex functions to give formal justification of Heaviside's calculus. In addition to his many contributions to this subject, he gave the formal derivation of the Heaviside expansion theorem and the correct interpretation of Heaviside's operational results. After Bromwich's work, notable contributions to rigorous formulation of operational calculus were made by J. R. Carson, B. van der Pol, G. Doetsch, and many others.

In concluding our discussion on the historical development of operational calculus, we should add a note of caution against the controversial evaluation of Heaviside's work. From an applied mathematical point of view, Heaviside's operational calculus was an important achievement. In support of his statement, an assessment of Heaviside's work made by E. T. Whittaker in Heaviside's obituary is recorded below: "Looking back..., we should place the operational calculus with Poincaré's discovery of automorphic functions and Ricci's discovery of the tensor calculus as the three most important mathematical advances of the last quarter of the nineteenth century." Although Heaviside paid little attention to questions of mathematical rigor, he recognized that operational calculus is one of the most effective and useful mathematical methods in applied mathematical sciences. This has led naturally to rigorous mathematical analysis of integral transforms. Indeed, the *Fourier* or *Laplace transform* methods based on the rigorous mathematical foundation are essentially equivalent to the modern operational calculus.

There are many other integral transformations including the Mellin transform, the Hankel transform, the Hilbert transform and the Stieltjes transform which are widely used to solve initial and boundary value problems involving ordinary and partial differential equations and other problems in mathematics, science and engineering. Although, Mellin (1854–1933) presented an elaborate discussion of his transform and its inversion formula, it was G. Bernhard Riemann (1826–1866) who first recognized the Mellin transform and its inversion formula in his famous memoir on prime numbers. Hermann Hankel (1839–1873), a student of G. B. Riemann, introduced the Hankel transform with the Bessel function as its kernel, and this transform can easily be derived from the two-dimensional Fourier transform when circular symmetry is assumed. The Hankel transform arises naturally in solving boundary value problems in cylindrical polar coordinates.

Although the Hilbert transform was named after one of the greatest mathematicians of the twentieth century, David Hilbert (1862–1943), this transform and its properties are basically studied by G. H. Hardy (1877-1947) and E. C. Titchmarsh (1899-1963). The Dutch mathematician, T. J. Stieltjes (1856–1894) introduced the Stieltjes transform in his study of continued fractions. Both the Hilbert and Stieltjes transforms arise in many problems in mathematics, science and engineering. The former is used to solve problems in fluid mechanics, signal processing, and electronics, while the latter arises in solving the integral equations and moment problems.

We would like to conclude this section by making some comments on the history of the Radon transform, the Gabor transform and the wavelet transform. The Radon transform is introduced by Johann Radon (1887–1956) in 1917 and has enormous useful applications to medical imaging, and computer assisted tomography (CAT). The wavelet transform is discovered by Jean Morlet, a French geophysical engineer, as a new mathematical tool to study seismic signal analysis in 1982. It is one of the most versatile linear integral transformations and can be applied to solve a wide variety of problems in mathematics, science and engineering. The reader is referred to Chapter 19 of this book for more detailed information on wavelets and wavelet transforms.

## 1.2 Basic Concepts and Definitions

The *integral transform* of a function $f(x)$ defined in $a \le x \le b$ is denoted by $\mathscr{I}\{f(x)\} = F(k)$, and defined by

$$\mathscr{I}\{f(x)\} = F(k) = \int_a^b K(x,k) f(x) dx, \tag{1.2.1}$$

where $K(x,k)$, given function of two variables $x$ and $k$, is called the *kernel* of the transform. The operator $\mathscr{I}$ is usually called an *integral transform operator* or simply an *integral transformation*. The transform function $F(k)$ is often referred to as the *image* of the given object function $f(x)$, and $k$ is called the *transform variable*.

Similarly, the integral transform of a function of several variables is defined by

$$\mathscr{I}\{f(\mathbf{x})\} = F(\boldsymbol{\kappa}) = \int_S K(\mathbf{x}, \boldsymbol{\kappa}) f(\mathbf{x}) d\boldsymbol{x}, \tag{1.2.2}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{\kappa} = (k_1, k_2, \ldots, k_n)$, and $S \subset R^n$.

A mathematical theory of transformations of this type can be developed by using the properties of Banach spaces. From a mathematical point of view, such a program would be of great interest, but it may *not* be useful for practical applications. Our goal here is to study integral transforms as operational methods with special emphasis to applications.

The idea of the integral transform operator is somewhat similar to that of the well-known linear differential operator, $D \equiv \frac{d}{dx}$, which acts on a function $f(x)$ to produce another function $f'(x)$, that is,

$$Df(x) = f'(x). \tag{1.2.3}$$

Usually, $f'(x)$ is called the *derivative* or the image of $f(x)$ under the linear transformation $D$.

Evidently, there are a number of important integral transforms including *Fourier*, *Laplace*, *Hankel*, and *Mellin* transforms. They are defined by choosing different kernels $K(x,k)$ and different values for $a$ and $b$ involved in (1.2.1).

Obviously, $\mathscr{I}$ is a *linear operator* since it satisfies the property of *linearity*:

$$\begin{aligned}\mathscr{I}\{\alpha f(x)+\beta g(x)\} &= \int_a^b \{\alpha f(x)+\beta g(x)\}K(x,k)dx \\ &= \alpha\,\mathscr{I}\{f(x)\}+\beta\,\mathscr{I}\{g(x)\},\end{aligned} \tag{1.2.4}$$

where $\alpha$ and $\beta$ are arbitrary constants. In order to obtain $f(x)$ from a given $F(k)=\mathscr{I}\{f(x)\}$, we introduce the *inverse operator* $\mathscr{I}^{-1}$ such that

$$\mathscr{I}^{-1}\{F(k)\}=f(x). \tag{1.2.5}$$

Accordingly $\mathscr{I}^{-1}\mathscr{I}=\mathscr{I}\mathscr{I}^{-1}=\mathbf{1}$ which is the identity operator. It can be proved that $\mathscr{I}^{-1}$ is also a linear operator as follows

$$\begin{aligned}\mathscr{I}^{-1}\{\alpha\,F(k)+\beta\,G(k)\} &= \mathscr{I}^{-1}\{\alpha\,\mathscr{I}f(x)+\beta\,\mathscr{I}g(x)\} \\ &= \mathscr{I}^{-1}\{\mathscr{I}[\alpha\,f(x)+\beta\,g(x)]\} \\ &= \alpha\,f(x)+\beta\,g(x) \\ &= \alpha\,\mathscr{I}^{-1}\{F(k)\}+\beta\,\mathscr{I}^{-1}\{G(k)\}.\end{aligned}$$

It can also be proved that the integral transform is unique. In other words, if $\mathscr{I}\{f(x)\}=\mathscr{I}\{g(x)\}$, then $f(x)=g(x)$ under suitable conditions. This is known as the *uniqueness theorem.*

We close this section by adding the basic scope and applications of integral transformation from a general point of view. It follows from the above discussion that an integral transformation simply means a unique mathematical operation through which a real or complex-valued function $f$ is transformed into another new function $F=\mathscr{I}f$, or into a set of data that can be measured (or observed) experimentally. Thus, the importance of the integral transform is that it transforms a difficult mathematical problem to an relatively easy problem, which can easily be solved. In the study of initial-boundary value problem involving differential equations, the differential operators are replaced by much simpler algebraic operations involving $F$, which can readily be solved. The solution of the original problem is then obtained in the original variables by the inverse transformation. So, the next basic problem leads to the computation of the inverse integral transform exactly or approximately. Indeed, in order to make the integral transform method effective, it is essential to reconstruct $f$ from $\mathscr{I}f=F$ which is, in general, a difficult step in practice. However, this difficulty can be resolved in many different ways. In applications, often the transform function $F$ itself has some physical meaning and needs to be studied in its own right. For example, in electrical engineering problems, the original function $f(t)$ may represent a signal that is a function of time $t$. The Fourier transform $F(\omega)$ of $f(t)$ represents the frequency spectrum of the signal $f(t)$ and it is physically useful as the time representation of the signal itself. Indeed, it is often more important to work with $F$ rather than with $f$. Conversely, given the frequency spectrum, $F(\omega)$, the original signal $f(t)$ can be reconstructed by the inverse Fourier transform.

Other important and major examples include the Gabor transform and the wavelet transform both of which transform a signal $f(t)$ in the time-frequency domain ($t-\omega$ plane). In other words, these new transforms convey essential information about the nature and structure of a signal in the time-frequency domain simultaneously. In 1946, Dennis Gabor, a Hungarian-British physicist and engineer and a 1971 Nobel Prize winner in physics, introduced the *windowed Fourier transform* (or the *Gabor transform*) of a signal $f(t)$ with respect to a window function $g$, denoted by $\widetilde{f}_g(t,\omega)$ and defined by

$$\begin{aligned}\mathscr{G}[f](t,\omega)=\widetilde{f}_g(t,\omega)&=\int_{-\infty}^{\infty} f(\tau)\,g(\tau-t)\,e^{-i\omega t}d\tau\\ &=\left\langle f,\,\overline{g}_{t,\omega}\right\rangle, \qquad (1.2.6)\end{aligned}$$

where $f$ and $g\in L^2(\mathbb{R})$ with the inner product $\langle f,g\rangle$.

Gabor (1900–1979) first recognized the major weaknesses of the Fourier transform analysis of signals, and also realized the great importance of localized time and frequency concentrations in signal processing. All these motivated him to formulate a fundamental method of the Gabor transform for decomposition of signal in terms of elementary signals (or wave transforms). Gabor's pioneering approach has now become one of the standard models for time-frequency signal analysis. It is also important to point out that the Gabor transform $\widetilde{f}_g(t,\omega)$ is referred to as the canonical coherent state representation of $f$ in quantum mechanics. In the 1960s, the term "coherent states" was first used in quantum optics. For more information on the Gabor and the wavelet transforms and their basic properties, the reader is referred to Debnath (2002).

# 2

# *Fourier Transforms and Their Applications*

"The profound study of nature is the most fertile source of mathematical discoveries."

Joseph Fourier

"The theory of Fourier series and integrals has always had major difficulties and necessitated a large mathematical apparatus in dealing with questions of convergence. It engendered the development of methods of summation, although these did not lead to a completely satisfactory solution of the problem. .... For the Fourier transform, the introduction of distributions (hence, the space **S** ) is inevitable either in an explicit or hidden form. .... As a result one may obtain all that is desired from the point of view of the continuity and inversion of the Fourier transform."

Laurent Schwartz

## 2.1 Introduction

Many linear boundary value and initial value problems in applied mathematics, mathematical physics, and engineering science can be effectively solved by the use of the Fourier transform, the Fourier cosine transform, or the Fourier sine transform. These transforms are very useful for solving differential or integral equations for the following reasons. First, these equations are replaced by simple algebraic equations, which enable us to find the solution of the transform function. The solution of the given equation is then obtained in the original variables by inverting the transform solution. Second, the Fourier transform of the elementary source term is used for determination of the fundamental solution that illustrates the basic ideas behind the construction and implementation of Green's functions. Third, the transform solution combined with the convolution theorem provides an elegant representation of the solution for the boundary value and initial value problems.

We begin this chapter with a formal derivation of the Fourier integral formulas. These results are then used to define the Fourier, Fourier cosine, and Fourier sine transforms. This is followed by a detailed discussion of the basic operational properties of these transforms with examples. Special attention is given to convolution and its main properties. Sections 2.10 and 2.11 deal with applications of the Fourier transform to the solution of ordinary differential equations and integral equations. In Section 2.12, a wide variety of partial differential equations are solved by the use of the Fourier transform method. The technique that is developed in this and other sections can be applied with little or no modification to different kinds of initial and boundary value problems that are encountered in applications. The Fourier cosine and sine transforms are introduced in Section 2.13. The properties and applications of these transforms are discussed in Sections 2.14 and 2.15. This is followed by evaluation of definite integrals with the aid of Fourier transforms. Section 2.17 is devoted to applications of Fourier transforms in mathematical statistics. The multiple Fourier transforms and their applications are discussed in Section 2.18.

## 2.2 The Fourier Integral Formulas

A function $f(x)$ is said to satisfy *Dirichlet's conditions* in the interval $-a < x < a$, if

(i) $f(x)$ has only a finite number of finite discontinuities in $-a < x < a$ and has no infinite discontinuities.

(ii) $f(x)$ has only a finite number of maxima and minima in $-a < x < a$.

From the theory of Fourier series we know that if $f(x)$ satisfies the Dirichlet conditions in $-a < x < a$, it can be represented as the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \exp(in\pi x/a), \tag{2.2.1}$$

where the coefficients are

$$a_n = \frac{1}{2a}\int_{-a}^{a} f(\xi)\exp(-in\pi\xi/a)d\xi. \tag{2.2.2}$$

This representation is evidently periodic of period $2a$ in the interval. However, the right hand side of (2.2.1) cannot represent $f(x)$ *outside* the interval $-a < x < a$ unless $f(x)$ is periodic of period $2a$. Thus, problems on finite intervals lead to Fourier series, and problems on the whole line $-\infty < x < \infty$ lead to the

Fourier integrals. We now attempt to find an integral representation of a non-periodic function $f(x)$ in $(-\infty, \infty)$ by letting $a \to \infty$. As the interval grows $(a \to \infty)$ the values $k_n = \frac{n\pi}{a}$ become closer together and form a dense set. If we write $\delta k = (k_{n+1} - k_n) = \frac{\pi}{a}$ and substitute coefficients $a_n$ into (2.2.1), we obtain

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (\delta k) \left[ \int_{-a}^{a} f(\xi) \exp(-i\xi k_n) d\xi \right] \exp(ixk_n). \tag{2.2.3}$$

In the limit as $a \to \infty$, $k_n$ becomes a continuous variable $k$ and $\delta k$ becomes $dk$. Consequently, the sum can be replaced by the integral in the limit and (2.2.3) reduces to the result

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\xi) e^{-ik\xi} d\xi \right] e^{ikx} dk. \tag{2.2.4}$$

This is known as the celebrated *Fourier integral formula*. Although the above arguments do not constitute a rigorous proof of (2.2.4), the formula is correct and valid for functions that are piecewise continuously differentiable in every finite interval and is absolutely integrable on the whole real line.

A function $f(x)$ is said to be *absolutely integrable* on $(-\infty, \infty)$ if

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty \tag{2.2.5}$$

exists.

It can be shown that the formula (2.2.4) is valid under more general conditions. The result is contained in the following theorem:

***THEOREM 2.2.1***

If $f(x)$ satisfies Dirichlet's conditions in $(-\infty, \infty)$, and is absolutely integrable on $(-\infty, \infty)$, then the Fourier integral (2.2.4) converges to the function $\frac{1}{2}[f(x+0) + f(x-0)]$ at a finite discontinuity at $x$. In other words,

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \left[ \int_{-\infty}^{\infty} f(\xi) e^{-ik\xi} d\xi \right] dk. \tag{2.2.6}$$

This is usually called the *Fourier integral theorem.*

If the function $f(x)$ is continuous at point $x$, then $f(x+0) = f(x-0) = f(x)$, then (2.2.6) reduces to (2.2.4).

The Fourier integral theorem was originally stated in Fourier's famous treatise entitled *La Théorie Analytique da la Chaleur* (1822), and its deep significance was recognized by mathematicians and mathematical physicists. Indeed, this theorem is one of the most monumental results of modern mathematical analysis and has widespread physical and engineering applications.

We express the exponential factor $\exp[ik(x-\xi)]$ in (2.2.4) in terms of trigonometric functions and use the even and odd nature of the cosine and the sine functions respectively as functions of $k$ so that (2.2.4) can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty f(\xi)\cos k(x-\xi)d\xi. \tag{2.2.7}$$

This is another version of the *Fourier integral formula.* In many physical problems, the function $f(x)$ vanishes very rapidly as $|x| \to \infty$, which ensures the existence of the repeated integrals as expressed.

We now assume that $f(x)$ is an even function and expand the cosine function in (2.2.7) to obtain

$$f(x) = f(-x) = \frac{2}{\pi}\int_0^\infty \cos kx\, dk \int_0^\infty f(\xi)\cos k\xi\, d\xi. \tag{2.2.8}$$

This is called the *Fourier cosine integral formula.*

Similarly, for an odd function $f(x)$, we obtain the *Fourier sine integral formula*

$$f(x) = -f(-x) = \frac{2}{\pi}\int_0^\infty \sin\, kx\, dk \int_0^\infty f(\xi)\sin k\xi\, d\xi. \tag{2.2.9}$$

These integral formulas were discovered independently by Cauchy in his work on the propagation of waves on the surface of water.

## 2.3 Definition of the Fourier Transform and Examples

We use the Fourier integral formula (2.2.4) to give a formal definition of the Fourier transform.

***DEFINITION 2.3.1*** *The Fourier transform of $f(x)$ is denoted by $\mathscr{F}\{f(x)\} = F(k)$, $k \in \mathbb{R}$, and defined by the integral*

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-ikx} f(x)dx, \tag{2.3.1}$$

*where $\mathscr{F}$ is called the Fourier transform operator or the Fourier transformation and the factor $\frac{1}{\sqrt{2\pi}}$ is obtained by splitting the factor $\frac{1}{2\pi}$ involved in*

*(2.2.4). This is often called the complex Fourier transform. A sufficient condition for $f(x)$ to have a Fourier transform is that $f(x)$ is absolutely integrable on $(-\infty, \infty)$. The convergence of the integral (2.3.1) follows at once from the fact that $f(x)$ is absolutely integrable. In fact, the integral converges uniformly with respect to $k$.*

Thus, the definition of the Fourier transform is restricted to absolutely integrable functions. This restriction is too strong for many physical applications. Many simple and common functions, such as constant function, trigonometric functions $\sin ax$, $\cos ax$, exponential functions, and $x^n H(x)$ do not have Fourier transforms, even though they occur frequently in applications. The integral in (2.3.1) fails to converge when $f(x)$ is one of the above elementary functions. This is a very unsatisfactory feature of the theory of Fourier transforms. However, this unsatisfactory feature can be resolved by means of a natural extension of the definition of the Fourier transform of a generalized function, $f(x)$ in (2.3.1). We follow Lighthill (1958) and Jones (1982) to discuss briefly the theory of the Fourier transforms of good functions.

*The inverse Fourier transform, denoted by $\mathscr{F}^{-1}\{F(k)\} = f(x)$, is defined by*

$$\mathscr{F}^{-1}\{F(k)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, F(k)\, dk, \tag{2.3.2}$$

*where $\mathscr{F}^{-1}$ is called the inverse Fourier transform operator.*

Clearly, both $\mathscr{F}$ and $\mathscr{F}^{-1}$ are linear integral operators. In applied mathematics, $x$ usually represents a space variable and $k(=\frac{2\pi}{\lambda})$ is a wavenumber variable where $\lambda$ is the wavelength. However, in electrical engineering, $x$ is replaced by the time variable $t$ and $k$ is replaced by the frequency variable $\omega(=2\pi\nu)$ where $\nu$ is the frequency in cycles per second. The function $F(\omega) = \mathscr{F}\{f(t)\}$ is called the *spectrum* of the *time signal function* $f(t)$. In electrical engineering literature, the Fourier transform pairs are defined slightly differently by

$$\mathscr{F}\{f(t)\} = F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi\nu i t} dt, \tag{2.3.3}$$

and

$$\mathscr{F}^{-1}\{F(\nu)\} = f(t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i\nu t} d\nu = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega, \tag{2.3.4}$$

where $\omega = 2\pi\nu$ is called the *angular frequency*. The Fourier integral formula implies that any function of time $f(t)$ that has a Fourier transform can be equally specified by its spectrum. Physically, the signal $f(t)$ is represented as an integral superposition of an infinite number of sinusoidal oscillations with different frequencies $\omega$ and complex amplitudes $\frac{1}{2\pi}F(\omega)$. Equation (2.3.4) is called the *spectral resolution* of the signal $f(t)$, and $\frac{F(\omega)}{2\pi}$ is called the *spectral density*. In summary, the Fourier transform maps a function ( or signal) of time $t$ to a function of frequency $\omega$. In the same way as the Fourier series expansion of a periodic function decomposes the function into harmonic components, the Fourier transform generates a function (or signal) of a continuous variable whose value represents the frequency content of the original signal. This led to the successful use of the Fourier transform to analyze the form of time-varying signals in electrical engineering and seismology.

Next we give examples of Fourier transforms.

***Example 2.3.1***

Find the Fourier transform of $\exp(-ax^2)$. In fact, we prove

$$F(k) = \mathscr{F}\{\exp(-ax^2)\} = \frac{1}{\sqrt{2a}}\exp\left(-\frac{k^2}{4a}\right), \quad a > 0. \tag{2.3.5}$$

Here we have, by definition,

$$\begin{aligned} F(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx-ax^2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left[-a\left(x+\frac{ik}{2a}\right)^2 - \frac{k^2}{4a}\right]dx \\ &= \frac{1}{\sqrt{2\pi}}\exp(-k^2/4a)\int_{-\infty}^{\infty} e^{-ay^2}\,dy = \frac{1}{\sqrt{2a}}\exp\left(-\frac{k^2}{4a}\right), \end{aligned}$$

in which the change of variable $y = x + \frac{ik}{2a}$ is used. The above result is correct, but the change of variable can be justified by the method of complex analysis because $(ik/2a)$ is complex. If $a = \frac{1}{2}$

$$\mathscr{F}\{e^{-x^2/2}\} = e^{-k^2/2}. \tag{2.3.6}$$

This shows $\mathscr{F}\{f(x)\} = f(k)$. Such a function is said to be *self-reciprocal* under the Fourier transformation. Graphs of $f(x) = \exp(-ax^2)$ and its Fourier transform is shown in Figure 2.1 for $a = 1$. ▯

***Example 2.3.2***

Find the Fourier transform of $\exp(-a|x|)$, i.e.,

$$\mathscr{F}\{\exp(-a|x|)\} = \sqrt{\frac{2}{\pi}}\cdot\frac{a}{(a^2+k^2)}, \quad a > 0. \tag{2.3.7}$$

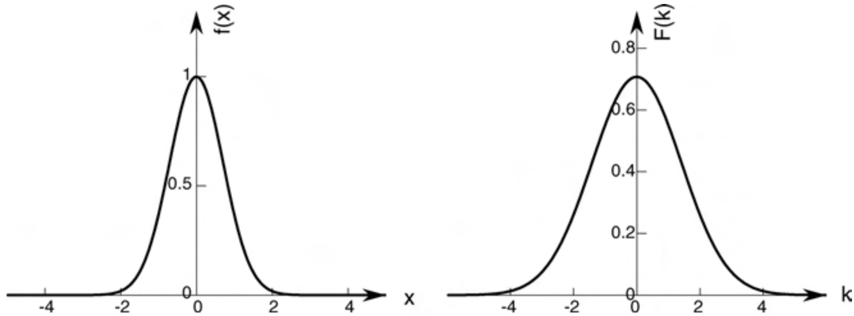


**Figure 2.1** Graphs of $f(x) = \exp(-ax^2)$ and $F(k)$ with $a = 1$.

Here we can write

$$\begin{aligned}\mathscr{F}\left\{e^{-a|x|}\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} e^{-(a+ik)x} dx + \int_{-\infty}^{0} e^{(a-ik)x} dx\right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a+ik} + \frac{1}{a-ik}\right] = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)}.\end{aligned}$$

We note that $f(x) = \exp(-a|x|)$ decreases rapidly at infinity, it is not differentiable at $x = 0$. Graphs of $f(x) = \exp(-a|x|)$ and its Fourier transform is displayed in Figure 2.2 for $a = 1$. ▯

***Example 2.3.3***

Find the Fourier transform of

$$f(x) = \left(1 - \frac{|x|}{a}\right) H\left(1 - \frac{|x|}{a}\right),$$

where $H(x)$ is the *Heaviside unit step function* defined by

$$H(x) = \left\{\begin{array}{ll} 1, & x > 0 \\ 0, & x < 0 \end{array}\right\}. \tag{2.3.8}$$

Or, more generally,

$$H(x-a) = \left\{\begin{array}{ll} 1, & x > a \\ 0, & x < a \end{array}\right\}, \tag{2.3.9}$$

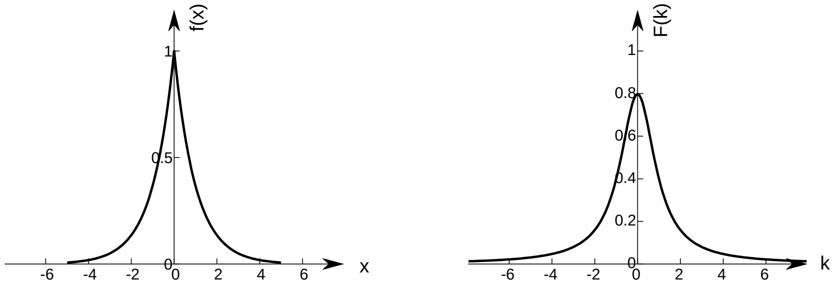


**Figure 2.2** Graphs of $f(x) = \exp(-a|x|)$ and $F(k)$ with $a = 1$.

where $a$ is a fixed real number. So the Heaviside function $H(x-a)$ has a finite discontinuity at $x = a$.

$$\begin{aligned}
\mathscr{F}\{f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx} \left(1 - \frac{|x|}{a}\right) dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{a} \left(1 - \frac{x}{a}\right) \cos kx \, dx \\
&= \frac{2a}{\sqrt{2\pi}} \int_{0}^{1} (1-x) \cos(akx) dx = \frac{2a}{\sqrt{2\pi}} \int_{0}^{1} (1-x) \frac{d}{dx} \left(\frac{\sin akx}{ak}\right) dx \\
&= \frac{2a}{\sqrt{2\pi}} \int_{0}^{1} \frac{\sin(akx)}{ak} dx = \frac{a}{\sqrt{2\pi}} \int_{0}^{1} \frac{d}{dx} \left[ \frac{\sin^2\left(\frac{akx}{2}\right)}{\left(\frac{ak}{2}\right)^2} \right] dx \\
&= \frac{a}{\sqrt{2\pi}} \frac{\sin^2\left(\frac{ak}{2}\right)}{\left(\frac{ak}{2}\right)^2}. \qquad (2.3.10)
\end{aligned}$$

▯

***Example 2.3.4***

Find the Fourier transform of the characteristic function $\chi_{[-a,a]}(x)$, where

$$\chi_{[-a,a]}(x) = H(a - |x|) = \left\{ \begin{array}{ll} 1, & |x| < a \\ 0, & |x| > a \end{array} \right\}. \qquad (2.3.11)$$

We have

$$F_a(k)=\mathscr{F}\{\chi_{[-a,a]}(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\chi_{[-a,a]}(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ikx}\,dx = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right). \quad (2.3.12)$$

Graphs of $f(x)=\chi_{[-a,a]}(x)$ and its Fourier transform are shown in Figure 2.3 for $a=1$.

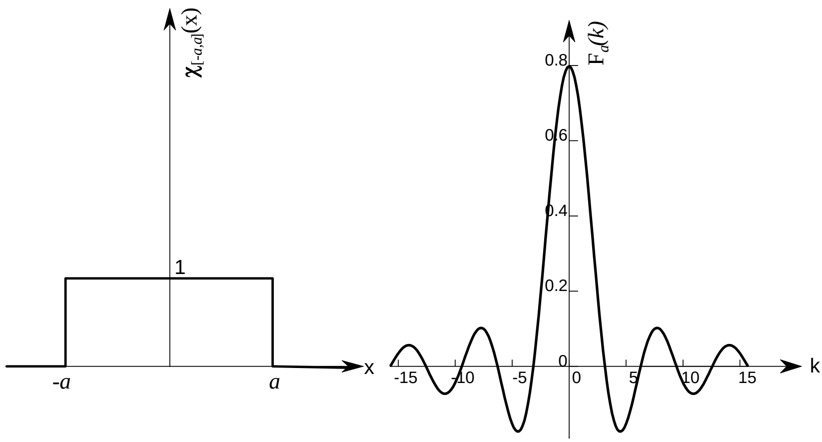


**Figure 2.3** Graphs of $\chi_{[-a,a]}(x)$ and $F_a(k)$ with $a=1$.

□

## 2.4 Fourier Transforms of Generalized Functions

The natural way to define the Fourier transform of a generalized function, is to treat $f(x)$ in (2.3.1) as a generalized function. The advantage of this is that every generalized function has a Fourier transform and an inverse Fourier transform, and that the ordinary functions whose Fourier transforms are of interest form a subset of the generalized functions. We would not go into great detail, but refer to the famous books of Lighthill (1958) and Jones (1982) for the introduction to the subject of generalized functions.

A *good function*, $g(x)$ is a function in $C^{\infty}(\mathbb{R})$ that decays sufficiently rapidly that $g(x)$ and all of its derivatives decay to zero faster than $|x|^{-N}$ as $|x| \to \infty$ for all $N > 0$.

**DEFINITION 2.4.1** *Suppose a real or complex valued function $g(x)$ is defined for all $x \in \mathbb{R}$ and is infinitely differentiable everywhere, and suppose that each derivative tends to zero as $|x| \to \infty$ faster that any positive power of $\left(x^{-1}\right)$, or in other words, suppose that for each positive integer $N$ and $n$,*

$$\lim_{|x|\to\infty} x^N g^{(n)}(x) = 0,$$

*then $g(x)$ is called a good function.*

Usually, the class of good functions is represented by $\mathbb{S}$. The good functions play an important role in Fourier analysis because the inversion, convolution, and differentiation theorems as well as many others take simple forms with no problem of convergence. The rapid decay and infinite differentiability properties of good functions lead to the fact that the Fourier transform of a good function is also a good function.

Good functions also play an important role in the theory of generalized functions. A good function of bounded support is a special type of good function that also plays an important part in the theory of generalized functions. Good functions also have the following important properties. The sum (or difference) of two good functions is also a good function. The product and convolution of two good functions are good functions. The derivative of a good function is a good function; $x^n g(x)$ is a good function for all non-negative integers $n$ whenever $g(x)$ is a good function. A good function belongs to $L^p$ (a class of $p^{\text{th}}$ power Lebesgue integrable functions) for every $p$ in $1 \le p \le \infty$. The integral of a good function is not necessarily good. However, if $\phi(x)$ is a good function, then the function $g$ defined for all $x$ by

$$g(x) = \int_{-\infty}^{x} \phi(t)\, dt$$

is a good function if and only if $\int_{-\infty}^{\infty} \phi(t)\, dt$ exists.

Good functions are not only continuous, but are also uniformly continuous in $\mathbb{R}$ and absolutely continuous in $\mathbb{R}$. However, a good function cannot be necessarily represented by a Taylor series expansion in every interval. As an example, consider a good function of bounded support

$$g(x) = \left\{ \begin{array}{ll} \exp[-(1-x^2)^{-1}], & \text{if } |x| < 1 \\ 0, & \text{if } |x| \ge 1 \end{array} \right\}.$$

The function $g$ is infinitely differentiable at $x = \pm 1$, as it must be in order to be good. It does not have a Taylor series expansion in every interval, because a Taylor expansion based on the various derivatives of $g$ for any point having $|x| > 1$ would lead to zero value for all $x$.

For example, $\exp(-x^2)$, $x\exp(-x^2)$, $\left(1+x^2\right)^{-1}\exp(-x^2)$, and $\operatorname{sech}^2 x$ are good functions, while $\exp(-|x|)$ is not differentiable at $x=0$, and the function $\left(1+x^2\right)^{-1}$ is not a good function as it decays too slowly as $|x| \to \infty$.

A sequence of good functions, $\{f_n(x)\}$ is called *regular* if, for any good function $g(x)$,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx \tag{2.4.1}$$

exists. For example, $f_n(x) = \frac{1}{n}\,\phi(x)$ is a regular sequence for any good function $\phi(x)$, if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \lim_{n\to\infty}\frac{1}{n}\int_{-\infty}^{\infty}\phi(x)\, g(x)\, dx = 0\,.$$

Two regular sequences of good functions are equivalent if, for any good function $g(x)$, the limit (2.4.1) exists and is the same for each sequence.

A *generalized function*, $f(x)$, is a regular sequence of good functions, and two generalized functions are equal if their defining sequences are equivalent. Generalized functions are, therefore, only defined in terms of their action on integrals of good functions if

$$\langle f,\, g\rangle = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \lim_{n\to\infty}\langle f_n,\, g\rangle \tag{2.4.2}$$

for any good function, $g(x)$, where the symbol $\langle f,\, g\rangle$ is used to denote the action of the generalized function $f(x)$ on the good function $g(x)$, or $\langle f,\, g\rangle$ represents the number that $f$ associates with $g$. If $f(x)$ is an ordinary function such that $\left(1+x^2\right)^{-N} f(x)$ is integrable in $(-\infty,\, \infty)$ for some $N$, then the generalized function $f(x)$ equivalent to the ordinary function is defined as any sequence of good functions $\{f_n(x)\}$ such that, for any good function $g(x)$,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx \tag{2.4.3}$$

For example, the generalized function equivalent to zero can be represented by either of the sequences $\left\{\frac{\phi(x)}{n}\right\}$ and $\left\{\frac{\phi(x)}{n^2}\right\}$.

The unit function, $I(x)$, is defined by

$$\int_{-\infty}^{\infty} I(x)\, g(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx \tag{2.4.4}$$

for any good function $g(x)$. A very important and useful good function that defines the unit function is $\left\{\exp\left(-\frac{x^2}{4n}\right)\right\}$. Thus, the unit function is the generalized function that is equivalent to the ordinary function $f(x) = 1$.

The *Heaviside function*, $H(x)$, is defined by

$$\int_{-\infty}^{\infty} H(x)\, g(x)\, dx = \int_{0}^{\infty} g(x)\, dx. \tag{2.4.5}$$

The generalized function $H(x)$ is equivalent to the ordinary unit function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \tag{2.4.6}$$

since generalized functions are defined through the action on integrals of good functions, the value of $H(x)$ at $x = 0$ does not have significance here.

The *sign function*, sgn(x), is defined by

$$\int_{-\infty}^{\infty} \mathrm{sgn(x)\, g(x)\, dx} = \int_{0}^{\infty} g(x)\, dx - \int_{-\infty}^{0} g(x)\, dx \tag{2.4.7}$$

for any good function $g(x)$. Thus, sgn(x) can be identified with the ordinary function

$$\mathrm{sgn(x)} = \begin{cases} -1, & x < 0, \\ +1, & x > 0. \end{cases} \tag{2.4.8}$$

In fact, $\mathrm{sgn(x) = 2\, H(x) - I(x)}$ can be seen as follows:

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{sgn(x)\, g(x)\, dx} &= \int_{-\infty}^{\infty} [2\, H(x) - I(x)]\; g(x)\, dx \\
&= 2\int_{-\infty}^{\infty} H(x)\, g(x)\, dx - \int_{-\infty}^{\infty} I(x)\, g(x)\, dx \\
&= 2\int_{0}^{\infty} g(x)\, dx - \int_{-\infty}^{\infty} g(x)\, dx \\
&= \int_{0}^{\infty} g(x)\, dx - \int_{-\infty}^{0} g(x)\, dx
\end{aligned}$$

In 1926, Dirac introduced the delta function, $\delta(x)$, having the following properties

$$\begin{gathered} \delta(x) = 0, \qquad x \neq 0, \\ \int_{-\infty}^{\infty} \delta(x) dx = 1. \end{gathered} \tag{2.4.9}$$

The Dirac delta function, $\delta(x)$ is defined so that for any good function $\phi(x)$,

$$\int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx = \phi(0).$$

There is no ordinary function equivalent to the delta function.

The properties (2.4.9) cannot be satisfied by any ordinary functions in classical mathematics. Hence, the delta function is not a function in the classical sense. However, it can be treated as a function in the generalized sense, and in fact, $\delta(x)$ is called a *generalized function* or *distribution*. The concept of the delta function is clear and simple in modern mathematics. It is very useful in physics and engineering. Physically, the delta function represents a point mass, that is a particle of unit mass located at the origin. In this context, it may be called a *mass-density* function. This leads to the result for a point particle that can be considered as the limit of a sequence of continuous distributions which become more and more concentrated. Even though $\delta(x)$ is not a function in the classical sense, it can be approximated by a sequence of ordinary functions. As an example, we consider the sequence

$$\delta_n(x) = \sqrt{\frac{n}{\pi}}\,\exp(-nx^2), \quad n = 1, 2, 3, \ldots. \tag{2.4.10}$$

Clearly, $\delta_n(x) \to 0$ as $n \to \infty$ for any $x \neq 0$ and $\delta_n(0) \to \infty$ as $n \to \infty$ as shown in Figure 2.4. Also, for all $n = 1, 2, 3, \ldots,$

$$\int_{-\infty}^{\infty} \delta_n(x)dx = 1$$

and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x)dx = \int_{-\infty}^{\infty} \delta(x)dx = 1$$

as expected. So the delta function can be considered as the limit of a sequence of ordinary functions, and we write

$$\delta(x) = \lim_{n\to\infty} \sqrt{\frac{n}{\pi}}\exp(-nx^2). \tag{2.4.11}$$

Sometimes, the delta function $\delta(x)$ is defined by its fundamental property

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a), \tag{2.4.12}$$

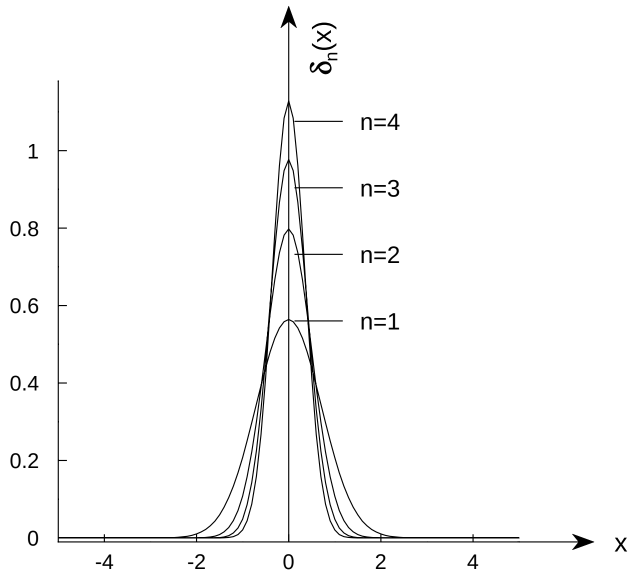


**Figure 2.4** The sequence of delta functions, $\delta_n(x)$.

where $f(x)$ is continuous in any interval containing the point $x = a$. Clearly,

$$\int_{-\infty}^{\infty} f(a)\delta(x-a)\,dx = f(a)\int_{-\infty}^{\infty} \delta(x-a)\,dx = f(a). \tag{2.4.13}$$

Thus, (2.4.12) and (2.4.13) lead to the result

$$f(x)\delta(x-a) = f(a)\delta(x-a). \tag{2.4.14}$$

The following results are also true

$$x\,\delta(x) = 0 \tag{2.4.15}$$

$$\delta(x-a) = \delta(a-x). \tag{2.4.16}$$

Result (2.4.16) shows that $\delta(x)$ is an even function.

Clearly, the result

$$\int_{-\infty}^{x} \delta(y)\,dy = \left\{\begin{matrix} 1, & x>0 \\ 0, & x<0 \end{matrix}\right\} = H(x)$$

shows that

$$\frac{d}{dx}H(x) = \delta(x). \tag{2.4.17}$$

The Fourier transform of the Dirac delta function is

$$\mathscr{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\delta(x)\,dx = \frac{1}{\sqrt{2\pi}}. \tag{2.4.18}$$

Hence,

$$\delta(x) = \mathscr{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk. \tag{2.4.19}$$

This is an integral representation of the *delta function* extensively used in quantum mechanics. Also, (2.4.19) can be rewritten as

$$\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dx. \tag{2.4.20}$$

The Dirac delta function, $\delta(x)$, is defined so that for any good function $g(x)$,

$$\langle \delta,\, g\rangle = \int_{-\infty}^{\infty} \delta(x)\,g(x)\,dx = g(0). \tag{2.4.21}$$

Derivatives of generalized functions are defined by the derivatives of any equivalent sequences of good functions. We can integrate by parts using any member of the sequences and assuming $g(x)$ vanishes at infinity. We can obtain this definition as follows:

$$\begin{aligned}\langle f',\, g\rangle &= \int_{-\infty}^{\infty} f'(x)\,g(x)\,dx \\ &= [f(x)\,g(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\,g'(x)\,dx = -\langle f,\, g'\rangle\,.\end{aligned}$$

The derivative of a generalized function $f$ is the generalized function $f'$ defined by

$$\langle f',\, g\rangle = -\langle f,\, g'\rangle \tag{2.4.22}$$

for any good function $g$.

The differential calculus of generalized functions can easily be developed with locally integrable functions. To every locally integrable function $f$, there corresponds a *generalized function* (or *distribution*) defined by

$$\langle f,\, \phi\rangle = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx \tag{2.4.23}$$

where $\phi$ is a test function in $\mathbb{R}\to\mathbb{C}$ with bounded support ($\phi$ is infinitely differentiable with its derivatives of all orders exist and are continuous).

The derivative of a generalized function $f$ is the generalized function $f'$ defined by

$$\langle f',\, \phi\rangle = -\,\langle f,\, \phi'\rangle \tag{2.4.24}$$

for all test functions $\phi$. This definition follows from the fact that

$$\begin{aligned}\langle f',\, \phi\rangle &= \int_{-\infty}^{\infty} f'(x)\,\phi(x)\,dx \\ &= [f(x)\,\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\,\phi'(x)\,dx = -\,\langle f,\, \phi'\rangle\end{aligned}$$

which was obtained from integration by parts and using the fact that $\phi$ vanishes at infinity.

It is easy to check that $H'(x) = \delta(x)$, for

$$\begin{aligned}\langle H',\, \phi\rangle &= \int_{-\infty}^{\infty} H'(x)\,\phi(x)\,dx = -\int_{-\infty}^{\infty} H(x)\,\phi'(x)\,dx \\ &= -\int_{0}^{\infty} \phi'(x)\,dx = -\,[\phi(x)]_0^{\infty} = \phi(0) = \langle\delta,\, \phi\rangle\,.\end{aligned}$$

Another result is

$$\langle\delta',\, \phi\rangle = -\int_{-\infty}^{\infty} \delta(x)\,\phi'(x)\,dx = -\phi'(0)\,.$$

It is easy to verify

$$f(x)\,\delta(x) = f(0)\,\delta(x)\,.$$

We next define $|x| = x\,\text{sgn(x)}$ and calculate its derivative as follows. We have

$$\begin{aligned}\frac{d}{dx}|x| = \frac{d}{dx}\{x\,\text{sgn(x)}\} &= x\,\frac{d}{dx}\{\text{sgn(x)}\} + \text{sgn(x)}\,\frac{\text{dx}}{\text{dx}} \\ &= x\,\frac{d}{dx}\{2\,H(x) - I(x)\} + \text{sgn(x)} \\ &= 2x\,\delta(x) + \text{sgn(x)} = \text{sgn(x)}\end{aligned} \tag{2.4.25}$$

which is, by $\text{sgn(x)} = 2\,\text{H(x)} - \text{I(x)}$ and $x\,\delta(x) = 0$.

Similarly, we can show that

$$\frac{d}{dx}\{\text{sgn(x)}\} = 2H'(x) = 2\delta(x). \tag{2.4.26}$$

If we can show that (2.3.1) holds for good functions, it follows that it holds for generalized functions.

***THEOREM 2.4.1***
The Fourier transform of a good function is a good function.

**PROOF** The Fourier transform of a good function $f(x)$ exists and is given by

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx. \tag{2.4.27}$$

Differentiating $F(k)$ $n$ times and integrating $N$ times by parts, we get

$$\begin{aligned} \left|F^{(n)}(k)\right| &\leq \left|\frac{(-1)^N}{(-ik)^N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{d^N}{dx^N} \{(-ix)^n f(x)\}\, dx\right| \\ &\leq \frac{1}{|k|^N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left|\frac{d^N}{dx^N} \{x^n f(x)\}\right| dx. \end{aligned}$$

Evidently, all derivatives tend to zero as fast as $|k|^{-N}$ as $|k| \to \infty$ for any $N > 0$ and hence, $F(k)$ is a good function. ∎

***THEOREM 2.4.2***
If $f(x)$ is a good function with the Fourier transform (2.4.27), then the inverse Fourier transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k)\, dk. \tag{2.4.28}$$

**PROOF** For any $\epsilon > 0$, we have

$$\mathscr{F}\left\{e^{-\epsilon x^2} F(-x)\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx - \epsilon x^2} \left\{\int_{-\infty}^{\infty} e^{ixt} f(t)\, dt\right\} dx.$$

Since $f$ is a good function, the order of integration can be interchanged to obtain

$$\mathscr{F}\left\{e^{-\epsilon x^2} F(-x)\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, dt \int_{-\infty}^{\infty} e^{-i(k-t)x - \epsilon x^2}\, dx$$

which is, by similar calculation used in Example 2.3.1,

$$= \frac{1}{\sqrt{4\pi\epsilon}} \int_{-\infty}^{\infty} \exp\left[-\frac{(k-t)^2}{4\epsilon}\right] f(t)\, dt\,.$$

Using the fact that

$$\frac{1}{\sqrt{4\pi\epsilon}} \int_{-\infty}^{\infty} \exp\left[-\frac{(k-t)^2}{4\epsilon}\right] dt = 1,$$

we can write

$$\mathscr{F}\left\{e^{-\epsilon x^2}F(-x)\right\} - f(k)\,.\,1$$
$$= \frac{1}{\sqrt{4\pi\epsilon}} \int_{-\infty}^{\infty} [f(t) - f(k)] \exp\left[-\frac{(k-t)^2}{4\epsilon}\right] dt. \quad (2.4.29)$$

Since $f$ is a good function, we have

$$\left|\frac{f(t) - f(k)}{t - k}\right| \leq \max_{x \in \mathbb{R}} |f'(x)|\,.$$

It follows from (2.4.29) that

$$\left|\mathscr{F}\left\{e^{-\epsilon x^2}F(-x)\right\} - f(k)\right|$$
$$\leq \frac{1}{\sqrt{4\pi\epsilon}} \max_{x\in\mathbb{R}} |f'(x)| \int_{-\infty}^{\infty} |t-k|\, \exp\left[-\frac{(t-k)^2}{4\epsilon}\right] dt$$
$$= \frac{1}{\sqrt{4\pi\epsilon}} \max_{x\in\mathbb{R}} |f'(x)|\; 4\epsilon \int_{-\infty}^{\infty} |\alpha|\;\, e^{-\alpha^2} d\alpha \to 0$$

as $\epsilon \to 0$, where $\alpha = \frac{t-k}{2\sqrt{\epsilon}}$.
Consequently,

$$f(k) = \mathscr{F}\{F(-x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, F(-x)\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, F(x)\, dx$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dx \int_{-\infty}^{\infty} e^{-i\xi x}\, f(\xi)\, d\xi.$$

Interchanging $k$ with $x$, this reduces to the Fourier integral formula (2.2.4) and hence, the theorem is proved. ∎

***Example 2.4.1***
The Fourier transform of a constant function $c$ is

$$\mathscr{F}\{c\} = \sqrt{2\pi}.c.\delta(k). \quad (2.4.30)$$

In the ordinary sense

$$\mathscr{F}\{c\} = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, dx$$

is not a well defined (divergent) integral. However, treated as a generalized function, $c = c\,I(x)$ and we consider $\left\{\exp\left(-\frac{x^2}{4n}\right)\right\}$ as an equivalent sequence to the unit function, $I(x)$. Thus,

$$\mathscr{F}\left\{c\,\exp\left(-\frac{x^2}{4n}\right)\right\} = \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-ikx - \frac{x^2}{4n}\right)dx$$

which is, by Example 2.3.1,

$$= c\sqrt{2n}\,\exp(-nk^2) = \sqrt{2\pi}.c.\sqrt{\frac{n}{\pi}}\,\exp(-nk^2)$$
$$= \sqrt{2\pi}.c.\delta_n(k) = \sqrt{2\pi}.c.\delta(k) \quad \text{as} \quad n \to \infty,$$

since $\{\delta_n(k)\} = \left\{\sqrt{\frac{n}{\pi}}\exp\left(-nk^2\right)\right\}$ is a sequence equivalent to the delta function defined by (2.4.10).

□

***Example 2.4.2***
Show that

$$\mathscr{F}\{e^{-ax}H(x)\} = \frac{1}{\sqrt{2\pi}(ik+a)}, \quad a > 0. \tag{2.4.31}$$

We have, by definition,

$$\mathscr{F}\{e^{-ax}H(x)\} = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\exp\{-x(ik+a)\}dx = \frac{1}{\sqrt{2\pi}(ik+a)}.$$

□

***Example 2.4.3***
By considering the function (see Figure 2.5)

$$f_a(x) = e^{-ax}H(x) - e^{ax}H(-x), \quad a > 0, \tag{2.4.32}$$

find the Fourier transform of sgn(x). In Figure 2.5, the vertical axis (y-axis) represents $f_a(x)$ and the horizontal axis represents the x-axis.

We have, by definition,

$$\mathscr{F}\{f_a(x)\} = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\{(a-ik)x\}dx$$
$$+\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\exp\{-(a+ik)x\}dx$$
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+ik} - \frac{1}{a-ik}\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{(-ik)}{a^2+k^2}.$$

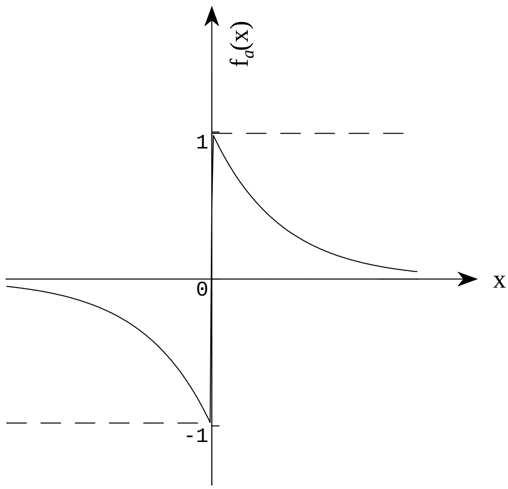


**Figure 2.5** Graph of the function $f_a(x)$.

In the limit as $a \to 0$, $f_a(x) \to \text{sgn(x)}$ and then

$$\mathscr{F}\{\text{sgn(x)}\} = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\text{ik}}.$$

Or,

$$\mathscr{F}\left\{\sqrt{\frac{\pi}{2}} i \, \text{sgn(x)}\right\} = \frac{1}{k}.$$

$\square$

## 2.5 Basic Properties of Fourier Transforms

***THEOREM 2.5.1***
If $\mathscr{F}\{f(x)\} = F(k)$, then

| | | | |
|---|---|---|---|
| (a) | (Shifting) | $\mathscr{F}\{f(x-a)\} = e^{-ika}\mathscr{F}\{f(x)\},$ | (2.5.1) |
| (b) | (Scaling) | $\mathscr{F}\{f(ax)\} = \dfrac{1}{\lvert a\rvert}\, F(\dfrac{k}{a}),$ | (2.5.2) |
| (c) | (Conjugate) | $\mathscr{F}\{\overline{f(-x)}\} = \overline{\mathscr{F}\{f(x)\}},$ | (2.5.3) |
| (d) | (Translation) | $\mathscr{F}\{e^{iax} f(x)\} = F(k-a),$ | (2.5.4) |
| (e) | (Duality) | $\mathscr{F}\{F(x)\} = f(-k),$ | (2.5.5) |
| (f) | (Composition) | $\displaystyle\int_{-\infty}^{\infty} F(k)g(k)e^{ikx}dk = \int_{-\infty}^{\infty} f(\xi)G(\xi - x)d\xi,$ where $G(k) = \mathscr{F}\{g(x)\}.$ | (2.5.6) |

**PROOF** (a) We obtain, from the definition,

$$\begin{aligned}\mathscr{F}\{f(x-a)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x-a)dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik(\xi+a)} f(\xi)d\xi, \qquad (x-a=\xi) \\ &= e^{-ika}\mathscr{F}\{f(x)\}.\end{aligned}$$

The proofs of results (b)–(d) follow easily from the definition of the Fourier transform. We give a proof of the duality (e) and composition (f).

We have, by definition,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} F(k)dk = \mathscr{F}^{-1}\{F(k)\}.$$

Interchanging $x$ and $k$, and then replacing $k$ by $-k$, we obtain

$$f(-k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(x)dx = \mathscr{F}\{F(x)\}.$$

To prove (f), we have

$$\int_{-\infty}^{\infty} F(k)g(k)\, e^{ikx} dk = \int_{-\infty}^{\infty} g(k)\, e^{ikx} dk \;\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\xi} f(\xi) d\xi$$

$$= \int_{-\infty}^{\infty} f(\xi) d\xi \;\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(\xi - x)} g(k) dk$$

$$= \int_{-\infty}^{\infty} f(\xi) G(\xi - x) d\xi.$$

In particular, when $x = 0$

$$\int_{-\infty}^{\infty} F(k)g(k) dk = \int_{-\infty}^{\infty} f(\xi)G(\xi)d\xi.$$

▮

***THEOREM 2.5.2***

If $f(x)$ is piecewise continuously differentiable and absolutely integrable, then

(i) $F(k)$ is bounded for $-\infty < k < \infty$,

(ii) $F(k)$ is continuous for $-\infty < k < \infty$.

**PROOF** It follows from the definition that

$$|F(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ikx}||f(x)|dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|dx = \frac{c}{\sqrt{2\pi}},$$

where $c = \int_{-\infty}^{\infty} |f(x)|dx =$ constant. This proves result (i).

To prove (ii), we have

$$|F(k+h) - F(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ihx} - 1||f(x)|dx$$

$$\leq \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} |f(x)|dx.$$

Since $\lim_{h\to 0}|e^{-ihx}-1|=0$ for all $x\in\mathbb{R}$, we obtain

$$\lim_{h\to 0}|F(k+h)-F(k)|\leq \lim_{h\to 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|e^{-ihx}-1||f(x)|dx=0.$$

This shows that $F(k)$ is continuous. ∎

***THEOREM 2.5.3***
(*Riemann-Lebesgue Lemma*). If $F(k)=\mathscr{F}\{f(x)\}$, then

$$\lim_{|k|\to\infty}|F(k)|=0. \tag{2.5.7}$$

**PROOF** Since $e^{-ikx}=-e^{-ikx-i\pi}$, we have

$$\begin{aligned}F(k)&=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ik(x+\frac{\pi}{k})}f(x)dx\\&=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}f\left(x-\frac{\pi}{k}\right)dx.\end{aligned}$$

Hence,

$$\begin{aligned}F(k)&=\frac{1}{2}\left\{\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty}e^{-ikx}f(x)dx-\int_{-\infty}^{\infty}e^{-ikx}f\left(x-\frac{\pi}{k}\right)dx\right]\right\}\\&=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\left[f(x)-f\left(x-\frac{\pi}{k}\right)\right]dx.\end{aligned}$$

Therefore,

$$|F(k)|\leq\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|f(x)-f\left(x-\frac{\pi}{k}\right)\right|dx.$$

Thus, we obtain

$$\lim_{|k|\to\infty}|F(k)|\leq\frac{1}{2\sqrt{2\pi}}\lim_{|k|\to\infty}\int_{-\infty}^{\infty}\left|f(x)-f\left(x-\frac{\pi}{k}\right)\right|dx=0.$$

∎

***THEOREM 2.5.4***
If $f(x)$ is continuously differentiable and $f(x) \to 0$ as $|x| \to \infty$, then

$$\mathscr{F}\{f'(x)\} = (ik)\mathscr{F}\{f(x)\} = ik\, F(k). \tag{2.5.8}$$

**PROOF** We have, by definition,

$$\mathscr{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f'(x) dx$$

which is, integrating by parts,

$$= \frac{1}{\sqrt{2\pi}} \left[f(x)e^{-ikx}\right]_{-\infty}^{\infty} + \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$
$$= (ik)F(k).$$

If $f(x)$ is continuously $n$-times differentiable and $f^{(k)}(x) \to 0$ *as* $|x| \to \infty$ for $k = 1, 2, \ldots, (n-1)$, then the Fourier transform of the $n$th derivative is

$$\mathscr{F}\{f^{(n)}(x)\} = (ik)^n \mathscr{F}\{f(x)\} = (ik)^n F(k). \tag{2.5.9}$$

A repeated application of Theorem 2.5.4 to higher derivatives gives the result.

The operational results similar to those of (2.5.8) and (2.5.9) hold for partial derivatives of a function of two or more independent variables. For example, if $u(x,t)$ is a function of space variable $x$ and time variable $t$, then

$$\mathscr{F}\left\{\frac{\partial u}{\partial x}\right\} = ik\, U(k,t), \quad \mathscr{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -k^2\, U(k,t)\,,$$
$$\mathscr{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{dU}{dt}, \quad \mathscr{F}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = \frac{d^2 U}{dt^2}\,,$$

where $U(k,t) = \mathscr{F}\,\{u(x,t)\}$. ∎

***DEFINITION 2.5.1*** *The convolution of two integrable functions $f(x)$ and $g(x)$, denoted by $(f * g)(x)$, is defined by*

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi) g(\xi) d\xi, \tag{2.5.10}$$

*provided the integral in (2.5.10) exists, where the factor $\frac{1}{\sqrt{2\pi}}$ is a matter of choice. In the study of convolution, this factor is often omitted as this factor* *does not affect the properties of convolution. We will include or exclude the factor* $\frac{1}{\sqrt{2\pi}}$ *freely in this book.*

We give some examples of convolution.

***Example 2.5.1***

Find the convolution of

(a) $f(x) = \cos x$ and $g(x) = \exp(-a|x|)$, $a > 0$,

(b) $f(x) = \chi_{[a,b]}(x)$ and $g(x) = x^2$,

where $\chi_{[a,b]}(x)$ is the characteristic function of the interval $[a,b] \subseteq \mathbb{R}$ defined by

$$\chi_{[a,b]}(x) = \left\{ \begin{matrix} 1, & a \le x \le b \\ 0, & \text{otherwise} \end{matrix} \right\}.$$

(a) We have, by definition,

$$\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{\infty} f(x-\xi)\ g(\xi) d\xi = \int_{-\infty}^{\infty} \cos(x-\xi)\ e^{-a|\xi|} d\xi \\
&= \int_{-\infty}^{0} \cos(x-\xi)\ e^{a\xi} d\xi + \int_{0}^{\infty} \cos(x-\xi)\ e^{-a\xi} d\xi \\
&= \int_{0}^{\infty} \cos(x+\xi)\ e^{-a\xi} d\xi + \int_{0}^{\infty} \cos(x-\xi)\ e^{-a\xi} d\xi \\
&= 2\cos x \int_{0}^{\infty} \cos\xi\ e^{-a\xi} d\xi \quad = \quad \frac{2a\cos x}{(1+a^2)}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{\infty} f(x-\xi)\ g(\xi) d\xi = \int_{-\infty}^{\infty} \chi_{[a,b]}(x-\xi)\ g(\xi) d\xi \\
&= \int_a^b \xi^2\ d\xi = \frac{1}{3}\left(b^3 - a^3\right).
\end{aligned}$$

□

***THEOREM 2.5.5***

(*Convolution Theorem*). If $\mathscr{F}\{f(x)\} = F(k)$ and $\mathscr{F}\{g(x)\} = G(k)$, then

$$\mathscr{F}\{f(x) * g(x)\} = F(k)G(k), \tag{2.5.11}$$

or,

$$f(x) * g(x) = \mathscr{F}^{-1}\{F(k)G(k)\}, \tag{2.5.12}$$

or, equivalently,

$$\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi = \int_{-\infty}^{\infty} e^{ikx}F(k)G(k)dk. \tag{2.5.13}$$

**PROOF** We have, by the definition of the Fourier transform,

$$\begin{aligned}
\mathscr{F}\{f(x) * g(x)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dx \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik\xi}g(\xi)d\xi \int_{-\infty}^{\infty} e^{-ik(x-\xi)}f(x-\xi)dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik\xi}g(\xi)d\xi \int_{-\infty}^{\infty} e^{-ik\eta}f(\eta)d\eta = G(k)F(k),
\end{aligned}$$

where, in this proof, the factor $\frac{1}{\sqrt{2\pi}}$ is included in the definition of the convolution. This completes the proof. ∎

The convolution has the following algebraic properties:

$$f * g = g * f \quad \text{(Commutative)}, \tag{2.5.14}$$

$$f * (g * h) = (f * g) * h \quad \text{(Associative)}, \tag{2.5.15}$$

$$(\alpha f + \beta g) * h = \alpha\,(f * h) + \beta\,(g * h) \quad \text{(Distributive)}, \tag{2.5.16}$$

$$f * \sqrt{2\pi}\delta = f = \sqrt{2\pi}\delta * f \quad \text{(Identity)}, \tag{2.5.17}$$

where $\alpha$ and $\beta$ are constants.

We give proofs of (2.5.15) and (2.5.16). If $f * (g * h)$ exists, then

$$\begin{aligned}
[f * (g * h)](x) &= \int_{-\infty}^{\infty} f(x-\xi)(g * h)(\xi)d\xi \\
&= \int_{-\infty}^{\infty} f(x-\xi) \int_{-\infty}^{\infty} g(\xi - t)h(t)\,dt \;\; d\xi
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi - t) d\xi \right] h(t) dt$$

$$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x-t-\eta) \;\; g(\eta) d\eta \right] h(t) dt \quad (\text{put } \xi - t = \eta)$$

$$= \int_{-\infty}^{\infty} \left[ (f * g)\,(x-t) \right] h(t) dt$$

$$= \left[ (f * g) * h \right](x),$$

where, in the above proof, under suitable assumptions, the interchange of the order of integration can be justified.

Similarly, we prove (2.5.16) using the right-hand side of (2.5.16), that is,

$$\begin{aligned} \alpha\,(f * h) + \beta\,(g * h) &= \alpha \int_{-\infty}^{\infty} f(x-\xi) h(\xi) d\xi + \beta \int_{-\infty}^{\infty} g(x-\xi) h(\xi) d\xi \\ &= \int_{-\infty}^{\infty} \left[ \alpha f(x-\xi) + \beta g(x-\xi) \right] h(\xi) d\xi \\ &= \left[ (\alpha f + \beta g) * h \right](x). \end{aligned}$$

In view of the commutative property of the convolution, (2.5.13) can be written as

$$\int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi = \int_{-\infty}^{\infty} e^{ikx} F(k) G(k) dk. \tag{2.5.18}$$

This is valid for all real $x$, and hence, putting $x = 0$ gives

$$\int_{-\infty}^{\infty} f(\xi) g(-\xi) d\xi = \int_{-\infty}^{\infty} f(x) g(-x) dx = \int_{-\infty}^{\infty} F(k) G(k) dk. \tag{2.5.19}$$

We substitute $g(x) = \overline{f(-x)}$ to obtain

$$G(k) = \mathscr{F}\{g(x)\} = \mathscr{F}\left\{\overline{f(-x)}\right\} = \overline{\mathscr{F}\{f(x)\}} = \overline{F(k)}.$$

Evidently, (2.5.19) becomes

$$\int_{-\infty}^{\infty} f(x)\, \overline{f(x)} dx = \int_{-\infty}^{\infty} F(k)\, \overline{F(k)} dk \tag{2.5.20}$$

or,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 dk. \tag{2.5.21}$$

This is well known as *Parseval's relation.*

For square integrable functions $f(x)$ and $g(x)$, the *inner product* $\langle f,\, g\rangle$ is defined by

$$\langle f,\, g\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}dx \tag{2.5.22}$$

so the *norm* $\|f\|_2$ is defined by

$$\|f\|_2^2 = \langle f,\, f\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}dx = \int_{-\infty}^{\infty} |f(x)|^2 dx. \tag{2.5.23}$$

The function space $L^2(\mathbb{R})$ of all complex-valued Lebesgue square integrable functions with the inner product defined by (2.5.22) is a complete normed space with the norm (2.5.23). In terms of the norm, the Parseval relation takes the form

$$\|f\|_2 = \|F\|_2 = \|\mathscr{F} f\|_2. \tag{2.5.24}$$

This means that the Fourier transform action is *unitary*. Physically, the quantity $\|f\|_2$ is a measure of energy and $\|F\|_2$ represents the *power spectrum* of $f$.

***THEOREM 2.5.6***

(*General Parseval's Relation*). If $\mathscr{F}\{f(x)\} = F(k)$ and $\mathscr{F}\{g(x)\} = G(k)$ then

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}dx = \int_{-\infty}^{\infty} F(k)\,\overline{G(k)}dk. \tag{2.5.25}$$

**PROOF** We proceed formally to obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} F(k)\,\overline{G(k)}dk &= \int_{-\infty}^{\infty} dk \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky} f(y)\;\; dy \overline{\int_{-\infty}^{\infty} e^{-ikx} g(x)\;\; dx} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\;\; dy \int_{-\infty}^{\infty} \overline{g(x)}dx \int_{-\infty}^{\infty} e^{ik(x-y)}dk \\
&= \int_{-\infty}^{\infty} \overline{g(x)}\;\; dx \int_{-\infty}^{\infty} \delta(x-y) f(y)\;\; dy = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\;\; dx.
\end{aligned}$$

In particular, when $g(x) = f(x)$, the above result agrees with (2.5.20).

We now use an indirect method to obtain the Fourier transform of sgn(x), that is,

$$\mathscr{F}\{\mathrm{sgn(x)}\} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\mathrm{ik}}. \tag{2.5.26}$$

From (2.4.26), we find

$$\mathscr{F}\left\{\frac{d}{dx}\mathrm{sgn(x)}\right\} = \mathscr{F}\{2H'(x)\} = 2\mathscr{F}\{\delta(x)\} = \sqrt{\frac{2}{\pi}},$$

which is, by (2.5.8),

$$ik\;\mathscr{F}\{\mathrm{sgn(x)}\} = \sqrt{\frac{2}{\pi}},$$

or

$$\mathscr{F}\{\mathrm{sgn(x)}\} = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\mathrm{ik}}.$$

The Fourier transform of $H(x)$ follows from (2.4.30) and (2.5.26):

$$\begin{aligned}\mathscr{F}\{H(x)\} &= \frac{1}{2}\mathscr{F}\{1+\mathrm{sgn(x)}\} = \frac{1}{2}[\mathscr{F}\{1\} + \mathscr{F}\{\mathrm{sgn(x)}\}]\\ &= \sqrt{\frac{\pi}{2}}\left[\delta(k) + \frac{1}{i\pi k}\right]. \end{aligned} \tag{2.5.27}$$

■

## 2.6 Poisson's Summation Formula

A class of functions designated as $L^p(\mathbb{R})$ is of great importance in the theory of Fourier transformations, where $p(\geq 1)$ is any real number. We denote the vector space of all complex-valued functions $f(x)$ of the real variable $x$. If $f$ is a locally integrable function such that $|f|^p \in L(\mathbb{R})$, then we say $f$ is $p$-th power Lebesgue integrable. The set of all such functions is written $L^p(\mathbb{R})$. The number $||f||_p$ is called the $L^p$-norm of $f$ and is defined by

$$||f||_p = \left[\int_{-\infty}^{\infty} |f(x)|^p \; dx\right]^{\frac{1}{p}} < \infty. \tag{2.6.1}$$

Suppose $f$ is a Lebesgue integrable function on $\mathbb{R}$. Since $\exp(-ikx)$ is continuous and bounded, the product $\exp(-ikx)\,f(x)$ is locally integrable for any $k \in \mathbb{R}$. Also, $|\exp(-ikx)| \leq 1$ for all $k$ and $x$ on $\mathbb{R}$. Consider the inner product

$$\langle f, e^{ikx}\rangle = \int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx, \qquad k \in \mathbb{R}. \tag{2.6.2}$$

Clearly,

$$\left|\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx\right| \le \int_{-\infty}^{\infty} |f(x)|\,dx = ||f||_1 < \infty. \tag{2.6.3}$$

This means that integral (2.6.2) exists for all $k \in \mathbb{R}$, and was used to define the Fourier transform, $F(k) = \mathscr{F}\{f(x)\}$ without the factor $\frac{1}{\sqrt{2\pi}}$.

Although the theory of Fourier series is a very important subject, a detailed study is beyond the scope of this book. Without rigorous analysis, we can establish a simple relation between the Fourier transform of functions in $L^1(\mathbb{R})$ and the Fourier series of related periodic functions in $L^1(-a,\,a)$ of period $2a$. If $f(x) \in L^1(-a,\,a)$ and is defined by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}\,, \qquad (-a \le x \le a), \tag{2.6.4}$$

where the Fourier coefficients $c_n$ is given by

$$c_n = \frac{1}{2a}\int_{-a}^{a} f(x)\,e^{-ikx}\,dx. \tag{2.6.5}$$

***THEOREM 2.6.1***
If $f(x) \in L^1(\mathbb{R})$, then the series

$$\sum_{n=-\infty}^{\infty} f(x + 2na) \tag{2.6.6}$$

converges absolutely for almost all $x$ in $(-a,\,a)$ and its sum $g(x) \in L^1(-a,\,a)$ with $g(x + 2a) = g(x)$ for $x \in \mathbb{R}$.

If $a_n$ denotes the Fourier coefficient of a function $g$, then

$$a_n = \frac{1}{2a}\int_{-a}^{a} g(x)\,e^{-inx}\,dx = \frac{1}{2a}\int_{-\infty}^{\infty} f(x)\,e^{-inx}\,dx = \frac{1}{2a}F(n).$$

**PROOF** We have

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\int_{-a}^{a} |f(x+2na)|\,dx &= \lim_{N\to\infty}\sum_{n=-N}^{N}\int_{-a}^{a} |f(x+2na)|\,dx \\ &= \lim_{N\to\infty}\sum_{n=-N}^{N}\int_{(2n-1)a}^{(2n+1)a} |f(t)|\,dt \\ &= \lim_{N\to\infty}\int_{-(2N+1)a}^{(2N+1)a} |f(t)|\,dt \\ &= \int_{-\infty}^{\infty} |f(t)|\,dt < \infty.\end{aligned}$$

It follows from Lebesgue's theorem on monotone convergence that

$$\int_{-a}^{a}\left[\sum_{n=-\infty}^{\infty}|f(x+2na)|\right]dx=\sum_{n=-\infty}^{\infty}\int_{-a}^{a}|f(x+2na)|\,dx<\infty.$$

Hence, the series $\sum_{n=-\infty}^{\infty} f(x+2na)$ converges absolutely for almost all $x$ in $(-a,\,a)$. If $g_N(x)=\sum_{n=-N}^{N} f(x+2na)$, $\lim_{N\to\infty} g_N(x)=g(x)$, where $g\in\mathbb{L}^1(-a,\,a)$, and $g(x+2a)=g(x)$.

Moreover,

$$\begin{aligned}||g||_1 &= \int_{-a}^{a}|g(x)|\,dx=\int_{-a}^{a}\left|\sum_{n=-\infty}^{\infty} f(x+2na)\right|dx\\ &\le \int_{-a}^{a}\sum_{n=-\infty}^{\infty}|f(x+2na)|\,dx\\ &= \sum_{n=-\infty}^{\infty}\int_{-a}^{a}|f(x+2na)|\,dx\\ &= \int_{-\infty}^{\infty}|f(x)|\,dx=||f||_1.\end{aligned}$$

$\blacksquare$

We consider the Fourier series of $g(x)$ given by

$$g(x)=\sum_{m=-\infty}^{\infty} c_m\exp(im\pi x/a), \tag{2.6.7}$$

where the coefficients $c_m$ for $m=0,\pm1,\pm2,...$ are given by

$$c_m=\frac{1}{2a}\int_{-a}^{a} g(x)\exp(-im\pi x/a)dx. \tag{2.6.8}$$

We replace $g(x)$ by the limit of the sum

$$g(x)=\lim_{N\to\infty}\sum_{n=-N}^{N} f(x+2na), \tag{2.6.9}$$

so that (2.6.8) reduces to

$$\begin{aligned} c_m &= \frac{1}{2a} \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-a}^{a} f(x+2na)\exp(-im\pi x/a)\,dx \\ &= \frac{1}{2a} \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{(2n-1)a}^{(2n+1)a} f(y)\exp(-im\pi y/a)\;\; dy \\ &= \frac{1}{2a} \lim_{N\to\infty} \int_{-(2N+1)a}^{(2N+1)a} f(x)\exp(-im\pi x/a)\;\; dx \\ &= \frac{\sqrt{2\pi}}{2a} F\left(\frac{m\pi}{a}\right), \end{aligned} \tag{2.6.10}$$

where $F\left(\frac{m\pi}{a}\right)$ is the discrete Fourier transform of $f(x)$.

Evidently,

$$\sum_{n=-\infty}^{\infty} f(x+2na) = g(x) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{2\pi}}{2a} F\left(\frac{n\pi}{a}\right)\; \exp(in\pi x/a). \tag{2.6.11}$$

We let $x=0$ in (2.6.11) to obtain the *Poisson summation formula*

$$\sum_{n=-\infty}^{\infty} f(2na) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{2\pi}}{2a}\, F\left(\frac{n\pi}{a}\right). \tag{2.6.12}$$

When $a=\pi$, this formula becomes

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} F(n). \tag{2.6.13}$$

When $2a=1$, formula (2.6.12) becomes

$$\sum_{n=-\infty}^{\infty} f(n) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} F(2n\pi). \tag{2.6.14}$$

To obtain a more general formula, we assume that $a$ is a given positive constant, and write $g(x) = f(ax)$ for all $x$. Then

$$f\left(a.\frac{2\pi n}{a}\right) = g\left(\frac{2\pi n}{a}\right),$$

and we define the Fourier transform of $f(x)$ without the factor $\frac{1}{\sqrt{2\pi}}$ so that

$$\begin{aligned}F(n) &= \int_{-\infty}^{\infty} e^{-inx} f(x)\, dx = \int_{-\infty}^{\infty} e^{-inx} f\left(a.\frac{x}{a}\right)\, dx \\ &= \int_{-\infty}^{\infty} e^{-inx}\, g\left(\frac{x}{a}\right)\, dx \\ &= a\int_{-\infty}^{\infty} e^{-i(an)y}\, g(y)\, dy \\ &= a\, G(an).\end{aligned}$$

Consequently, equality (2.6.13) reduces to

$$\sum_{n=-\infty}^{\infty} g\left(\frac{2\pi n}{a}\right) = \frac{a}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} G(an). \tag{2.6.15}$$

Putting $b = \frac{2\pi}{a}$ in (2.6.15) gives

$$\sum_{n=-\infty}^{\infty} g(bn) = \sqrt{2\pi}\, b^{-1} \sum_{n=-\infty}^{\infty} G(2\pi b^{-1} n). \tag{2.6.16}$$

When $b = 2\pi$, result (2.6.16) becomes (2.6.13). We apply these formulas to prove the following series

(a) $$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2+b^2)} = \frac{\pi}{b}\coth(\pi b), \tag{2.6.17}$$

(b) $$\sum_{n=-\infty}^{\infty} \exp(-\pi n^2 t) = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi n^2}{t}\right), \tag{2.6.18}$$

(c) $$\sum_{n=-\infty}^{\infty} \frac{1}{(x+n\pi)^2} = \operatorname{cosec}^2(x). \tag{2.6.19}$$

To prove (a), we write $f(x) = (x^2+b^2)^{-1}$ so that $F(k) = \sqrt{\frac{\pi}{2}}\,\frac{1}{b}\exp(-b|k|)$. We now use (2.6.14) to derive

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{1}{(n^2+b^2)} &= \frac{\pi}{b}\sum_{n=-\infty}^{\infty} \exp(-2|n|\pi b) \\ &= \frac{\pi}{b}\left[\sum_{n=0}^{\infty} \exp(-2n\pi b) + \sum_{n=1}^{\infty} \exp(2n\pi b)\right]\end{aligned}$$

which is, by writing $r = \exp(-2\pi b)$,

$$\begin{aligned}&= \frac{\pi}{b}\left[\sum_{n=0}^{\infty} r^n + \sum_{n=1}^{\infty}\left(\frac{1}{r}\right)^n\right] = \frac{\pi}{b}\left(\frac{r}{1-r} + \frac{1}{1-r}\right) \\ &= \frac{\pi}{b}\left(\frac{1+r}{1-r}\right) = \frac{\pi}{b}\coth(\pi b).\end{aligned}$$

It follows from (2.6.14) that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2+b^2)} = \frac{\pi}{b} \frac{\left(1+e^{-2\pi b}\right)}{\left(1-e^{-2\pi b}\right)}.$$

Or,

$$2\sum_{n=1}^{\infty} \frac{1}{(n^2+b^2)} + \frac{1}{b^2} = \frac{\pi}{b} \frac{\left(1+e^{-2\pi b}\right)}{\left(1-e^{-2\pi b}\right)}.$$

It turns out that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{(n^2+b^2)} &= \frac{\pi}{2b}\left[\frac{\left(1+e^{-2\pi b}\right)}{\left(1-e^{-2\pi b}\right)} - \frac{1}{\pi b}\right] \\
&= \frac{\pi^2}{x}\left[\frac{\left(1+e^{-x}\right)}{\left(1-e^{-x}\right)} - \frac{2}{x}\right], \qquad (2\pi b = x) \\
&= \frac{\pi^2}{x^2}\left[\frac{x\left(1+e^{-x}\right) - 2\left(1-e^{-x}\right)}{\left(1-e^{-x}\right)}\right] \\
&= \left(\frac{\pi}{x}\right)^2 \left[\frac{x^3\left(\frac{1}{2}-\frac{1}{3}\right) - \frac{x^4}{12} + \ldots.}{x - \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots.}\right].
\end{aligned}$$

In the limit as $b \to 0\,(x \to 0)$, we obtain the well-known result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{2.6.20}$$

To prove (b), we assume $f(x) = \exp(-\pi t x^2)$ so that $F(k) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{k^2}{4\pi t}\right)$. Thus, the Poisson formula (2.6.14) gives

$$\sum_{n=-\infty}^{\infty} \exp(-\pi t n^2) = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} \exp(-\pi n^2/t).$$

This identity plays an important role in number theory and in the theory of elliptic functions. The *Jacobi theta function* $\Theta(s)$ is defined by

$$\Theta(s) = \sum_{n=-\infty}^{\infty} \exp(-\pi s n^2), \qquad s > 0, \tag{2.6.21}$$

so that (2.6.16) gives the *functional equation* for the theta function

$$\sqrt{s}\,\Theta(s) = \Theta\left(\frac{1}{s}\right). \tag{2.6.22}$$

The theta function $\Theta(s)$ also extends to complex values of $s$ when $Re(s) > 0$ and the functional equation is still valid for complex $s$. The theta function is closely related to the *Riemann zeta function* $\zeta(s)$ defined for $Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2.6.23}$$

An integral representation of $\zeta(s)$ can be found from the result

$$\int_0^{\infty} x^{s-1}\, e^{-nx} dx = \frac{\Gamma(s)}{n^s}, \qquad Re(s) > 0,$$

where the *gamma function* $\Gamma(s)$ is defined by

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt, \qquad Re(s) > 0.$$

Summing both sides of this result and interchanging the order of summation and integration, which is permissible for $Re(s) > 1$, gives

$$\Gamma(s)\,\zeta(s) = \int_0^{\infty} x^{s-1} \frac{dx}{e^x - 1}, \qquad Re(s) > 1. \tag{2.6.24}$$

It turns out that $\zeta(s)$, $\Theta(s)$, and $\Gamma(s)$ are related by the following identity:

$$\zeta(s)\Gamma(s/2) = \frac{1}{2}\pi^{s/2} \int_0^{\infty} x^{s/2-1} \left[\Theta(x) - 1\right] dx, \qquad Re(s) > 1. \tag{2.6.25}$$

Considering the complex integral in a suitable closed contour $C$

$$I = \frac{1}{2\pi i} \int_C \frac{z^{s-1}}{e^{-z} - 1} dz,$$

and using the Cauchy residue theorem with all zeros of $(e^{-z} - 1)$ at $z = 2\pi i n$, $n = \pm 1, \pm 2, ..., \pm N$ gives

$$I = -2 \sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} (2\pi n)^{s-1}.$$

To prove (c), we use the Fourier transform of the function $f(x) = (1 - |x|)$ $H\,(1 - |x|)$ to obtain the result. In the limit as $N \to \infty$, the sum of the residues is convergent so that the integral gives the relation

$$2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \zeta(1-s) = \frac{\zeta(s)}{\Gamma(1-s)}. \tag{2.6.26}$$

In view of another relation for the gamma function, $\Gamma(1+z)\Gamma(-z) = -\frac{\pi}{\sin \pi z}$, the relation (2.6.26) leads to a famous functional relation for $\zeta(s)$ in the form

$$\pi^s \zeta(1-s) = 2^{1-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s). \tag{2.6.27}$$

## 2.7 The Shannon Sampling Theorem

An analog signal $f(t)$ is a continuous function of time $t$ defined in $-\infty < t < \infty$, with the exception of perhaps a countable number of jump discontinuities. Almost all analog signals $f(t)$ of interest in engineering have finite energy. By this we mean that $f \in L^2(-\infty, \infty)$. The norm of $f$ defined by

$$||f|| = \left[\int_{-\infty}^{\infty} |f(x)|^2 \; dx\right]^{\frac{1}{2}} \tag{2.7.1}$$

represents the square root of the total energy content of the signal $f(t)$. The *spectrum* of a signal $f(t)$ is represented by its Fourier transform $F(\omega)$, where $\omega$ is called the *frequency*. The frequency is measured by $\nu = \frac{\omega}{2\pi}$ in terms of Hertz.

A continuous signal $f(t)$ is called *band limited* if its Fourier transform $F(\omega)$ is zero except in a finite interval, that is, if

$$F_a(\omega) = 0 \qquad \text{for} \quad |\omega| > a. \tag{2.7.2}$$

Then $a(> 0)$ is called the *cutoff frequency.*

In particular, if

$$F(\omega) = \left\{\begin{array}{ll} 1, & |\omega| \le a \\ 0, & |\omega| > a \end{array}\right\} \tag{2.7.3}$$

then $F(\omega)$ is called a *gate function* and is denoted by $F_a(\omega)$, and the band limited signal is denoted by $f_a(t)$. If $a$ is the smallest value for which (2.7.2) holds, it is called the *bandwidth* of the signal. Even if an analog signal $f(t)$ is not band-limited, we can reduce it to a band-limited signal by what is called an *ideal low-pass filtering.* To reduce $f(t)$ to a band-limited signal $f_a(t)$ with bandwidth less than or equal to $a$, we consider

$$F_a(\omega) = \left\{\begin{array}{ll} F(\omega), & |\omega| \le a \\ 0, & |\omega| > a \end{array}\right\} \tag{2.7.4}$$

and find the low-pass filter function $f_a(t)$ by the inverse Fourier transform

$$f_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} F_a(\omega) d\omega = \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega t} F_a(\omega) d\omega. \tag{2.7.5}$$

This function $f_a(t)$ is called the *Shannon sampling function.* When $a = \pi$, $f_\pi(t)$ is called the *Shannon scaling function.* The band-limited signal $f_a(t)$ is given by

$$f_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega t} d\omega = \frac{\sin at}{\pi t}. \tag{2.7.6}$$

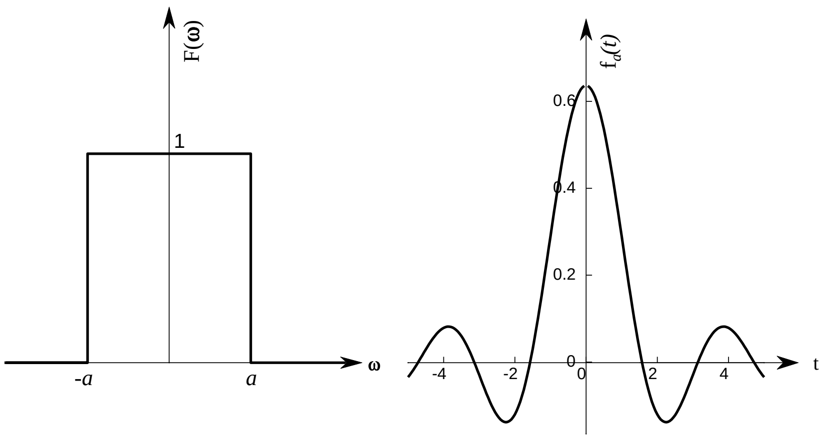


**Figure 2.6** The gate function and its Fourier transform.

Both $F(\omega)$ and $f_a(t)$ are shown in Figure 2.6 for $a=2$.

Consider the limit as $a \to \infty$ of the Fourier integral for $-\infty < \omega < \infty$

$$1 = \lim_{a\to\infty} \int_{-\infty}^{\infty} e^{-i\omega t} f_a(t)dt = \lim_{a\to\infty} \int_{-\infty}^{\infty} e^{-i\omega t} \frac{\sin at}{\pi t} dt$$

$$= \int_{-\infty}^{\infty} e^{-i\omega t} \left[ \lim_{a\to\infty} \frac{\sin at}{\pi t} \right] dt = \int_{-\infty}^{\infty} e^{-i\omega t} \delta(t) dt.$$

Clearly, the delta function $\delta(t)$ can be thought of as the limit of the sequence of functions $f_a(t)$. More precisely,

$$\delta(t) = \lim_{a\to\infty} \left( \frac{\sin at}{\pi t} \right). \tag{2.7.7}$$

We next consider the band-limited signal

$$f_a(t) = \frac{1}{2\pi} \int_{-a}^{a} F(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, F_a(\omega)\, e^{i\omega t} d\omega,$$

which is, by the Convolution Theorem,

$$f_a(t) = \int_{-\infty}^{\infty} f(\tau) f_a(t-\tau) d\tau = \int_{-\infty}^{\infty} \frac{\sin a(t-\tau)}{\pi(t-\tau)} f(\tau) d\tau. \tag{2.7.8}$$

This integral represents the *sampling integral representation* of the band-limited signal $f_a(t)$.

**Example 2.7.1**

(*Synthesis and Resolution of a Signal; Physical Interpretation of Convolution*). In electrical engineering problems, a time-dependent electric, optical or electromagnetic *pulse* is usually called a *signal*. Such a signal can be considered as a superposition of plane waves of all real frequencies so that it can be represented by the inverse Fourier transform

$$f(t) = \mathscr{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega, \tag{2.7.9}$$

where $F(\omega) = \mathscr{F}\{f(t)\}$, the factor $(1/2\pi)$ is introduced because the angular frequency $\omega$ is related to linear frequency $\nu$ by $\omega = 2\pi\nu$, and negative frequencies are introduced for mathematical convenience so that we can avoid dealing with the cosine and sine functions separately. Clearly, $F(\omega)$ can be represented by the Fourier transform of the signal $f(t)$ as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt. \tag{2.7.10}$$

This represents the *resolution* of the signal into its angular frequency components, and (2.7.9) gives a *synthesis* of the signal from its individual components.

Consider a simple electrical device such as an amplifier with an input signal $f(t)$, and an output signal $g(t)$. For an input of a single frequency $\omega$, $f(t) = e^{i\omega t}$. The amplifer will change the amplitude and may also change the phase so that the output can be expressed in terms of the input, the amplitude and the phase modifying function $\Phi(\omega)$ as

$$g(t) = \Phi(\omega)f(t), \tag{2.7.11}$$

where $\Phi(\omega)$ is usually known as the *transfer function* and is, in general, a complex function of the real variable $\omega$. This function is generally independent of the presence or absence of any other frequency components. Thus, the total output may be found by integrating over the entire input as modified by the amplifier

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\omega)F(\omega)\, e^{i\omega t}d\omega. \tag{2.7.12}$$

Thus, the total output signal can readily be calculated from any given input signal $f(t)$. On the other hand, the transfer function $\Phi(\omega)$ is obviously characteristic of the amplifier device and can, in general, be obtained as the Fourier transform of some function $\phi(t)$ so that

$$\Phi(\omega) = \int_{-\infty}^{\infty} \phi(t)e^{-i\omega t}dt. \tag{2.7.13}$$

The Convolution Theorem 2.5.5 allows us to rewrite (2.7.12) as

$$g(t) = \mathscr{F}^{-1}\{\Phi(\omega)F(\omega)\} = f(t) * \phi(t) = \int_{-\infty}^{\infty} f(\tau)\phi(t-\tau)d\tau. \tag{2.7.14}$$

Physically, this result represents an output signal $g(t)$ as the integral superposition of an input signal $f(t)$ modified by $\phi(t-\tau)$. Linear translation invariant systems, such as *sensors* and *filters*, are modeled by the convolution equations $g(t) = f(t) * \phi(t)$, where $\phi(t)$ is the system impulse response function. In fact (2.7.14) is the most general mathematical representation of an output (effect) function in terms of an input (cause) function modified by the amplifier where $t$ is the time variable. Assuming the principle of causality, that is, every effect has a cause, we must require $\tau < t$. The principle of causality is imposed by requiring

$$\phi(t-\tau) = 0 \qquad \text{when } \tau > t. \tag{2.7.15}$$

Consequently, (2.7.14) gives

$$g(t) = \int_{-\infty}^{t} f(\tau)\phi(t-\tau)d\tau. \tag{2.7.16}$$

In order to determine the significance of $\phi(t)$, we use an impulse function $f(\tau) = \delta(\tau)$ so that (2.7.16) becomes

$$g(t) = \int_{-\infty}^{t} \delta(\tau)\phi(t-\tau)d\tau = \phi(t)H(t). \tag{2.7.17}$$

This recognizes $\phi(t)$ as the output corresponding to a unit impulse at $t = 0$, and the Fourier transform of $\phi(t)$ is

$$\Phi(\omega) = \mathscr{F}\{\phi(t)\} = \int_{0}^{\infty} \phi(t)e^{-i\omega t}dt, \tag{2.7.18}$$

with $\phi(t) = 0$ for $t < 0$. ▯

***Example 2.7.2***

(*The Series Sampling Expansion of a Bandlimited Signal*). Consider a band-limited signal $f_a(t)$ with Fourier transform $F(\omega) = 0$ for $|\omega| > a$. We write the Fourier series expansion of $F(\omega)$ on the interval $-a < \omega < a$ in terms of the orthogonal set of functions $\left\{\exp\left(-\frac{in\pi\omega}{a}\right)\right\}$ in the form

$$F(\omega) = \sum_{n=-\infty}^{\infty} a_n \exp\left(-\frac{in\pi}{a}\omega\right), \tag{2.7.19}$$

where the Fourier coefficients $a_n$ are given by

$$a_n = \frac{1}{2a}\int_{-a}^{a} F(\omega)\exp\left(\frac{in\pi}{a}\omega\right)d\omega = \frac{1}{2a}\ f_a\left(\frac{n\pi}{a}\right). \tag{2.7.20}$$

Thus, the Fourier series expansion (2.7.19) becomes

$$F(\omega) = \frac{1}{2a}\sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\exp\left(-\frac{in\pi}{a}\omega\right). \tag{2.7.21}$$

The signal function $f_a(t)$ is obtained by multiplying (2.7.21) by $e^{i\omega t}$ and integrating over $(-a, a)$ so that

$$\begin{aligned}
f_a(t) &= \int_{-a}^{a} F(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2a}\int_{-a}^{a} e^{i\omega t}d\omega\left[\sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\exp\left(-\frac{in\pi}{a}\omega\right)\right] \\
&= \frac{1}{2a}\sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\int_{-a}^{a}\exp\left[i\omega\left(t-\frac{n\pi}{a}\right)\right]d\omega \\
&= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\frac{\sin a\left(t-\frac{n\pi}{a}\right)}{a\left(t-\frac{n\pi}{a}\right)} \\
&= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\frac{\sin\left(at-n\pi\right)}{\left(at-n\pi\right)}.
\end{aligned} \tag{2.7.22}$$

This result is the main content of the sampling theorem. It simply states that a band-limited signal $f_a(t)$ can be reconstructed from the infinite set of discrete samples of $f_a(t)$ at $t=0, \pm\frac{\pi}{a}, \ldots$ . In practice, a discrete set of samples is useful in the sense that most systems receive discrete samples $\{f(t_n)\}$ as an input. The sampling theorem can be realized physically. Modern telephone equipment employs sampling to send messages over wires. In fact, it seems that sampling is audible on some transoceanic cable calls.

Result (2.7.22) can be obtained from the convolution theorem by using discrete input samples

$$\sum_{n=-\infty}^{\infty}\frac{\pi}{a}f_a\left(\frac{n\pi}{a}\right)\delta\left(t-\frac{n\pi}{a}\right) = f(t). \tag{2.7.23}$$

Hence, the sampling expansion (2.7.8) gives the band-limited signal

$$\begin{aligned} f_a(t) &= \int_{-\infty}^{\infty} \frac{\sin a(t-\tau)}{\pi(t-\tau)} \left[ \sum_{n=-\infty}^{\infty} \frac{\pi}{a} f_a\left(\frac{n\pi}{a}\right) \delta\left(\tau - \frac{n\pi}{a}\right) \right] d\tau \\ &= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right) \int_{-\infty}^{\infty} \frac{\sin a(t-\tau)}{a(t-\tau)} \delta\left(\tau - \frac{n\pi}{a}\right) d\tau \\ &= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right) \frac{\sin a\left(t - \frac{n\pi}{a}\right)}{a\left(t - \frac{n\pi}{a}\right)}. \end{aligned} \tag{2.7.24}$$

□

In general, the output can be best described by taking the Fourier transform of (2.7.14) so that

$$G(\omega) = F(\omega)\Phi(\omega), \tag{2.7.25}$$

where $\Phi(\omega)$ is called the *transfer function* of the system. Thus, the output can be calculated from (2.7.25) by the Fourier inversion formula

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, \Phi(\omega)\, e^{i\omega t}\, d\omega, \tag{2.7.26}$$

Obviously, the transfer function $\Phi(\omega)$ is a characteristic of a linear system. A linear system is a *filter* if it possesses signals of certain frequencies and attenuates others. If the transfer function

$$\Phi(\omega) = 0 \qquad |\omega| \geq \omega_0, \tag{2.7.27}$$

then $\phi(t)$, the Fourier inverse of $\Phi(\omega)$, is called a *low-pass filter.*

On the other hand, if the transfer function

$$\Phi(\omega) = 0 \qquad |\omega| \leq \omega_1, \tag{2.7.28}$$

then $\phi(t)$ is a *high-pass filter.* A *bandpass filter* possesses a band $\omega_0 \leq |\omega| \leq \omega_1$. It is often convenient to express the system transfer function $\Phi(\omega)$ in the complex form

$$\Phi(\omega) = A(\omega) \exp[-i\theta(\omega)], \tag{2.7.29}$$

where $A(\omega)$ is called the *amplitude* and $\theta(\omega)$ is called the *phase* of the transfer function. Obviously, the system impulse response $\phi(t)$ is given by the inverse Fourier transform

$$\phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) \exp[i\{\omega t - \theta(\omega)\}]\, d\omega. \tag{2.7.30}$$

For a unit step function as the input $f(t) = H(t)$, we have

$$F(\omega) = \hat{H}(\omega) = \left(\pi\delta(\omega) + \frac{1}{i\omega}\right),$$

where $\hat{H}(\omega) = \mathscr{F}\{H(t)\}$ and the associated output $g(t)$ is then given by

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\omega)\hat{H}(\omega)e^{i\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\pi\delta(\omega) + \frac{1}{i\omega}\right) A(\omega)\exp[i\{\omega t - \theta(\omega)\}]\,d\omega \\ &= \frac{1}{2}A(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{A(\omega)}{\omega}\exp\left[i\left\{\omega t - \theta(\omega) - \frac{\pi}{2}\right\}\right]d\omega\,. \end{aligned} \quad (2.7.31)$$

We next give another characterization of a filter in terms of the amplitude of the transfer function.

A filter is called *distortionless* if its output $g(t)$ to an arbitrary input $f(t)$ has the same form as the input, that is,

$$g(t) = A_0 f(t - t_0). \quad (2.7.32)$$

Evidently,

$$G(\omega) = A_0 e^{-i\omega t_0}\,F(\omega) = \Phi(\omega)\,F(\omega)$$

where

$$\Phi(\omega) = A_0 e^{-i\omega t_0}$$

represents the transfer function of the distortionless filter. It has a constant amplitude $A_0$ and a linear phase shift $\theta(\omega) = \omega t_0$.

However, in general, the amplitude $A(\omega)$ of a transfer function is not constant, and the phase $\theta(\omega)$ is not a linear function.

A filter with constant amplitude, $|\theta(\omega)| = A_0$ is called an *all-pass filter*. It follows from Parseval's formula that the energy of the output of such a filter is proportional to the energy of its input.

A filter whose amplitude is constant for $|\omega| < \omega_0$ and zero for $|\omega| > \omega_0$ is called an *ideal low-pass filter*. More explicitly, the amplitude is given by

$$A(\omega) = A_0\hat{H}(\omega_0 - |\omega|) = A_0\hat{\chi}_{\omega_0}(\omega)\,, \quad (2.7.33)$$

where $\hat{\chi}_{\omega_0}(\omega)$ is a rectangular pulse. So, the transfer function of the low-pass filter is

$$\Phi(\omega) = A_0\hat{\chi}_{\omega_0}(\omega)\exp(-i\omega t_0)\,. \quad (2.7.34)$$

Finally, the *ideal high-pass filter* is characterized by its amplitude given by

$$A(\omega) = A_0\hat{H}(|\omega| - \omega_0) = A_0\hat{\chi}_{\omega_0}(\omega)\,, \quad (2.7.35)$$

where $A_0$ is a constant. Its transfer function is given by

$$\Phi(\omega) = A_0 \left[1 - \hat{\chi}_{\omega_0}(\omega)\right] \exp(-i\omega t_0)\,. \tag{2.7.36}$$

***Example 2.7.3***

(*Bandwidth and Bandwidth Equation*). The Fourier spectrum of a signal (or waveform) gives an indication of the frequencies that exist during the total duration of the signal (or *waveform*). From the knowledge of the frequencies that are present, we can calculate the average frequency and the spread about that average. In particular, if the signal is represented by $f(t)$, we can define its Fourier spectrum by

$$F(\nu) = \int_{-\infty}^{\infty} e^{-2\pi i\nu t}\, f(t)\, dt. \tag{2.7.37}$$

Using $|F(\nu)|^2$ for the density in frequency, the average frequency is denoted by $<\nu>$ and defined by

$$<\nu> = \int_{-\infty}^{\infty} \nu\, |F(\nu)|^2\, d\nu. \tag{2.7.38}$$

The bandwidth is then the *root mean square* (RMS) deviation at about the average, that is,

$$B^2 = \int_{-\infty}^{\infty} (\nu - <\nu>)^2\, d\nu. \tag{2.7.39}$$

Expressing the signal in terms of its amplitude and phase

$$f(t) = a(t)\, \exp\{i\theta t\}, \tag{2.7.40}$$

the instantaneous frequency, $\nu(t)$ is the frequency at a particular time defined by

$$\nu(t) = \frac{1}{2\pi}\, \theta'(t). \tag{2.7.41}$$

Substituting (2.7.37) and (2.7.40) into (2.7.38) gives

$$<\nu> = \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta'(t)\, a^2(t)\, dt = \int_{-\infty}^{\infty} \nu(t)\, a^2(t)\, dt. \tag{2.7.42}$$

This formula states that the average frequency is the average value of the instantaneous frequency weighted by the square of the amplitude of the signal.

We next derive the bandwidth equation in terms of the amplitude and phase of the signal in the form

$$B^2 = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \left[\frac{a'(t)}{a(t)}\right]^2 a^2(t)\, dt + \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\, \theta'(t) - <\nu>\right]^2 a^2(t)\, dt. \tag{2.7.43}$$

A straightforward but lengthy way to derive it is to substitute (2.7.40) into (2.7.39) and simplify. However, we give an elegant derivation of (2.7.43) by representing the frequency by the operator

$$\nu = \frac{1}{2\pi i}\,\frac{d}{dt}. \tag{2.7.44}$$

We calculate the average by sandwiching the operator between the complex conjugate of the signal and the signal. Thus,

$$\begin{aligned} <\nu> &= \int_{-\infty}^{\infty} \nu\,|F(\nu)|^2\,d\nu = \int_{-\infty}^{\infty} \bar{f}(t)\left[\frac{1}{2\pi i}\,\frac{d}{dt}\right] f(t)\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} a(t)\left\{-ia'(t) + a(t)\theta'(t)\right\}\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{1}{2}i\left[\frac{d}{dt}a^2(t)\right]\,dt + \frac{1}{2\pi}\int_{-\infty}^{\infty} a^2(t)\theta'(t)\,dt \qquad (2.7.45) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta'(t)a^2(t)\,dt \qquad (2.7.46) \end{aligned}$$

provided the first integral in (2.7.44) vanishes if $a(t)\to 0$ as $|t|\to\infty$.

It follows from the definition (2.7.39) of the bandwidth that

$$\begin{aligned} B^2 &= \int_{-\infty}^{\infty} (\nu - <\nu>)^2\,|F(\nu)|^2\,d\nu \\ &= \int_{-\infty}^{\infty} \bar{f}(t)\left[\frac{1}{2\pi i}\frac{d}{dt} - <\nu>\right]^2 f(t)\,dt \\ &= \int_{-\infty}^{\infty} \left|\left[\frac{1}{2\pi i}\frac{d}{dt} - <\nu>\right] f(t)\right|^2\,dt \\ &= \int_{-\infty}^{\infty} \left|\frac{1}{2\pi i}\,\frac{a'(t)}{a(t)} + \frac{1}{2\pi}\theta'(t) - <\nu>\right|^2\,a^2(t)\,dt \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\left[\frac{a'(t)}{a(t)}\right]^2 a^2(t)\,dt + \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\,\theta'(t) - <\nu>\right]^2 a^2(t)\,dt. \end{aligned}$$

This completes the derivation. ▯

Physically, the second term in equation (2.7.43) gives averages of all of the deviations of the instantaneous frequency from the average frequency. In electrical engineering literature, the spread of frequency about the *instantaneous frequency*, which is defined as an average of the frequencies that exist at a particular time, is called *instantaneous bandwidth*, given by

$$\sigma^2_{\nu/t} = \frac{1}{(2\pi)^2}\left[\frac{a'(t)}{a(t)}\right]^2. \tag{2.7.47}$$

In the case of a chirp with a Gaussian envelope

$$f(t)=\left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}}\exp\left[-\frac{1}{2}\alpha t^2+\frac{1}{2}i\beta\alpha t^2+2\pi i\nu_0 t\right], \tag{2.7.48}$$

where its *Fourier spectrum* is given by

$$F(\nu)=(\alpha\pi)^{\frac{1}{4}}\left(\frac{1}{\alpha-i\beta}\right)^{\frac{1}{2}}\exp\left[-2\pi^2(\nu-\nu_0)^2/(\alpha-i\beta)\right]. \tag{2.7.49}$$

The *energy density spectrum* of the signal is

$$|F(\nu)|^2=2\left(\frac{\alpha\pi}{\alpha^2+\beta^2}\right)^{\frac{1}{2}}\exp\left[-\frac{4\alpha\pi^2(\nu-\nu_0)^2}{\alpha^2+\beta^2}\right]. \tag{2.7.50}$$

Finally, the average frequency $<\nu>$ and the bandwidth square are respectively given by

$$<\nu>=\nu_0 \quad \text{and} \quad B^2=\frac{1}{8\pi^2}\left(\alpha+\frac{\beta^2}{\alpha}\right). \tag{2.7.51}$$

A large bandwidth can be achieved in two very qualitatively different ways. The amplitude modulation can be made large by taking $\alpha$ large, and the frequency modulation can be small by letting $\beta\to 0$. It is possible to make the frequency modulation large by making $\beta$ large and $\alpha$ very small. These two extreme situations are physically very different even though they produce the same bandwidth.

***Example 2.7.4***

Find the transfer function and the corresponding *impulse response function* of the RLC circuit governed by the differential equation

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=e(t) \tag{2.7.52}$$

where $q(t)$ is the charge, $R$, $L$, $C$ are constants, and $e(t)$ is the given voltage (input).

Equation (2.7.25) provides the definition of the *transfer function* in the frequency domain

$$\Phi(\omega)=\frac{G(\omega)}{F(\omega)}=\frac{\mathscr{F}\{g(t)\}}{\mathscr{F}\{f(t)\}}, \tag{2.7.53}$$

where $\phi(t)=\mathscr{F}^{-1}\{\Phi(\omega)\}$ is called the *impulse response function*.

Taking the Fourier transfrom of (2.7.52) gives

$$\left(-L\omega^2+Ri\omega+\frac{1}{C}\right)Q(\omega)=E(\omega). \tag{2.7.54}$$

Thus, the transfer function is

$$\Phi(\omega) = \frac{Q(\omega)}{E(\omega)} = \frac{-C}{LC\omega^2 - iRC\omega - 1}$$
$$= \frac{i}{2L\beta}\left[\frac{1}{\omega - i(\alpha+\beta)} - \frac{1}{\omega - i(\alpha-\beta)}\right], \qquad (2.7.55)$$

where

$$\alpha = \frac{R}{2L} \quad \text{and} \quad \beta = \left[\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right]^{\frac{1}{2}}. \qquad (2.7.56)$$

The inverse Fourier transform of (2.7.55) yields the impulse response function

$$\phi(t) = \frac{1}{2\beta L}\left(e^{\beta t} - e^{-\beta t}\right)e^{-\alpha t}H(t). \qquad (2.7.57)$$

$\Box$

## 2.8 Gibbs' Phenomenon

We now examine the so-called the *Gibbs jump phenomenon* which deals with the limiting behavior of a band-limited signal $f_{\omega_0}(t)$ represented by the sampling integral representation (2.7.8) at a point of discontinuity of $f(t)$. This phenomenon reveals the intrinsic overshoot near a jump discontinuity of a function associated with the Fourier series. More precisely, the partial sums of the Fourier series overshoot the function near the discontinuity, and the overshoot continues no matter how many terms are taken in the partial sum. However, the Gibbs phenomenon does not occur if the partial sums are replaced by the Cesaro means, the average of the partial sums.

In order to demonstrate the Gibbs phenomenon, we rewrite (2.7.8) in the convolution form

$$f_{\omega_0}(t) = \int_{-\infty}^{\infty} f(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau = (f * \delta_{\omega_0})(t), \qquad (2.8.1)$$

where

$$\delta_{\omega_0}(t) = \frac{\sin\omega_0 t}{\pi t}. \qquad (2.8.2)$$

Clearly, at every point of continuity of $f(t)$, we have

$$\begin{aligned}
\lim_{\omega_0\to\infty} f_{\omega_0}(t) &= \lim_{\omega_0\to\infty} (f*\delta_{\omega_0})(t) = \lim_{\omega_0\to\infty}\int_{-\infty}^{\infty} f(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau \\
&= \int_{-\infty}^{\infty} f(\tau)\left[\lim_{\omega_0\to\infty}\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\right]d\tau \\
&= \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,d\tau = f(t)\,. \qquad (2.8.3)
\end{aligned}$$

We now consider the limiting behavior of $f_{\omega_0}(t)$ at the point of discontinuity $t=t_0$. To simplify the calculation, we set $t_0=0$ so that we can write $f(t)$ as a sum of a continuous function, $f_c(t)$ and a suitable step function

$$f(t) = f_c(t) + [f(0+) - f(0-)]\;H(t). \qquad (2.8.4)$$

Replacing $f(t)$ by the right hand side of (2.8.4) in Equation (2.8.1) yields

$$\begin{aligned}
f_{\omega_0}(t) &= \int_{-\infty}^{\infty} f_c(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau \\
&\quad + [f(0+)-f(0-)]\int_{-\infty}^{\infty} H(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau \\
&= f_c(t) + [f(0+)-f(0-)]\;H_{\omega_0}(t)\,, \qquad (2.8.5)
\end{aligned}$$

where

$$\begin{aligned}
H_{\omega_0}(t) &= \int_{-\infty}^{\infty} H(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau = \int_0^{\infty}\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau \\
&= \int_{-\infty}^{\omega_0 t}\frac{\sin x}{\pi x}\,dx \qquad (\text{putting } \omega_0(t-\tau)=x) \\
&= \left(\int_{-\infty}^{0} + \int_0^{\omega_0 t}\right)\left(\frac{\sin x}{\pi x}\right)dx = \left(\int_0^{\infty} + \int_0^{\omega_0 t}\right)\left(\frac{\sin x}{\pi x}\right)dx \\
&= \frac{1}{2} + \frac{1}{\pi}\,si(\omega_0 t)\,, \qquad (2.8.6)
\end{aligned}$$

and the function $si(t)$ is defined by

$$si(t) = \int_0^t \frac{\sin x}{x}\,dx\,. \qquad (2.8.7)$$

Note that

$$H_{\omega_0}\left(\frac{\pi}{\omega_0}\right) = \frac{1}{2} + \int_0^{\pi}\frac{\sin x}{\pi x}\,dx > 1\,, \quad H_{\omega_0}\left(-\frac{\pi}{\omega_0}\right) = \frac{1}{2} - \int_0^{\pi}\frac{\sin x}{\pi x}\,dx < 0\,.$$

Clearly, for a fixed $\omega_0$, $\frac{1}{\pi}si(\omega_0 t)$ attains its maximum at $t=\frac{\pi}{\omega_0}$ in $(0,\infty)$ and minimum at $t=-\frac{\pi}{\omega_0}$, since for a larger $t$ the integrand oscillates with decreasing amplitudes. The function $H_{\omega_0}(t)$ is shown in Figure 2.7 since $H_{\omega_0}(0)=\frac{1}{2}$

and $f_c(0) = f(0-)$ and

$$f_{\omega_0}(0) = f_c(0) + \frac{1}{2}\left[f(0+) - f(0-)\right] = \frac{1}{2}\left[f(0+) + f(0-)\right] .$$

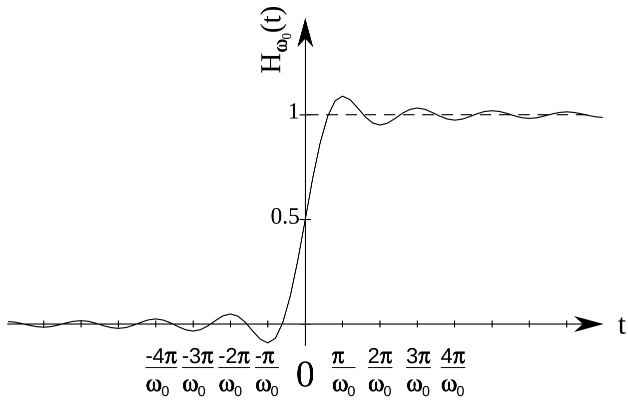


**Figure 2.7** Graph of $H_{\omega_0}(t)$.

Thus, the graph of $H_{\omega_0}(t)$ shows that as $\omega_0$ increases, the time scale changes, and the ripples remain the same. In the limit $\omega_0 \to \infty$, the convergence of $H_{\omega_0}(t) = (H * \delta_{\omega_0})(t)$ to $H(t)$ exhibits the intrinsic overshoot leading to the classical Gibbs phenomenon.

***Example 2.8.1***

(*The Square Wave Function and the Gibbs Phenomenon*). Consider the single-pulse square function defined by

$$f(x) = \left\{\begin{array}{ll} 1, & -a < x < a \\ \frac{1}{2}, & x = \pm a \\ 0, & |x| > a \end{array}\right\} .$$

The graph of $f(x)$ is given in Figure 2.8.

Thus,

$$F(k) = \mathscr{F}\{f(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right) .$$

▯

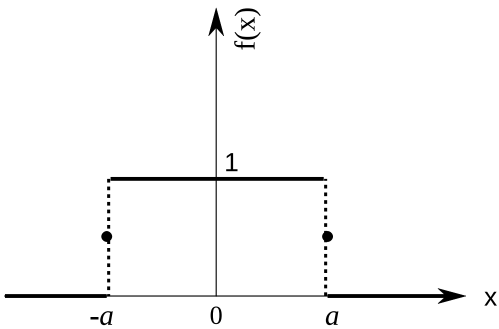


**Figure 2.8** The square wave function.

We next define a function $f_\lambda(x)$ by the integral

$$f_\lambda(x) = \int_{-\lambda}^{\lambda} F(k)\, e^{ikx}\, dk$$

.

As $|\lambda| \to \infty$, $f_\lambda(x)$ will tend pointwise to $f(x)$ for all $x$. Convergence occurs even at $x = \pm\, a$ because the function $f(x)$ is defined to have a value “half way up the step” at these points. Let us examine the behavior of $f_\lambda(x)$ as $|\lambda| \to \infty$ in a region just one side of one of the discontinuities, that is, for $x \in (0, a)$. For a fixed $\lambda$, the difference, $f_\lambda(x) - f(x)$, oscillates above and below the value 0 as $x \to a$, attaining a maximum positive value at some point, say $x = x_\lambda$. Then the quantity $f_\lambda(x_\lambda) - f(x_\lambda)$ is called the *overshoot*.

As $|\lambda| \to \infty$, so the period of the oscillations tends to zero and so also $x_\lambda \to a$; however, the value of the overshoot $f_\lambda(x_\lambda) - f(x_\lambda)$ does not tend to zero but instead tends to a finite limit. The existence of this non-zero, finite, limiting value for the overshoot is known as the *Gibbs phenomenon.* This phenomenon also occurs in an almost identical manner in the Fourier synthesis of periodic functions using Fourier series.

## 2.9 Heisenberg’s Uncertainty Principle

If $f \in L^2(\mathbb{R})$, then $f$ and $F(k) = \mathscr{F}\,\{f(x)\}$ cannot both be essentially localized. In other words, it is not possible that the widths of the graphs of $|f(x)|^2$ and $|F(k)|^2$ can both be made arbitrarily small. This fact underlines the Heisenberg uncertainty principle in quantum mechanics and the bandwidth theorem in signal analysis. If $|f(x)|^2$ and $|F(k)|^2$ are interpreted as weighting functions, then the weighted means (averages) $<x>$ and $<k>$ of $x$ and $k$ are given by

$$<x> = \frac{1}{||f||_2^2} \int_{-\infty}^{\infty} x\,|f(x)|^2\,dx, \tag{2.9.1}$$

$$<k> = \frac{1}{||F||_2^2} \int_{-\infty}^{\infty} k\,|F(k)|^2\,dk. \tag{2.9.2}$$

Corresponding measures of the widths of these weight functions are given by the second moments about the respective means. Usually, it is convenient to define widths $\triangle x$ and $\triangle k$ by

$$(\triangle x)^2 = \frac{1}{||f||_2^2} \int_{-\infty}^{\infty} (x-<x>)^2\;|f(x)|^2\,dx, \tag{2.9.3}$$

$$(\triangle k)^2 = \frac{1}{||F||_2^2} \int_{-\infty}^{\infty} (k-<k>)^2\;|F(k)|^2\,dk. \tag{2.9.4}$$

The essence of the *Heisenberg principle* and the bandwidth theorems lies in the fact that the product $(\triangle x)(\triangle k)$ will never less than $\frac{1}{2}$. Indeed,

$$(\triangle x)(\triangle k) \geq \frac{1}{2}, \tag{2.9.5}$$

where equality in (2.9.5) holds only if $f(x)$ is a Gaussian function given by $f(x)=C\exp(-ax^2)$, $a>0$.

We next state the Heisenberg inequality theorem as follows:

***THEOREM 2.9.1***

(*Heisenberg Inequality*). If $f(x)$, $x\,f(x)$ and $k\,F(k)$ belong to $L^2(\mathbb{R})$ and $\sqrt{x}|f(x)$ $|\to 0$ as $|x|\to\infty$, then

$$(\triangle x)^2(\triangle k)^2 \geq \frac{1}{4}, \tag{2.9.6}$$

where $(\triangle x)^2$ and $(\triangle k)^2$ are defined by (2.9.3) and (2.9.4) respectively. Equality in (2.9.6) holds only if $f(x)$ is a *Gaussian function* given by $f(x)=C\,e^{-ax^2}$, $a>0$.

**PROOF** If the averages are $<x>$ and $<k>$, then the average location of $\exp(-i<k>x)f(x+<x>)$ is zero. Hence, it is sufficient to prove the theorem around the zero mean values, that is, $<x>=<k>=0$. Since $||f||_2=||F||_2$, we have

$$||f||_2^4(\triangle x)^2(\triangle k)^2 = \int_{-\infty}^{\infty} |xf(x)|^2 dx \int_{-\infty}^{\infty} |kF(k)|^2 dk.$$

Using $ikF(k) = \mathscr{F}\{f'(x)\}$ and the Parseval formula $||f'(x)||_2 = ||ikF(k)||_2$, we obtain

$$\begin{aligned}
||f||_2^4(\triangle x)^2(\triangle k)^2 &= \int_{-\infty}^{\infty} |xf(x)|^2 dx \int_{-\infty}^{\infty} |f'(x)|^2 dx \\
&\geq \left| \int_{-\infty}^{\infty} \left\{ xf(x)\, \overline{f'(x)} \right\} dx \right|^2, \ \text{(see Debnath (2002))} \\
&\geq \left| \int_{-\infty}^{\infty} x.\frac{1}{2} \left\{ f'(x)\, \overline{f(x)} + \overline{f'(x)}\, f(x) \right\} \right|^2 \\
&= \frac{1}{4} \left[ \int_{-\infty}^{\infty} x \left( \frac{d}{dx} |f|^2 \right) dx \right]^2 \\
&= \frac{1}{4} \left\{ \left[ x|f(x)|^2 \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} |f|^2 dx \right\}^2 = \frac{1}{4} ||f||_2^4 .
\end{aligned}$$

in which $\sqrt{x} f(x) \to 0$ as $|x| \to \infty$ was used to eliminate the integrated term. This completes the proof.
If we assume $f'(x)$ is proportional to $x\, f(x)$, that is, $f'(x) = b\, x\, f(x)$, where $b$ is a constant of proportionality, this leads to the *Gaussian signals*

$$f(x) = C \exp(-ax^2),$$

where $C$ is a constant of integration and $a = -\frac{b}{2} > 0$. ∎

In 1924, Heisenberg first formulated the uncertainty principle between the position and momentum in quantum mechanics. This principle has an important interpretation as an uncertainty of both the position and momentum of a particle described by a wave function $\psi \in L^2(\mathbb{R})$. In other words, it is not possible to determine the position and momentum of a particle exactly and simultaneously.
In signal processing, time and frequency concentrations of energy of a signal $f$ are also governed by the Heisenberg uncertainty principle. The average or expectation values of time $t$ and frequency $\omega$, are respectively defined by

$$< t > = \frac{1}{||f||_2^2} \int_{-\infty}^{\infty} t|f(t)|^2 dt, \qquad < \omega > = \frac{1}{||F||_2^2} \int_{-\infty}^{\infty} \omega |F(\omega)|^2 d\omega, \quad (2.9.7)$$

where the energy of a signal $f(t)$ is well localized in time, and its Fourier transform $F(\omega)$ has an energy concentrated in a small frequency domain.
The variances around these average values are given respectively by

$$\begin{aligned}
\sigma_t^2 &= \frac{1}{||f||_2^2} \int_{-\infty}^{\infty} (t - < t >)^2 |f(t)|^2 dt, \\
\sigma_\omega^2 &= \frac{1}{2\pi ||F||_2^2} \int_{-\infty}^{\infty} (\omega - < \omega >)^2 |F(\omega)|^2 d\omega.
\end{aligned} \quad (2.9.8)$$

Remarks:

1. In a time-frequency analysis of signals, the measure of the resolution of a signal $f$ in the time or frequency domain is given by $\sigma_t$ and $\sigma_\omega$. Then, the joint resolution is given by the product $(\sigma_t)$ $(\sigma_\omega)$ which is governed by the Heisenberg uncertainty principle. In other words, the product $(\sigma_t)$ $(\sigma_\omega)$ cannot be arbitrarily small and is always greater than the minimum value $\frac{1}{2}$ which is attained for the Gaussian signal.

2. In many applications in science and engineering, signals with a high concentration of energy in the time and frequency domains are of special interest. The uncertainty principle can also be interpreted as a measure of this concentration of the second moment of $f^2(t)$ and its energy spectrum $F^2(\omega)$.

## 2.10 Applications of Fourier Transforms to Ordinary Differential Equations

We consider the $n$th order linear ordinary differential equation with constant coefficients

$$Ly(x) = f(x), \tag{2.10.1}$$

where $L$ is the $n$th order differential operator given by

$$L \equiv a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0, \tag{2.10.2}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, $D \equiv \frac{d}{dx}$ and $f(x)$ is a given function.

Application of the Fourier transform to both sides of (2.10.1) gives

$$[a_n(ik)^n + a_{n-1}(ik)^{n-1} + \cdots + a_1(ik) + a_0]Y(k) = F(k),$$

where $\mathscr{F}\{y(x)\} = Y(k)$ and $\mathscr{F}\{f(x)\} = F(k)$.

Or, equivalently

$$P(ik)Y(k) = F(k),$$

where

$$P(z) = \sum_{r=0}^{n} a_r z^r.$$

Thus,

$$Y(k) = \frac{F(k)}{P(ik)} = F(k)Q(k), \tag{2.10.3}$$

where $Q(k) = \frac{1}{P(ik)}$.

Applying the Convolution Theorem 2.5.5 to (2.10.3) gives the formal solution

$$y(x) = \mathscr{F}^{-1}\{F(k)\,Q(k)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) q(x-\xi) d\xi, \qquad (2.10.4)$$

provided $q(x) = \mathscr{F}^{-1}\{Q(k)\}$ is known explicitly.

In order to give a physical interpretation of the solution (2.10.4), we consider the differential equation with a suddenly applied impulse function $f(x) = \delta(x)$ so that

$$L\{G(x)\} = \delta(x). \qquad (2.10.5)$$

The solution of this equation can be written from the inversion of (2.10.3) in the form

$$G(x) = \mathscr{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\,Q(k)\right\} = \frac{1}{\sqrt{2\pi}}\,q(x). \qquad (2.10.6)$$

Thus, the solution (2.10.4) takes the form

$$y(x) = \int_{-\infty}^{\infty} f(\xi) G(x-\xi) d\xi. \qquad (2.10.7)$$

Clearly, $G(x)$ behaves like a *Green's function*, that is, it is the response to a *unit impulse*. In any physical system, $f(x)$ usually represents the *input function*, while $y(x)$ is referred to as the *output* obtained by the superposition principle. The Fourier transform of $\{\sqrt{2\pi}G(x)\} = q(x)$ is called the *admittance*. In order to find the reponse to a given input, we determine the Fourier transform of the input function, multiply the result by the admittance, and then apply the inverse Fourier transform to the product so obtained.

We illustrate these ideas by solving a simple problem in the electrical circuit theory.

### *Example 2.10.1*

(*Electric Current in a Simple Circuit*). The current $I(t)$ in a simple circuit containing the resistance $R$ and inductance $L$ satisfies the equation

$$L\,\frac{dI}{dt} + RI = E(t), \qquad (2.10.8)$$

where $E(t)$ is the applied electromagnetic force and $R$ and $L$ are constants.

With $E(t) = E_0 \exp(-a|t|)$, we use the Fourier transform with respect to time $t$ to obtain

$$(ikL+R)\hat{I}(k) = E_0\sqrt{\frac{2}{\pi}}\,\frac{a}{(a^2+k^2)}.$$

Or,

$$\hat{I}(k) = \frac{aE_0}{iL}\sqrt{\frac{2}{\pi}}\frac{1}{\left(k - \frac{Ri}{L}\right)\,(k^2+a^2)},$$

where $\mathscr{F}\{I(t)\} = \hat{I}(k)$. The inverse Fourier transform gives

$$I(t) = \frac{aE_0}{i\pi L}\int_{-\infty}^{\infty}\frac{\exp(ikt)dk}{\left(k - \frac{Ri}{L}\right)(k^2+a^2)}. \tag{2.10.9}$$

This integral can be evaluated by the Cauchy Residue Theorem. For $t > 0$

$$\begin{aligned}
I(t) &= \frac{aE_0}{i\pi L}\cdot 2\pi i\left[\text{Residue at}\,k = \frac{Ri}{L} + \text{Residue at}\,k = ia\right] \\
&= \frac{2aE_0}{L}\left[\frac{e^{-\frac{R}{L}t}}{\left(a^2 - \frac{R^2}{L^2}\right)} - \frac{e^{-at}}{2a\left(a - \frac{R}{L}\right)}\right] \\
&= E_0\left[\frac{e^{-at}}{R - aL} - \frac{2aLe^{-\frac{R}{L}t}}{R^2 - a^2L^2}\right].
\end{aligned} \tag{2.10.10}$$

Similarly, for $t < 0$, the Residue Theorem gives

$$\begin{aligned}
I(t) &= -\frac{aE_0}{i\pi L}\cdot 2\pi i[\text{Residue at}\,k = -ia] \\
&= -\frac{2aE_0}{L}\left[\frac{-Le^{at}}{(aL+R)2a}\right] = \frac{E_0e^{at}}{(aL+R)}.
\end{aligned} \tag{2.10.11}$$

At $t = 0$, the current is continuous and therefore,

$$I(0) = \lim_{t\to 0} I(t) = \frac{E_0}{R + aL}.$$

If $E(t) = \delta(t)$, then $\hat{E}(k) = \frac{1}{\sqrt{2\pi}}$ and the solution is obtained by using the inverse Fourier transform

$$I(t) = \frac{1}{2\pi iL}\int_{-\infty}^{\infty}\frac{e^{ikt}}{k - \frac{iR}{L}}dk,$$

which is, by the Theorem of Residues,

$$\begin{aligned}
&= \frac{1}{L}[\text{Residue at}\,k = iR/L] \\
&= \frac{1}{L}\exp\left(-\frac{Rt}{L}\right).
\end{aligned} \tag{2.10.12}$$

Thus, the current tends to zero as $t \to \infty$ as expected. ▯

***Example 2.10.2***
Find the solution of the ordinary differential equation

$$-\frac{d^2u}{dx^2}+a^2u=f(x), \quad -\infty<x<\infty \tag{2.10.13}$$

by the Fourier transform method.

Application of the Fourier transform to (2.10.13) gives

$$U(k)=\frac{F(k)}{k^2+a^2}.$$

This can readily be inverted by the Convolution Theorem 2.5.5 to obtain

$$u(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi, \tag{2.10.14}$$

where $g(x)=\mathscr{F}^{-1}\left\{\frac{1}{k^2+a^2}\right\}=\frac{1}{a}\sqrt{\frac{\pi}{2}}\exp(-a|x|)$ by Example 2.3.2. Thus, the final solution is

$$u(x)=\frac{1}{2a}\int_{-\infty}^{\infty} f(\xi)e^{-a|x-\xi|}\,d\xi. \tag{2.10.15}$$

$\square$

***Example 2.10.3***
(*The Bernoulli-Euler Beam Equation*). We consider the vertical deflection $u(x)$ of an infinite beam on an elastic foundation under the action of a prescribed vertical load $W(x)$. The deflection $u(x)$ satisfies the ordinary differential equation

$$EI\frac{d^4u}{dx^4}+\kappa u=W(x), \quad -\infty<x<\infty. \tag{2.10.16}$$

where $EI$ is the flexural rigidity and $\kappa$ is the foundation modulus of the beam. We find the solution assuming that $W(x)$ has a compact support and $u, u', u'', u'''$ all tend to zero as $|x|\to\infty$.

We first rewrite (2.10.16) as

$$\frac{d^4u}{dx^4}+a^4u=w(x) \tag{2.10.17}$$

where $a^4=\kappa/EI$ and $w(x)=W(x)/EI$. Use of the Fourier transform to (2.10.17) gives

$$U(k)=\frac{W(k)}{k^4+a^4}.$$

The inverse Fourier transform gives the solution

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{W(k)}{k^4 + a^4} e^{ikx}\, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^4 + a^4} dk \int_{-\infty}^{\infty} w(\xi) e^{-ik\xi}\, d\xi$$

$$= \int_{-\infty}^{\infty} w(\xi) G(\xi, x)\, d\xi, \tag{2.10.18}$$

where

$$G(\xi, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik(x-\xi)}}{k^4 + a^4} dk = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos k(x-\xi)\, dk}{k^4 + a^4}. \tag{2.10.19}$$

The integral can be evaluated by the Theorem of Residues or by using the table of Fourier integrals. We simply state the result

$$G(\xi, x) = \frac{1}{2a^3} \exp\left(-\frac{a}{\sqrt{2}}|x - \xi|\right) \sin\left[\frac{a(x-\xi)}{\sqrt{2}} + \frac{\pi}{4}\right]. \tag{2.10.20}$$

In particular, we find the explicit solution due to a concentrated load of unit strength acting at some point $x_0$, that is, $w(x) = \delta(x - x_0)$. Then the solution for this case becomes

$$u(x) = \int_{-\infty}^{\infty} \delta(\xi - x_0) G(x, \xi)\, d\xi = G(x, x_0). \tag{2.10.21}$$

Thus, the kernel $G(x, \xi)$ involved in the solution (2.10.18) has the physical significance of being the deflection, as a function of $x$, due to a unit point load acting at $\xi$. Thus, the deflection due to a point load of strength $w(\xi)\, d\xi$ at $\xi$ is $w(\xi)\, d\xi \cdot G(x, \xi)$, and hence, (2.10.18) represents the superposition of all such incremental deflections.

The reader is referred to a more general dynamic problem of an infinite Bernoulli-Euler beam with damping and elastic foundation that has been solved by Stadler and Shreeves (1970), and also by Sheehan and Debnath (1972). These authors used the Fourier-Laplace transform method to determine the steady state and the transient solutions of the beam problem. ▯

## 2.11 Solutions of Integral Equations

The method of Fourier transforms can be used to solve simple integral equations of the convolution type. We illustrate the method by examples.

We first solve the *Fredholm integral equation* with convolution kernel in the form

$$\int_{-\infty}^{\infty} f(t)g(x-t)\,dt + \lambda f(x) = u(x), \tag{2.11.1}$$

where $g(x)$ and $u(x)$ are given functions and $\lambda$ is a known parameter.

Application of the Fourier transform to (2.11.1) gives

$$\sqrt{2\pi}F(k)G(k) + \lambda F(k) = U(k).$$

Or,

$$F(k) = \frac{U(k)}{\sqrt{2\pi}G(k) + \lambda}. \tag{2.11.2}$$

The inverse Fourier transform leads to a formal solution

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{U(k)e^{ikx}\,dk}{\sqrt{2\pi}G(k) + \lambda}. \tag{2.11.3}$$

In particular, if $g(x) = \frac{1}{x}$ so that

$$G(k) = -i\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}\,\mathrm{k},$$

then the solution becomes

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{U(k)e^{ikx}\,dk}{\lambda - i\pi\,\mathrm{sgn}\,\mathrm{k}}. \tag{2.11.4}$$

If $\lambda = 1$ and $g(x) = \frac{1}{2}\left(\frac{x}{|x|}\right)$ so that $G(k) = \frac{1}{\sqrt{2\pi}}\frac{1}{(ik)}$, solution (2.11.3) reduces to the form

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ik)\frac{U(k)e^{ikx}\,dk}{(1+ik)} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathscr{F}\{u'(x)\}\mathscr{F}\{\sqrt{2\pi}\,e^{-x}\}e^{ikx}\,dk \\ &= u'(x) * \sqrt{2\pi}\,e^{-x} = \int_{-\infty}^{\infty} u'(\xi)\exp(\xi - x)\,d\xi. \end{aligned} \tag{2.11.5}$$

***Example 2.11.1***
Find the solution of the integral equation

$$\int_{-\infty}^{\infty} f(x-\xi)f(\xi)\,d\xi = \frac{1}{x^2+a^2}. \tag{2.11.6}$$

Application of the Fourier transform gives

$$\sqrt{2\pi}F(k)\,F(k) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-a|k|}}{a}.$$

Or,

$$F(k) = \frac{1}{\sqrt{2a}}\exp\left\{-\frac{1}{2}a|k|\right\}. \tag{2.11.7}$$

The inverse Fourier transform gives the solution

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{2a}}\int_{-\infty}^{\infty}\exp\left(ikx-\frac{1}{2}a|k|\right)\,dk \\
&= \frac{1}{2\sqrt{\pi a}}\left[\int_0^{\infty}\exp\left\{-k\left(\frac{a}{2}+ix\right)\right\}\,dk + \int_0^{\infty}\exp\left\{-k\left(\frac{a}{2}-ix\right)\right\}\,dk\right] \\
&= \frac{1}{2\sqrt{\pi a}}\left[\frac{4a}{(4x^2+a^2)}\right] = \sqrt{\frac{a}{\pi}}\cdot\frac{2}{(4x^2+a^2)}.
\end{aligned}$$

□

***Example 2.11.2***
Solve the integral equation

$$\int_{-\infty}^{\infty}\frac{f(t)\,dt}{(x-t)^2+a^2} = \frac{1}{(x^2+b^2)},\ b>a>0. \tag{2.11.8}$$

Taking the Fourier transform, we obtain

$$\sqrt{2\pi}\,F(k)\,\mathscr{F}\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\,\frac{e^{-b|k|}}{b},$$

or,

$$\sqrt{2\pi}\,F(k)\sqrt{\frac{\pi}{2}}\cdot\frac{e^{-a|k|}}{a} = \sqrt{\frac{\pi}{2}}\,\frac{e^{-b|k|}}{b}.$$

Thus,

$$F(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{a}{b}\right)\exp\{-|k|(b-a)\}. \tag{2.11.9}$$

The inverse Fourier transform leads to the solution

$$\begin{aligned}
f(x) &= \frac{a}{2\pi b}\int_{-\infty}^{\infty} \exp[ikx - |k|(b-a)]dk \\
&= \frac{a}{2\pi b}\left[\int_{0}^{\infty} \exp[-k\{(b-a)+ix\}]dk + \int_{0}^{\infty} \exp[-k\{(b-a)-ix\}]\right] dk \\
&= \frac{a}{2\pi b}\left[\frac{1}{(b-a)+ix} + \frac{1}{(b-a)-ix}\right] \\
&= \left(\frac{a}{\pi b}\right)\frac{(b-a)}{(b-a)^2 + x^2}. \qquad (2.11.10)
\end{aligned}$$

▯

***Example 2.11.3***
Solve the integral equation

$$f(t) + 4\int_{-\infty}^{\infty} e^{-a|x-t|} f(t)dt = g(x). \qquad (2.11.11)$$

Application of the Fourier transform gives

$$\begin{aligned}
F(k) + 4\sqrt{2\pi}F(k)\cdot \frac{2a}{\sqrt{2\pi}(a^2+k^2)} &= G(k) \\
F(k) = \frac{(a^2+k^2)}{a^2+k^2+8a}\, G(k). \qquad (2.11.12)
\end{aligned}$$

The inverse Fourier transform gives

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{(a^2+k^2)G(k)}{a^2+k^2+8a} e^{ikx} dk. \qquad (2.11.13)$$

In particular, if $a = 1$ and $g(x) = e^{-|x|}$ so that $G(k) = \sqrt{\frac{2}{\pi}}\frac{1}{1+k^2}$, then solution (2.11.13) becomes

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2+3^2}\, dk. \qquad (2.11.14)$$

For $x > 0$, we use a semicircular closed contour in the lower half of the complex plane to evaluate (2.11.14). It turns out that

$$f(x) = \frac{1}{3}\, e^{-3x}. \qquad (2.11.15)$$

Similarly, for $x<0$, a semicircular closed contour in the upper half of the complex plane is used to evaluate (2.11.14) so that

$$f(x)=\frac{1}{3}\,e^{3x}, \quad x<0. \tag{2.11.16}$$

Thus, the final solution is

$$f(x)=\frac{1}{3}\exp(-3|x|). \tag{2.11.17}$$

□

## 2.12 Solutions of Partial Differential Equations

In this section we illustrate how the Fourier transform method can be used to obtain the solution of boundary value and initial value problems for linear partial differential equations of different kinds.

***Example 2.12.1***
(*Dirichlet's Problem in the Half-Plane*). We consider the solution of the Laplace equation in the half-plane

$$u_{xx}+u_{yy}=0, \quad -\infty<x<\infty, \quad y\geq 0, \tag{2.12.1}$$

with the boundary conditions

$$u(x,0)=f(x), \qquad -\infty<x<\infty, \tag{2.12.2}$$

$$u(x,y)\to 0 \text{ as } |x|\to\infty, \quad y\to\infty. \tag{2.12.3}$$

We introduce the Fourier transform with respect to $x$

$$U(k,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}u(x,y)dx \tag{2.12.4}$$

so that (2.12.1)–(2.12.3) becomes

$$\frac{d^2U}{dy^2}-k^2U=0, \tag{2.12.5}$$

$$U(k,0)=F(k), \quad U(k,y)\to 0 \qquad \text{as } y\to\infty. \tag{2.12.6ab}$$

Thus, the solution of this transformed system is

$$U(k,y)=F(k)e^{-|k|y}. \tag{2.12.7}$$

Application of the Convolution Theorem 2.5.5 gives the solution

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi, \tag{2.12.8}$$

where

$$g(x) = \mathscr{F}^{-1}\{e^{-|k|y}\} = \sqrt{\frac{2}{\pi}} \frac{y}{(x^2+y^2)}. \tag{2.12.9}$$

Consequently, the solution (2.12.8) becomes

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi) d\xi}{(x-\xi)^2+y^2}, \quad y>0. \tag{2.12.10}$$

This is the well-known *Poisson integral formula* in the half-plane. It is noted that

$$\lim_{y\to 0+} u(x,y) = \int_{-\infty}^{\infty} f(\xi) \left[\lim_{y\to 0+} \frac{y}{\pi} \cdot \frac{1}{(x-\xi)^2+y^2}\right] d\xi = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi) d\xi, \tag{2.12.11}$$

where Cauchy's definition of the delta function is used, that is,

$$\delta(x-\xi) = \lim_{y\to 0+} \frac{y}{\pi} \cdot \frac{1}{(x-\xi)^2+y^2}. \tag{2.12.12}$$

This may be recognized as a solution of the Laplace equation for a dipole source at $(x,y) = (\xi, 0)$.

In particular, when

$$f(x) = T_0 H(a-|x|) \tag{2.12.13}$$

the solution (2.12.10) reduces to

$$\begin{aligned} u(x,y) &= \frac{yT_0}{\pi} \int_{-a}^{a} \frac{d\xi}{(\xi-x)^2+y^2} \\ &= \frac{T_0}{\pi}\left[\tan^{-1}\left(\frac{x+a}{y}\right) - \tan^{-1}\left(\frac{x-a}{y}\right)\right] \\ &= \frac{T_0}{\pi} \tan^{-1}\left(\frac{2ay}{x^2+y^2-a^2}\right). \end{aligned} \tag{2.12.14}$$

The curves in the upper half-plane for which the steady state temperature is constant are known as *isothermal curves.* In this case, these curves represent a family of circular arcs

$$x^2+y^2-\alpha y = a^2 \tag{2.12.15}$$

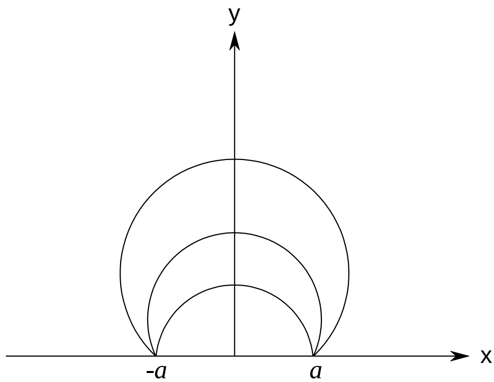


**Figure 2.9** A family of circular arcs.

with centers on the $y$-axis and the fixed end points on the $x$-axis at $x = \pm a$. The graphs of the arcs are is displayed in Figure 2.9.

Another special case deals with

$$f(x) = \delta(x). \tag{2.12.16}$$

The solution for this case follows from (2.12.10) and is

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\delta(\xi)d\xi}{(x-\xi)^2 + y^2} = \frac{y}{\pi} \frac{1}{(x^2+y^2)}. \tag{2.12.17}$$

Further, we can readily deduce the solution of the *Neumann problem* in the half-plane from the solution of the Dirichlet problem. ▯

***Example 2.12.2***

(*Neumann's Problem in the Half-Plane*). Find a solution of the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0, \tag{2.12.18}$$

with the boundary condition

$$u_y(x,0) = f(x), \quad -\infty < x < \infty. \tag{2.12.19}$$

This condition specifies the normal derivative on the boundary, and physically, it describes the fluid flow or, heat flux at the boundary.

We define a new function $v(x,y) = u_y(x,y)$ so that

$$u(x,y) = \int^{y} v(x,\eta)d\eta, \tag{2.12.20}$$

where an arbitrary constant can be added to the right-hand side. Clearly, the function $v$ satisfies the Laplace equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}=\frac{\partial}{\partial y}(u_{xx}+u_{yy})=0,$$

with the boundary condition

$$v(x,0)=u_y(x,0)=f(x) \text{ for } -\infty<x<\infty.$$

Thus, $v(x,y)$ satisfies the Laplace equation with the Dirichlet condition on the boundary. Obviously, the solution is given by (2.12.10); that is,

$$v(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{(x-\xi)^2+y^2}. \tag{2.12.21}$$

Then the solution $u(x,y)$ can be obtained from (2.12.20) in the form

$$\begin{aligned} u(x,y) &= \int^{y} v(x,\eta)d\eta=\frac{1}{\pi}\int^{y}\eta\,d\eta\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{(x-\xi)^2+\eta^2} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}f(\xi)d\xi\int^{y}\frac{\eta\,d\eta}{(x-\xi)^2+\eta^2}, \quad y>0 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}f(\xi)\log[(x-\xi)^2+y^2]d\xi, \end{aligned} \tag{2.12.22}$$

where an arbitrary constant can be added to this solution. In other words, the solution of any Neumann problem is uniquely determined up to an arbitrary constant. ▯

***Example 2.12.3***
(*The Cauchy Problem for the Diffusion Equation*). We consider the initial value problem for a one-dimensional diffusion equation with no sources or sinks

$$u_t=\kappa u_{xx}, \quad -\infty<x<\infty, \quad t>0, \tag{2.12.23}$$

where $\kappa$ is a diffusivity constant with the initial condition

$$u(x,0)=f(x), \quad -\infty<x<\infty. \tag{2.12.24}$$

We solve this problem using the Fourier transform in the space variable $x$ defined by (2.12.4). Application of this transform to (2.12.23)–(2.12.24) gives

$$U_t=-\kappa k^2U, \quad t>0, \tag{2.12.25}$$

$$U(k,0)=F(k). \tag{2.12.26}$$

The solution of the transformed system is

$$U(k,t) = F(k)\, e^{-\kappa k^2 t}. \tag{2.12.27}$$

The inverse Fourier transform gives the solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp[(ikx - \kappa k^2 t)]dk$$

which is, by the Convolution Theorem 2.5.5,

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi, \tag{2.12.28}$$

where

$$g(x) = \mathscr{F}^{-1}\{e^{-\kappa k^2 t}\} = \frac{1}{\sqrt{2\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right), \text{ by (2.3.5).}$$

Thus, solution (2.12.28) becomes

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right] d\xi. \tag{2.12.29}$$

The integrand involved in the solution consists of the initial value $f(x)$ and *Green's function* (or, *elementary solution*) $G(x-\xi,t)$ of the diffusion equation for the infinite interval:

$$G(x-\xi,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right]. \tag{2.12.30}$$

So, in terms of $G(x-\xi,t)$, solution (2.12.29) can be written as

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi) G(x-\xi,t) d\xi \tag{2.12.31}$$

so that, in the limit as $t \to 0+$, this formally becomes

$$u(x,0) = f(x) = \int_{-\infty}^{\infty} f(\xi) \lim_{t\to 0+} G(x-\xi,t) d\xi.$$

The limit of $G(x-\xi,t)$ represents the Dirac delta function

$$\delta(x-\xi) = \lim_{t\to 0+} \frac{1}{2\sqrt{\pi\kappa t}} \exp\left[-\frac{(x-\xi)^2}{4\,\kappa t}\right]. \tag{2.12.32}$$

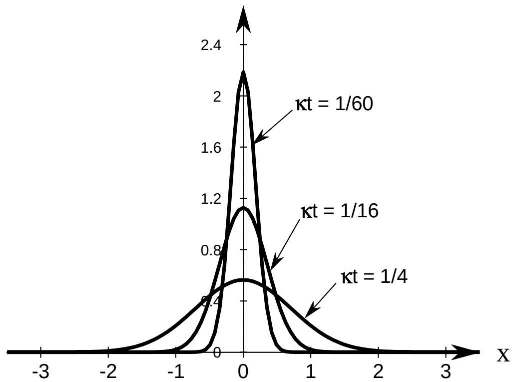


**Figure 2.10** Graphs of $G(x,t)$ against $x$.

Graphs of $G(x,t)$ are shown in Figure 2.10 for different values of $\kappa t$.

It is important to point out that the integrand in (2.12.31) consists of the initial temperature distribution $f(x)$ and Green's function $G(x-\xi,t)$ which represents the temperature response along the rod at time $t$ due to an initial unit impulse of heat at $x=\xi$. The physical meaning of the solution (2.12.31) is that the initial temperature distribution $f(x)$ is decomposed into a spectrum of impulses of magnitude $f(\xi)$ at each point $x=\xi$ to form the resulting temperature $f(\xi)G(x-\xi,t)$. Thus, the resulting temperature is integrated to find solution (2.12.31). This is called the *principle of integral superposition.*

We make the change of variable

$$\frac{\xi-x}{2\sqrt{\kappa t}}=\zeta, \quad d\zeta=\frac{d\xi}{2\sqrt{\kappa t}}$$

to express solution (2.12.29) in the form

$$u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2\sqrt{\kappa t}\,\zeta)\exp(-\zeta^2)d\zeta. \tag{2.12.33}$$

The integral solution (2.12.33) or (2.12.29) is called the *Poisson integral representation* of the temperature distribution. This integral is convergent for all time $t>0$, and the integrals obtained from (2.12.33) by differentiation under the integral sign with respect to $x$ and $t$ are uniformly convergent in the neighborhood of the point $(x,t)$. Hence, the solution $u(x,t)$ and its derivatives of all orders exist for $t>0$.

Finally, we consider a special case involving discontinuous initial condition in the form

$$f(x)=T_0H(x), \tag{2.12.34}$$

where $T_0$ is a constant. In this case, solution (2.12.29) becomes

$$u(x,t) = \frac{T_0}{2\sqrt{\pi\kappa t}} \int_0^\infty \exp\left[-\frac{(x-\xi)^2}{4\,\kappa t}\right] d\xi. \tag{2.12.35}$$

Introducing the change of variable $\eta = \frac{\xi - x}{2\sqrt{\kappa t}}$, we can express solution (2.12.35) in the form

$$\begin{aligned} u(x,t) &= \frac{T_0}{\sqrt{\pi}} \int_{-x/2\sqrt{\kappa t}}^{\infty} e^{-\eta^2}\, d\eta = \frac{T_0}{2}\, erfc\left(-\frac{x}{2\sqrt{\kappa t}}\right) \\ &= \frac{T_0}{2}\left[1 + erf\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]. \end{aligned} \tag{2.12.36}$$

The solution given by equation (2.12.36) with $T_0 = 1$ is shown in Figure 2.11.

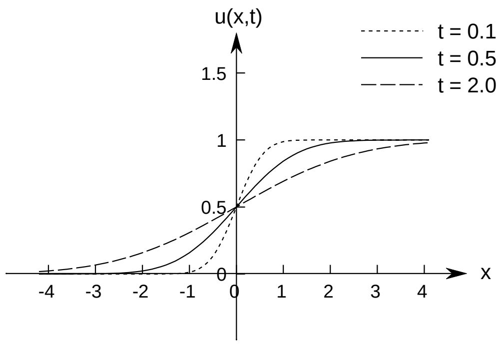


**Figure 2.11** The time development of solution (2.12.36).

▯

If $f(x) = \delta(x)$, then the fundamental solution (2.7.29) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi\,\kappa\, t}} \exp\left(-\frac{x^2}{4\kappa\, t}\right).$$

***Example 2.12.4***
(*The Cauchy Problem for the Wave Equation*). Obtain the d'Alembert solution of the initial value problem for the wave equation

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.37}$$

with the arbitrary but fixed initial data

$$u(x,0)=f(x), \quad u_t(x,0)=g(x), \quad -\infty<x<\infty. \tag{2.12.38ab}$$

Application of the Fourier transform $\mathscr{F}\{u(x,t)\}=U(k,t)$ to this system gives

$$\frac{d^2U}{dt^2}+c^2k^2U=0,$$
$$U(k,0)=F(k), \quad \left(\frac{dU}{dt}\right)_{t=0}=G(k).$$

The solution of the transformed system is

$$U(k,t)=A\ e^{ickt}+B\ e^{-ickt},$$

where $A$ and $B$ are constants to be determined from the transformed data so that $A+B=F(k)$ and $A-B=\frac{1}{ikc}G(k)$. Solving for $A$ and $B$, we obtain

$$U(k,t)=\frac{1}{2}F(k)(e^{ickt}+e^{-ickt})+\frac{G(k)}{2ick}(e^{ickt}-e^{-ickt}). \tag{2.12.39}$$

Thus, the inverse Fourier transform of (2.12.39) yields the solution

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\{e^{ik(x+ct)}+e^{ik(x-ct)}\}dk\right] \\ &\quad +\frac{1}{2c}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{G(k)}{ik}\{e^{ik(x+ct)}-e^{ik(x-ct)}\}dk\right]. \end{aligned} \tag{2.12.40}$$

We use the following results

$$f(x)=\mathscr{F}^{-1}\{F(k)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}F(k)dk,$$

$$g(x)=\mathscr{F}^{-1}\{G(k)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}G(k)dk,$$

to obtain the solution in the final form

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(k)dk\int_{x-ct}^{x+ct} e^{ik\xi}d\xi \\ &= \frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} d\xi\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik\xi}G(k)dk\right] \\ &= \frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi. \end{aligned} \tag{2.12.41}$$

This is the well known *d'Alembert's solution* of the wave equation.

The method and the form of the solution reveal several important features of the wave equation. First, the method of solution essentially proves the existence of the d'Alembert solution and the solution is unique provided $f(x)$ is twice continuously differentiable and $g(x)$ is continuously differentiable. Second, the terms involving $f(x \pm ct)$ in (2.12.41) show that disturbances are propagated along the characteristics with constant velocity $c$. Both terms combined together suggest that the value of the solution at position $x$ and at time $t$ depends only on the initial values of $f(x)$ at $x - ct$ and $x + ct$ and the values of $g(x)$ between these two points. The interval $(x - ct, x + ct)$ is called the *domain of dependence* of the variable $(x, t)$. Finally, the solution depends continuously on the initial data, that is, the problem is well posed. In other words, a small change in either $f(x)$ or $g(x)$ results in a correspondingly small change in the solution $u(x, t)$.

In particular, if $f(x) = \exp(-x^2)$ and $g(x) \equiv 0$, the time development of solution (2.12.41) with $c = 1$ is shown in Figure 2.12. In this case, the solution becomes

$$u(x,t) = \frac{1}{2}[e^{-(x-t)^2} + e^{-(x+t)^2}]. \tag{2.12.42}$$

As shown in Figure 2.12, the initial form $f(x) = \exp(-x^2)$ is found to split into two similar waves propagating in opposite direction with unit velocity.
□

***Example 2.12.5***
(*The Schrödinger Equation in Quantum Mechanics*). The time-dependent Schrödinger equation of a particle of mass $m$ is

$$i\hbar\,\psi_t = \left[V(x) - \frac{\hbar^2}{2m}\nabla^2\right]\psi = H\psi, \tag{2.12.43}$$

where $h = 2\pi\hbar$ is the *Planck constant*, $\psi(\mathbf{x}, t)$ is the wave function, $V(x)$ is the potential, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the three-dimensional *Laplacian*, and $H$ is the *Hamiltonian*.

If $V(\mathbf{x}) = \text{constant} = V$, we can seek a *plane wave solution* of the form

$$\psi(\mathbf{x}, t) = A\exp[i(\boldsymbol{\kappa}\cdot\mathbf{x} - \omega t)], \tag{2.12.44}$$

where $A$ is a constant amplitude, $\boldsymbol{\kappa} = (k, l, m)$ is the wavenumber vector, and $\omega$ is the frequency.

Substituting this solution into (2.12.43), we conclude that this solution is possible provided the following relation is satisfied:

$$i\hbar(-i\omega) = V - \frac{\hbar^2}{2m}(i\kappa)^2, \quad \kappa^2 = k^2 + l^2 + m^2.$$

Or,

$$\hbar\omega = V + \frac{\hbar^2\kappa^2}{2m}. \tag{2.12.45}$$

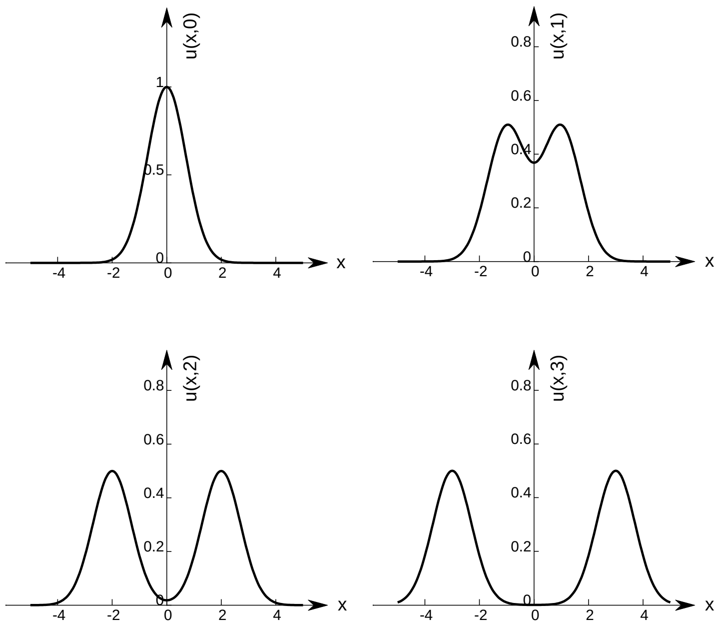


**Figure 2.12** The time development of solution (2.12.42).

This is called the *dispersion relation* and shows that the sum of the potential energy $V$ and the kinetic energy $\frac{(\hbar\kappa)^2}{2m}$ is equal to the total energy $\hbar\omega$. Further, the kinetic energy

$$K.E. = \frac{1}{2m}(\hbar\kappa)^2 = \frac{p^2}{2m}, \tag{2.12.46}$$

where $p = \hbar\kappa$ is the momentum of the particle.

The phase velocity, $C_p$ and the group velocity, $C_g$ of the wave are defined by

$$C_p = \frac{\omega}{\kappa}\,\hat{\boldsymbol{\kappa}}, \qquad C_g = \nabla_{\boldsymbol{\kappa}}\omega(\boldsymbol{\kappa}), \tag{2.12.47ab}$$

where $\boldsymbol{\kappa}$ is the wavenumber vector and $\kappa = |\boldsymbol{\kappa}|$ and $\hat{\boldsymbol{\kappa}}$ is the unit wavenumber vector.

In the one-dimensional case, the phase velocity is

$$C_p = \frac{\omega}{k} \tag{2.12.48}$$

and the group velocity is

$$C_g = \frac{\partial\omega}{\partial k} = \frac{\hbar k}{m} = \frac{p}{m} = \frac{mv}{v} = v. \tag{2.12.49}$$

This shows that the group velocity is equal to the classical particle velocity $v$.

We now use the Fourier transform method to solve the one-dimensional Schrödinger equation for a free particle ($V \equiv 0$), that is,

$$i\hbar\,\psi_t = -\frac{\hbar^2}{2m}\psi_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.50}$$

$$\psi(x,0) = \psi_0(x), \quad -\infty < x < \infty, \tag{2.12.51}$$

$$\psi(x,t) \to 0 \quad \text{as } |x| \to \infty. \tag{2.12.52}$$

Application of the Fourier transform to (2.12.50)–(2.12.52) gives

$$\Psi_t = -\frac{i\hbar k^2}{2m}\Psi, \qquad \Psi(k,0) = \Psi_0(k). \tag{2.12.53}$$

The solution of this transformed system is

$$\Psi(k,t) = \Psi_0(k)\exp(-i\alpha k^2 t), \quad \alpha = \frac{\hbar}{2m}. \tag{2.12.54}$$

The inverse Fourier transform gives the formal solution

$$\begin{aligned}
\psi(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi_0(k)\exp\{ik(x-\alpha kt)\}dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky}\psi(y,0)dy \int_{-\infty}^{\infty} \exp\{ik(x-\alpha kt)\}dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(y,0)dy \int_{-\infty}^{\infty} \exp\{ik(x-y-\alpha kt)\}dk.
\end{aligned} \tag{2.12.55}$$

We rewrite the integrand of the second integral in (2.12.55) as follows

$$\begin{aligned}
&\exp[ik(x-y-\alpha kt)] \\
&\quad = \exp\left[-i\alpha t\left\{k^2 - 2k\cdot\frac{x-y}{2\alpha t} + \left(\frac{x-y}{2\alpha t}\right)^2 - \left(\frac{x-y}{2\alpha t}\right)^2\right\}\right] \\
&\quad = \exp\left[-i\alpha t\left\{k - \frac{x-y}{2\alpha t}\right\}^2\right]\exp\left[\frac{i(x-y)^2}{4\alpha t}\right] \\
&\quad = \exp\left[\frac{i(x-y)^2}{4\alpha t}\right]\exp(-i\alpha t\xi^2), \quad \xi = k - \frac{x-y}{2\alpha t}.
\end{aligned}$$

Using this result in (2.12.55), we obtain

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[\frac{i(x-y)^2}{4\alpha t}\right]\psi(y,0)dy \int_{-\infty}^{\infty}\exp(-i\alpha t\xi^2)d\xi$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{2\alpha t}}(1-i)\int_{-\infty}^{\infty}\exp\left[\frac{i(x-y)^2}{4\alpha t}\right]\psi(y,0)dy$$

$$= \frac{(1-i)}{2\sqrt{2\alpha\pi t}}\int_{-\infty}^{\infty}\exp\left[\frac{i(x-y)^2}{4\alpha t}\right]\psi(y,0)dy. \tag{2.12.56}$$

This is the integral solution of the problem. ▯

***Example 2.12.6***

(*Slowing Down of Neutrons*). We consider the problem of slowing down neutrons in an infinite medium with a source of neutrons governed by

$$u_t = u_{xx} + \delta(x)\delta(t), \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.57}$$

$$u(x,0) = \delta(x), \quad -\infty < x < \infty, \tag{2.12.58}$$

$$u(x,t) \to 0 \qquad \text{as } |x| \to \infty \text{ for } t > 0, \tag{2.12.59}$$

where $u(x,t)$ represents the number of neutrons per unit volume per unit time, which reach the age $t$, and $\delta(x)\delta(t)$ is the source function.

Application of the Fourier transform method gives

$$\frac{dU}{dt} + k^2 U = \frac{1}{\sqrt{2\pi}}\,\delta(t),$$

$$U(k,0) = \frac{1}{\sqrt{2\pi}}.$$

The solution of this transformed system is

$$U(k,t) = \frac{1}{\sqrt{2\pi}}\, e^{-k^2 t},$$

and the inverse Fourier transform gives the solution

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}dk = \frac{1}{\sqrt{2\pi}}\mathscr{F}^{-1}\left\{e^{-k^2t}\right\}$$

$$= \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right). \tag{2.12.60}$$

▯

**Example 2.12.7**
(*One-Dimensional Wave Equation*). Obtain the solution of the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.61}$$

$$u(x,0) = 0, \quad u_t(x,0) = \delta(x), \quad -\infty < x < \infty. \tag{2.12.62ab}$$

Making reference to Example 2.12.4, we find $f(x) \equiv 0$ and $g(x) = \delta(x)$ so that $F(k) = 0$ and $G(k) = \frac{1}{\sqrt{2\pi}}$. The solution for $U(k,t)$ is given by

$$U(k,t) = \frac{1}{2c\sqrt{2\pi}} \left[ \frac{e^{ickt}}{ik} - \frac{e^{-ickt}}{ik} \right].$$

Thus, the inverse Fourier transform gives

$$\begin{aligned} u(x,t) &= \frac{1}{2c\sqrt{2\pi}} \mathscr{F}^{-1} \left\{ \frac{e^{ickt}}{ik} - \frac{e^{-ickt}}{ik} \right\} \\ &= \frac{1}{2c\sqrt{2\pi}} \left[ \sqrt{\frac{\pi}{2}} \{\text{sgn}(x+ct) - \text{sgn}(x-ct)\} \right] \\ &= \frac{1}{4c} [\text{sgn}(x+ct) - \text{sgn}(x-ct)] \\ &= \begin{cases} \dfrac{1-1}{4c} = 0, & |x| > ct > 0 \\ \dfrac{1+1}{4c} = \dfrac{1}{2c}, & |x| < ct. \end{cases} \end{aligned}$$

In other words, the solution can be written in the form

$$u(x,t) = \frac{1}{2c} H(c^2 t^2 - x^2).$$

□

**Example 2.12.8**
(*Linearized Shallow Water Equations in a Rotating Ocean*). The horizontal equations of motion of a uniformly rotating inviscid homogeneous ocean of constant depth $h$ are

$$u_t - f\upsilon = -g\,\eta_x, \tag{2.12.63}$$
$$\upsilon_t + fu = 0, \tag{2.12.64}$$
$$\eta_t + hu_x = 0, \tag{2.12.65}$$

where $f = 2\Omega \sin\theta$ is the Coriolis parameter, which is constant in the present problem, $g$ is the acceleration due to gravity, $\eta(x,t)$ is the free surface elevation, $u(x,t)$ and $\upsilon(x,t)$ are the velocity fields. The wave motion is generated by the prescribed free surface elevation at $t=0$ so that the initial conditions are

$$u(x,0)=0=v(x,0), \quad \eta(x,0)=\eta_0 H(a-|x|), \tag{2.12.66abc}$$

and the velocity fields and free surface elevation function vanish at infinity.

We apply the Fourier transform with respect to $x$ defined by

$$\mathscr{F}\{f(x,t)\}=F(k,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x,t)dx \tag{2.12.67}$$

to the system (2.12.63)–(2.12.65) so that the system becomes

$$\begin{aligned} \frac{dU}{dt}-fV &= -gikE \\ \frac{dV}{dt}+fU &= 0 \\ \frac{dE}{dt} &= -hikU \end{aligned}$$

$$U(k,0)=0=V(k,0), \quad E(k,0)=\sqrt{\frac{2}{\pi}}\,\eta_0\left(\frac{\sin ak}{k}\right), \tag{2.12.68abc}$$

where $E(k,t)=\mathscr{F}\{\eta(x,t)\}$.

Elimination of $U$ and $V$ from the transformed system gives a single equation for $E(k,t)$ as

$$\frac{d^3E}{dt^3}+\omega^2\frac{dE}{dt}=0, \tag{2.12.69}$$

where $\omega^2=(f^2+c^2k^2)$ and $c^2=gh$. The general solution of (2.12.69) is

$$E(k,t)=A+B\cos\omega t+C\sin\omega t, \tag{2.12.70}$$

where $A$, $B$, and $C$ are arbitrary constants to be determined from (2.12.68c) and

$$\left(\frac{d^2E}{dt^2}\right)_{t=0}=-c^2k^2E(k,0)=-c^2k^2\cdot\sqrt{\frac{2}{\pi}}\,\eta_0\,\frac{\sin ak}{k},$$

which gives

$$B=\sqrt{\frac{2}{\pi}}\,\eta_0\left(\frac{\sin ak}{k}\right)\cdot\left(\frac{c^2k^2}{\omega^2}\right).$$

Also $\left(\frac{dE}{dt}\right)_{t=0}=0$ gives $C\equiv 0$ and (2.12.68c) implies $A+B=\sqrt{\frac{2}{\pi}}\eta_0\frac{\sin ak}{k}$. Consequently, the solution (2.12.70) becomes

$$E(k,t)=\sqrt{\frac{2}{\pi}}\,\eta_0\left(\frac{\sin ak}{k}\right)\frac{f^2+c^2k^2\cos\omega t}{(f^2+c^2k^2)}. \tag{2.12.71}$$

Similarly

$$U(k,t) = \sqrt{\frac{2}{\pi}}\frac{\eta_0 \sin ak}{ih} \cdot \frac{c^2 \sin \omega t}{\sqrt{c^2k^2+f^2}}, \tag{2.12.72}$$

$$V(k,t) = \frac{1}{f}\left(\frac{dU}{dt} + gik\,E\right). \tag{2.12.73}$$

The inverse Fourier transform gives the formal solution for $\eta(x,t)$

$$\eta(x,t) = \left(\frac{\eta_0}{\pi}\right)\int_{-\infty}^{\infty} \frac{\sin ak}{k} \cdot \frac{f^2 + c^2k^2 \cos \omega t}{(f^2+c^2k^2)} e^{ikx} dk. \tag{2.12.74}$$

Similar integral expressions for $u(x,t)$ and $v(x,t)$ can be obtained. □

***Example 2.12.9***

(*Sound Waves Induced by a Spherical Body*). We consider propagation of sound waves in an unbounded fluid medium generated by an impulsive radial acceleration of a sphere of radius $a$. Such waves are assumed to be spherically symmetric and the associated velocity potential on the pressure field $p(r,t)$ satisfies the wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right)\right], \tag{2.12.75}$$

where $c$ is the speed of sound. The boundary condition required for the problem is

$$\frac{1}{\rho_0}\left(\frac{\partial p}{\partial r}\right) = -a_0\,\delta(t) \qquad \text{on } r = a, \tag{2.12.76}$$

where $\rho_0$ is the mean density of the fluid and $a_0$ is a constant.

Application of the Fourier transform of $p(r,t)$ with respect to time $t$ gives

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dP}{dr}\right) = -k^2P(r,\omega), \tag{2.12.77}$$

$$\frac{dP}{dr} = -\frac{a_0\rho_0}{\sqrt{2\pi}}, \quad \text{on } r = a, \tag{2.12.78}$$

where $\mathscr{F}\{p(r,t)\} = P(r,\omega)$ and $k^2 = \frac{\omega^2}{c^2}$.

The general solution of (2.12.77)–(2.12.78) is

$$P(r,\omega) = \frac{A}{r}e^{ikr} + \frac{B}{r}e^{-ikr}, \tag{2.12.79}$$

where $A$ and $B$ are arbitrary constants.

The inverse Fourier transform gives the solution

$$p(r,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{A}{r} e^{i(\omega t + kr)} + \frac{B}{r} e^{i(\omega t - kr)}\right] d\omega. \tag{2.12.80}$$

The first term of the integrand represents incoming spherical waves generated at infinity and the second term corresponds to outgoing spherical waves due to the impulsive radial acceleration of the sphere. Since there is no disturbance at infinity, we impose the *Sommerfeld radiation condition* at infinity to eliminate the incoming waves so that $A = 0$, and $B$ is calculated using (2.12.78). Thus, the inverse Fourier transform gives the formal solution

$$p(r,t) = \left(\frac{a_0 \rho_0 a^2}{2\pi r}\right) \int_{-\infty}^{\infty} \frac{\exp\left[i\omega \left\{t - \frac{r-a}{c}\right\}\right] d\omega}{\left(1 + \frac{i\omega a}{c}\right)}. \tag{2.12.81}$$

We next choose a closed contour with a semicircle in the upper half plane and the real $\omega$-axis. Using the Cauchy theory of residues, we calculate the residue contribution from the pole at $\omega = ic/a$. Finally, it turns out that the final solution is

$$u(r,t) = \left(\frac{\rho_0 a_0 ca}{r}\right) \exp\left[-\frac{c}{a}\left(t - \frac{r-a}{c}\right)\right] H\left(t - \frac{r-a}{c}\right). \tag{2.12.82}$$

□

**Example 2.12.10**
(*The Linearized Korteweg-de Vries Equation*). The linearized KdV equation for the free surface elevation $\eta(x,t)$ in an inviscid water of constant depth $h$ is

$$\eta_t + c\eta_x + \frac{ch^2}{6}\eta_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.83}$$

where $c = \sqrt{gh}$ is the shallow water speed.

Solve equation (2.12.83) with the initial condition

$$\eta(x,0) = f(x), \quad -\infty < x < \infty. \tag{2.12.84}$$

Application of the Fourier transform $\mathscr{F}\{\eta(x,t)\} = E(k,t)$ to the KdV system gives the solution for $E(k,t)$ in the form

$$E(k,t) = F(k) \exp\left[ikct\left(\frac{k^2h^2}{6} - 1\right)\right].$$

The inverse transform gives

$$\eta(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp\left[ik\left\{(x - ct) + \left(\frac{cth^2}{6}\right)k^2\right\}\right] dk. \tag{2.12.85}$$

In particular, if $f(x)=\delta(x)$, then (2.12.85) reduces to the Airy integral

$$\eta(x,t)=\frac{1}{\pi}\int_0^\infty \cos\left[k(x-ct)+\left(\frac{cth^2}{6}\right)k^3\right]dk \tag{2.12.86}$$

which is, in terms of the Airy function,

$$=\left(\frac{cth^2}{2}\right)^{-\frac{1}{3}} Ai\left[\left(\frac{cth^2}{2}\right)^{-\frac{1}{3}}(x-ct)\right], \tag{2.12.87}$$

where the Airy function $Ai(az)$ is defined by

$$Ai(az)=\frac{1}{2\pi a}\int_{-\infty}^\infty \exp\left[i\left(kz+\frac{k^3}{3a^3}\right)\right]dk=\frac{1}{\pi a}\int_0^\infty \cos\left(kz+\frac{k^3}{3a^3}\right)dk. \tag{2.12.88}$$

□

***Example 2.12.11***
(*Biharmonic Equation in Fluid Mechanics*). Usually, the biharmonic equation arises in fluid mechanics and in elasticity. The equation can readily be solved by using the Fourier transform method. We first derive a biharmonic equation from the *Navier-Stokes equations* of motion in a viscous fluid which is given by

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=\mathbf{F}-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{u}, \tag{2.12.89}$$

where $\mathbf{u}=(u,v,w)$ is the velocity field, $\boldsymbol{F}$ is the external force per unit mass of the fluid, $p$ is the pressure, $\rho$ is the density and $\nu$ is the kinematic viscosity of the fluid.

The conservation of mass of an incompressible fluid is described by the *continuity equation*

$$\text{div } \mathbf{u}=0. \tag{2.12.90}$$

In terms of some representative length scale $L$ and velocity scale $U$, it is convenient to introduce the nondimensional flow variables

$$\mathbf{x}'=\frac{\mathbf{x}}{L}, \quad t'=\frac{Ut}{L}, \quad \mathbf{u}'=\frac{\mathbf{u}}{U}, \quad p'=\frac{p}{\rho U^2}. \tag{2.12.91}$$

In terms of these nondimensional variables, equation (2.12.89) without the external force can be written, dropping the primes, as

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p+\frac{1}{R}\nabla^2\mathbf{u}, \tag{2.12.92}$$

where $R=UL/\nu$ is called the *Reynolds number*. Physically, it measures the ratio of inertial forces of the order $U^2/L$ to viscous forces of the order $\nu U/L^2$, and it has special dynamical significance. This is one of the most fundamental nondimensional parameters for the specification of the dynamical state of viscous flow fields.

In the absence of the external force, $\mathbf{F}=\mathbf{0}$, it is preferable to write the Navier-Stokes equations (2.12.89) in the form (since $\mathbf{u}\times\boldsymbol{\omega}=\frac{1}{2}\nabla u^2-\mathbf{u}\cdot\nabla\mathbf{u}$)

$$\frac{\partial\mathbf{u}}{\partial t}-\mathbf{u}\times\boldsymbol{\omega}=-\nabla\left(\frac{p}{\rho}+\frac{1}{2}u^2\right)-\nu\nabla^2\mathbf{u}, \tag{2.12.93}$$

where $\boldsymbol{\omega}=\text{curl}\,\mathbf{u}$ is the *vorticity vector* and $u^2=\boldsymbol{u}\cdot\boldsymbol{u}$.

We can eliminate the pressure $p$ from (2.12.93) by taking the curl of (2.12.93), giving

$$\frac{\partial\boldsymbol{\omega}}{\partial t}-\text{curl}(\mathbf{u}\times\boldsymbol{\omega})=\nu\nabla^2\boldsymbol{\omega} \tag{2.12.94}$$

which becomes, by div $\mathbf{u}=0$ and div $\boldsymbol{\omega}=0$,

$$\frac{\partial\boldsymbol{\omega}}{\partial t}=(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}-(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}+\nu\nabla^2\boldsymbol{\omega}. \tag{2.12.95}$$

This is universally known as the *vorticity transport equation*. The left hand-side represents the rate of change of vorticity. The first two terms on the right-hand side represent the rate of change of vorticity due to stretching and twisting of vortex lines. The last term describes the diffusion of vorticity by molecular viscosity.

In case of two-dimensional flow, $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}=0$, equation (2.12.95) becomes

$$\frac{D\boldsymbol{\omega}}{dt}=\frac{\partial\boldsymbol{\omega}}{\partial t}+(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}=\nu\nabla^2\boldsymbol{\omega}, \tag{2.12.96}$$

where $\mathbf{u}=(u,v,0)$ and $\boldsymbol{\omega}=(0,0,\zeta)$, and $\zeta=v_x-u_y$. Equation (2.12.96) shows that only convection and conduction occur. In terms of the stream function $\psi(x,y)$ where

$$u=\psi_y,\quad v=-\psi_x,\quad \boldsymbol{\omega}=-\nabla^2\psi, \tag{2.12.97}$$

which satisfy (2.12.90) identically, equation (2.12.96) assumes the form

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right)+\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\psi=\nu\nabla^4\psi. \tag{2.12.98}$$

In case of slow motion (velocity is small) or in case of a very viscous fluid ($\nu$ very large), the Reynolds number $R$ is very small. For a steady flow in such cases of an incompressible viscous fluid, $\frac{\partial}{\partial t}\equiv 0$, while $(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$ is negligible in comparison with the viscous term. Consequently, (2.12.98) reduces to the standard *biharmonic equation*

$$\nabla^4\psi=0. \tag{2.12.99}$$

Or, more explicitly,

$$\nabla^2(\nabla^2)\psi\equiv\psi_{xxxx}+2\psi_{xxyy}+\psi_{yyyy}=0. \tag{2.12.100}$$

We solve this equation in a semi-infinite viscous fluid bounded by an infinite horizontal plate at $y=0$, and the fluid is introduced normally with a prescribed velocity through a strip $-a<x<a$ of the plate. Thus, the required boundary conditions are

$$u\equiv\frac{\partial\psi}{\partial y}=0,\quad v\equiv\frac{\partial\psi}{\partial x}=H(a-|x|)f(x)\quad\text{on } y=0, \tag{2.12.101ab}$$

where $f(x)$ is a given function of $x$.

Furthermore, the fluid is assumed to be at rest at large distances from the plate, that is,

$$(\psi_x,\psi_y)\to(0,0)\quad\text{as } y\to\infty\qquad\text{for } -\infty<x<\infty. \tag{2.12.102}$$

To solve the biharmonic equation (2.12.100) with the boundary conditions (2.12.101ab) and (2.12.102), we introduce the Fourier transform with respect to $x$

$$\Psi(k,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\psi(x,y)dx. \tag{2.12.103}$$

Thus, the Fourier transformed problem is

$$\left(\frac{d^2}{dy^2}-k^2\right)^2\Psi(k,y)=0, \tag{2.12.104}$$

$$\frac{d\Psi}{dy}=0,\quad (ik)\Psi=F(k),\quad y=0, \tag{2.12.105ab}$$

where

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{-ikx}f(x)dx. \tag{2.12.106}$$

In view of the Fourier transform of (2.12.102), the bounded solution of (2.12.104) is

$$\Psi(k,y)=(A+B|k|y)\exp(-|k|y), \tag{2.12.107}$$

where $A$ and $B$ can be determined from (2.12.105ab) so that $A=B=(ik)^{-1}F(k)$. Consequently, the solution (2.12.107) becomes

$$\Psi(k,y)=(ik)^{-1}(1+|k|y)F(k)\exp(-|k|y). \tag{2.12.108}$$

The inverse Fourier transform gives the formal solution

$$\psi(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(k)G(k)\exp(ikx)dk, \tag{2.12.109}$$

where

$$G(k) = (ik)^{-1}(1 + |k|y)\exp(-|k|y)$$

so that

$$\begin{aligned} g(x) &= \mathscr{F}^{-1}\{G(k)\} = \mathscr{F}^{-1}\{(ik)^{-1}\exp(-|k|y)\} \\ &\qquad\qquad\qquad + y\,\mathscr{F}^{-1}\{(ik)^{-1}|k|\exp(-|k|y)\} \\ &= \mathscr{F}_s^{-1}\{k^{-1}\exp(-ky)\} + y\,\mathscr{F}_s^{-1}\{e^{-ky}\}, \end{aligned}$$

which is, by (2.13.7) and (2.13.8),

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{y}\right) + \sqrt{\frac{2}{\pi}}\frac{xy}{(x^2+y^2)}. \tag{2.12.110}$$

Using the Convolution Theorem 2.5.5 in (2.12.109) gives the final solution

$$\psi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x-\xi)\left[\tan^{-1}\left(\frac{\xi}{y}\right) + \frac{y\xi}{\xi^2+y^2}\right]d\xi. \tag{2.12.111}$$

In particular, if $f(x) = \delta(x)$, then solution (2.12.111) becomes

$$\psi(x,y) = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{x}{y}\right) + \frac{xy}{x^2+y^2}\right]. \tag{2.12.112}$$

The velocity fields $u$ and $v$ can be determined from (2.12.112). ▯

***Example 2.12.12***
(*Biharmonic Equation in Elasticity*). We derive the biharmonic equation in elasticity from the two-dimensional equilibrium equations and the compatibility condition. In two-dimensional elastic medium, the strain components $e_{xx}, e_{xy}, e_{yy}$ in terms of the displacement functions $(u, v, 0)$ are

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{2.12.113}$$

Differentiating these results gives the *compatibility condition*

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\,\frac{\partial^2 e_{xy}}{\partial x \partial y}. \tag{2.12.114}$$

In terms of the *Poisson ratio* $\nu$ and *Young's modulus* $E$ of the elastic material, the strain component in the $z$ direction is expressed in terms of stress components

$$Ee_{zz} = \sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}). \tag{2.12.115}$$

In the case of plane strain, $e_{zz} = 0$, so that

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}). \tag{2.12.116}$$

Substituting this result in other stress-strain relations, we obtain the strain components $e_{xx}, e_{xy}, e_{yy}$ that are related to stress components $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ by

$$Ee_{xx} = \sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}) = (1-\nu^2)\sigma_{xx} - \nu(1+\nu)\sigma_{yy}, \quad (2.12.117)$$
$$Ee_{yy} = \sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz}) = (1-\nu^2)\sigma_{yy} - \nu(1+\nu)\sigma_{xx}, \quad (2.12.118)$$
$$Ee_{xy} = (1+\nu)\sigma_{xy}. \quad (2.12.119)$$

Putting (2.12.117)-(2.12.119) into (2.12.114) gives

$$\frac{\partial^2}{\partial y^2}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] + \frac{\partial^2}{\partial x^2}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] = 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}. \quad (2.12.120)$$

The basic differential equations for the stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ in the medium under the action of body forces $X$ and $Y$ are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho X = \rho \frac{\partial^2 u}{\partial t^2}, \quad (2.12.121)$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho Y = \rho \frac{\partial^2 v}{\partial t^2}, \quad (2.12.122)$$

where $\rho$ is the mass density of the elastic material.

The equilibrium equations follow from (2.12.121)–(2.12.122) in the absence of the body forces ($X = Y = 0$) as

$$\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{xy} = 0, \quad (2.12.123)$$
$$\frac{\partial}{\partial x}\sigma_{xy} + \frac{\partial}{\partial y}\sigma_{yy} = 0. \quad (2.12.124)$$

It is obvious that the expressions

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2}, \quad \sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}, \quad \sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2} \quad (2.12.125)$$

satisfy the equilibrium equations for any arbitrary function $\chi(x,y)$. Substituting from equations (2.12.125) into the compatibility condition (2.12.120), we see that $\chi$ must satisfy the *biharmonic equation*

$$\frac{\partial^4 \chi}{\partial x^4} + 2\frac{\partial^4 \chi}{\partial x^2 \partial y^2} + \frac{\partial^4 \chi}{\partial y^4} = 0, \quad (2.12.126)$$

which may be written symbolically as

$$\nabla^4 \chi = 0. \quad (2.12.127)$$

The function $\chi$ was first introduced by Airy in 1862 and is known as the *Airy stress function.*

We determine the stress distribution in a semi-infinite elastic medium bounded by an infinite plane at $x=0$ due to an external pressure to its surface. The $x$-axis is normal to this plane and assumed positive in the direction into the medium. We assume that the external surface pressure $p$ varies along the surface so that the boundary conditions are

$$\sigma_{xx}=-p(y), \quad \sigma_{xy}=0 \quad \text{on } x=0 \quad \text{for all } y \text{ in } (-\infty,\infty). \tag{2.12.128}$$

We derive solutions so that stress components $\sigma_{xx}, \sigma_{yy}$, and $\sigma_{xy}$ all vanish as $x \to \infty$.

In order to solve the biharmonic equation (2.12.127), we introduce the Fourier transform $\tilde{\chi}(x,k)$ of the *Airy stress function* with respect to $y$ so that (2.12.127)–(2.12.128) reduce to

$$\left(\frac{d^2}{dx^2}-k^2\right)^2 \tilde{\chi}=0, \tag{2.12.129}$$

$$k^2\tilde{\chi}(0,k)=\tilde{p}(k), \quad (ik)\left(\frac{d\tilde{\chi}}{dx}\right)_{x=0}=0, \tag{2.12.130}$$

where $\tilde{p}(k)=\mathscr{F}\{p(y)\}$. The bounded solution of the transformed problem is

$$\tilde{\chi}(x,k)=(A+Bx)\exp(-|k|x), \tag{2.12.131}$$

where $A$ and $B$ are constants of integration to be determined from (2.12.130). It turns out that $A=\tilde{p}(k)/k^2$ and $B=\tilde{p}(k)/|k|$ and hence, the solution becomes

$$\tilde{\chi}(x,k)=\frac{\tilde{p}(k)}{k^2}\{1+|k|x\}\exp(-|k|x). \tag{2.12.132}$$

The inverse Fourier transform yields the formal solution

$$\chi(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\tilde{p}(k)}{k^2}(1+|k|x)\exp(iky-|k|x)dk. \tag{2.12.133}$$

The stress components are obtained from (2.12.125) in the form

$$\sigma_{xx}(x,y)=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k^2\tilde{\chi}(x,k)\exp(iky)dk, \tag{2.12.134}$$

$$\sigma_{xy}(x,y)=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(ik)\left(\frac{d\tilde{\chi}}{dx}\right)\exp(iky)dk, \tag{2.12.135}$$

$$\sigma_{yy}(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d^2\tilde{\chi}}{dx^2}\exp(iky)dk, \tag{2.12.136}$$

where $\tilde{\chi}(x,k)$ are given by (2.12.132). In particular, if $p(y) = P\delta(y)$ so that $\tilde{p}(k) = P(2\pi)^{-\frac{1}{2}}$. Consequently, from (2.12.133)–(2.12.136) we obtain

$$\chi(x,y) = \frac{P}{2\pi}\int_{-\infty}^{\infty} k^{-2}(1+|k|x)\exp(iky-|k|x)dk$$

$$= \frac{P}{\pi}\int_{0}^{\infty} k^{-2}(1+kx)\cos ky\exp(-kx)dk. \tag{2.12.137}$$

$$\sigma_{xx} = -\frac{P}{\pi}\int_{0}^{\infty}(1+kx)e^{-kx}\cos ky\,dk = -\frac{2Px^3}{\pi(x^2+y^2)^2}. \tag{2.12.138}$$

$$\sigma_{xy} = -\frac{Px}{\pi}\int_{0}^{\infty} k\sin ky\exp(-kx)dk = -\frac{2Px^2y}{\pi(x^2+y^2)^2}. \tag{2.12.139}$$

$$\sigma_{yy} = -\frac{P}{\pi}\int_{0}^{\infty}(1-kx)\exp(-kx)\cos ky\,dk = -\frac{2Pxy^2}{\pi(x^2+y^2)^2}. \tag{2.12.140}$$

Another physically realistic pressure distribution is

$$p(y) = PH(|a|-y), \tag{2.12.141}$$

where $P$ is a constant, so that

$$\tilde{p}(k) = \sqrt{\frac{2}{\pi}}\frac{P}{k}\sin ak. \tag{2.12.142}$$

Substituting this value for $\tilde{p}(k)$ into (2.12.133)–(2.12.136), we obtain the integral expression for $\chi, \sigma_{xx}, \sigma_{xy}$, and $\sigma_{yy}$.

It is noted here that if a point force of magnitude $P_0$ acts at the origin located on the boundary, then we put $P = (P_0/2a)$ in (2.12.142) and find

$$\tilde{p}(k) = \lim_{a\to 0}\sqrt{\frac{2}{\pi}}\frac{P_0}{2}\left(\frac{\sin ak}{ak}\right) = \frac{P_0}{\sqrt{2\pi}}. \tag{2.12.143}$$

Thus, the stress components can also be written in this case. □

## 2.13 Fourier Cosine and Sine Transforms with Examples

The Fourier cosine integral formula (2.2.8) leads to the *Fourier cosine transform* and its inverse defined by

$$\mathscr{F}_c\{f(x)\} = F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx\, f(x)dx, \tag{2.13.1}$$

$$\mathscr{F}_c^{-1}\{F_c(k)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx\, F_c(k)dk, \tag{2.13.2}$$

where $\mathscr{F}_c$ is the Fourier cosine transform operator and $\mathscr{F}_c^{-1}$ is its inverse operator.

Similarly, the Fourier sine integral formula (2.2.9) leads to the *Fourier sine transform* and its inverse defined by

$$\mathscr{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx f(x)dx, \tag{2.13.3}$$

$$\mathscr{F}_s^{-1}\{F_s(k)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx\ F_s(k)dk, \tag{2.13.4}$$

where $\mathscr{F}_s$ is the Fourier sine transform operator and $\mathscr{F}_s^{-1}$ is its inverse.

***Example 2.13.1***
Show that

$$\text{(a)}\ \mathscr{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)}, \quad (a>0). \tag{2.13.5}$$

$$\text{(b)}\ \mathscr{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{k}{(a^2+k^2)}, \quad (a>0). \tag{2.13.6}$$

We have

$$\begin{aligned}\mathscr{F}_c\{e^{-ax}\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos kx\, dx \\ &= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty [e^{-(a-ik)x} + e^{-(a+ik)x}]dx \\ \mathscr{F}_c\{e^{-ax}\} &= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left[\frac{1}{a-ik} + \frac{1}{a+ik}\right] = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)}.\end{aligned}$$

The proof of the other result is similar and hence left to the reader. ▯

***Example 2.13.2***
Show that

$$\mathscr{F}_s^{-1}\left\{\frac{1}{k}\exp(-sk)\right\} = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{s}\right). \tag{2.13.7}$$

We have the standard definite integral

$$\sqrt{\frac{\pi}{2}}\mathscr{F}_s^{-1}\{\exp(-sk)\} = \int_0^\infty \exp(-sk)\sin kx\,dk = \frac{x}{s^2+x^2}. \tag{2.13.8}$$

Integrating both sides with respect to $s$ from $s$ to $\infty$ gives

$$\begin{aligned}\int_0^\infty \frac{e^{-sk}}{k}\sin kx\,dk &= \int_s^\infty \frac{x\,ds}{x^2+s^2} = \left[\tan^{-1}\frac{s}{x}\right]_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{x}\right) = \tan^{-1}\left(\frac{x}{s}\right).\end{aligned} \tag{2.13.9}$$

Thus,

$$\begin{aligned}\mathscr{F}_s^{-1}\left\{\frac{1}{k}\exp(-sk)\right\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{k}\exp(-sk)\sin xk\,dk \\ &= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{s}\right).\end{aligned}$$

▯

***Example 2.13.3***
Show that

$$\mathscr{F}_s\{erfc(ax)\} = \sqrt{\frac{2}{\pi}}\frac{1}{k}\left[1-\exp\left(-\frac{k^2}{4a^2}\right)\right]. \tag{2.13.10}$$

We have

$$\begin{aligned}\mathscr{F}_s\{erfc(ax)\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty erfc(ax)\sin kx\,dx \\ &= \frac{2\sqrt{2}}{\pi}\int_0^\infty \sin kx\,dx \int_{ax}^\infty e^{-t^2}\,dt.\end{aligned}$$

Interchanging the order of integration, we obtain

$$\begin{aligned}\mathscr{F}_s\{erf(ax)\} &= \frac{2\sqrt{2}}{\pi}\int_0^\infty \exp(-t^2)dt\int_0^{t/a}\sin kx\,dx\\ &= \frac{2\sqrt{2}}{\pi k}\int_0^\infty \exp(-t^2)\left\{1-\cos\left(\frac{kt}{a}\right)\right\}dt\\ &= \frac{2\sqrt{2}}{\pi k}\left[\frac{\sqrt{\pi}}{2}-\frac{\sqrt{\pi}}{2}\exp\left(-\frac{k^2}{4a^2}\right)\right].\end{aligned}$$

Thus,

$$\mathscr{F}_s\{erfc(ax)\} = \sqrt{\frac{2}{\pi}}\frac{1}{k}\left[1-\exp\left(-\frac{k^2}{4a^2}\right)\right].$$

□

## 2.14 Properties of Fourier Cosine and Sine Transforms

***THEOREM 2.14.1***

If $\mathscr{F}_c\{f(x)\} = F_c(k)$ and $\mathscr{F}_s\{f(x)\} = F_s(k)$, then

$$\mathscr{F}_c\{f(ax)\} = \frac{1}{a}F_c\left(\frac{k}{a}\right), \quad a>0. \tag{2.14.1}$$

$$\mathscr{F}_s\{f(ax)\} = \frac{1}{a}F_s\left(\frac{k}{a}\right), \quad a>0. \tag{2.14.2}$$

Under appropriate conditions, the following properties also hold:

$$\mathscr{F}_c\{f'(x)\} = k\ F_s(k) - \sqrt{\frac{2}{\pi}}\,f(0), \tag{2.14.3}$$

$$\mathscr{F}_c\{f''(x)\} = -k^2\ F_c(k) - \sqrt{\frac{2}{\pi}}\,f'(0), \tag{2.14.4}$$

$$\mathscr{F}_s\{f'(x)\} = -k\ F_c(k), \tag{2.14.5}$$

$$\mathscr{F}_s\{f''(x)\} = -k^2\ F_s(k) + \sqrt{\frac{2}{\pi}}\,k\,f(0). \tag{2.14.6}$$

These results can be generalized for the cosine and sine transforms of higher order derivatives of a function. They are left as exercises.

***THEOREM 2.14.2***

(*Convolution Theorem for the Fourier Cosine Transform*). If $\mathscr{F}_c\{f(x)\} = F_c(k)$ *and* $\mathscr{F}_c\{g(x)\} = G_c(k)$, then

$$\mathscr{F}_c^{-1}\{F_c(k)G_c(k)\} = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(\xi)[g(x+\xi) + g(|x-\xi|)]d\xi. \tag{2.14.7}$$

Or, equivalently,

$$\int_0^\infty F_c(k)G_c(k)\cos kx\, dk = \frac{1}{2}\int_0^\infty f(\xi)[g(x+\xi) + g(|x-\xi|)]d\xi. \tag{2.14.8}$$

**PROOF** Using the definition of the inverse Fourier cosine transform, we have

$$\begin{aligned}\mathscr{F}_c^{-1}\{F_c(k)G_c(k)\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(k)G_c(k)\cos kx\, dk \\ &= \left(\frac{2}{\pi}\right)\int_0^\infty G_c(k)\cos kx\, dk \int_0^\infty f(\xi)\cos k\xi\, d\xi.\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{F}_c^{-1}\{F_c(k)G_c(k)\} &= \left(\frac{2}{\pi}\right)\int_0^\infty f(\xi)d\xi \int_0^\infty \cos kx \cos k\xi\, G_c(k)dk \\ &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)d\xi \sqrt{\frac{2}{\pi}}\int_0^\infty [\cos k(x+\xi) + \cos k(|x-\xi|)]G_c(k)dk \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\xi)[g(x+\xi) + g(|x-\xi|)]d\xi,\end{aligned}$$

in which the definition of the inverse Fourier cosine transform is used. This proves (2.14.7).

It also follows from the proof of Theorem 2.14.2 that

$$\int_0^\infty F_c(k)G_c(k)\cos kx\, dk = \frac{1}{2}\int_0^\infty f(\xi)[g(x+\xi) + g(|x-\xi|)]d\xi.$$

This proves result (2.14.8).

Putting $x = 0$ in (2.14.8), we obtain

$$\int_0^\infty F_c(k)G_c(k)dk = \int_0^\infty f(\xi)g(\xi)d\xi = \int_0^\infty f(x)g(x)dx.$$

Substituting $g(x) = \overline{f(x)}$ gives, since $G_c(k) = \overline{F_c(k)}$,

$$\int_0^\infty |F_c(k)|^2 dk = \int_0^\infty |f(x)|^2 dx. \tag{2.14.9}$$

This is the *Parseval relation* for the Fourier cosine transform.

Similarly, we obtain

$$\begin{aligned}
&\int_0^\infty F_s(k) G_s(k) \cos kx\, dk \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty G_s(k) \cos kx\, dk \int_0^\infty f(\xi) \sin k\xi\, d\xi
\end{aligned}$$

which is, by interchanging the order of integration,

$$\begin{aligned}
&= \sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi) d\xi \int_0^\infty G_s(k) \sin k\xi \cos kx\, dk \\
&= \frac{1}{2} \int_0^\infty f(\xi) d\xi \sqrt{\frac{2}{\pi}} \int_0^\infty G_s(k) [\sin k(\xi + x) + \sin k(\xi - x)] dk \\
&= \frac{1}{2} \int_0^\infty f(\xi) [g(\xi + x) + g(\xi - x)] d\xi,
\end{aligned}$$

in which the inverse Fourier sine transform is used. Thus, we find

$$\int_0^\infty F_s(k) G_s(k) \cos kx\, dk = \frac{1}{2} \int_0^\infty f(\xi) [g(\xi + x) + g(\xi - x)] d\xi. \tag{2.14.10}$$

Or, equivalently,

$$\mathscr{F}_c^{-1}\{F_s(k) G_s(k)\} = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(\xi) [g(\xi + x) + g(\xi - x)] d\xi. \tag{2.14.11}$$

Result (2.14.10) or (2.14.11) is also called the *Convolution Theorem* of the Fourier cosine transform.

Putting $x = 0$ in (2.14.10) gives

$$\int_0^\infty F_s(k) G_s(k) dk = \int_0^\infty f(\xi) g(\xi) d\xi = \int_0^\infty f(x) g(x) dx.$$

Replacing $g(x)$ by $\overline{f(x)}$ gives the *Parseval relation* for the Fourier sine transform

$$\int_0^\infty |F_s(k)|^2 dk = \int_0^\infty |f(x)|^2 dx. \tag{2.14.12}$$

∎

## 2.15 Applications of Fourier Cosine and Sine Transforms to Partial Differential Equations

***Example 2.15.1***

(*One-Dimensional Diffusion Equation on a Half Line*). Consider the initial-boundary value problem for the one-dimensional diffusion equation in $0 < x < \infty$ with no sources or sinks:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0, \tag{2.15.1}$$

where $\kappa$ is a constant, with the initial condition

$$u(x, 0) = 0, \quad 0 < x < \infty, \tag{2.15.2}$$

and the boundary conditions

$$\text{(a) } u(0, t) = f(t), \quad t \geq 0, \quad u(x, t) \to 0 \quad \text{as } x \to \infty, \tag{2.15.3}$$

or,

$$\text{(b) } u_x(0, t) = f(t), \quad t \geq 0, \quad u(x, t) \to 0 \quad \text{as } x \to \infty. \tag{2.15.4}$$

This problem with the boundary conditions (2.15.3) is solved by using the Fourier sine transform

$$U_s(k, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx \, u(x, t) \, dx.$$

Application of the Fourier sine transform gives

$$\frac{dU_s}{dt} = -\kappa \, k^2 \, U_s(k, t) + \sqrt{\frac{2}{\pi}} \, \kappa \, k \, f(t), \tag{2.15.5}$$

$$U_s(k, 0) = 0. \tag{2.15.6}$$

The bounded solution of this differential system with $U_s(k,0)=0$ is

$$U_s(k,t)=\sqrt{\frac{2}{\pi}}\,\kappa\,k\int_0^t f(\tau)\exp[-\kappa(t-\tau)k^2]\,d\tau. \tag{2.15.7}$$

The inverse transform gives the solution

$$\begin{aligned}u(x,t) &= \sqrt{\frac{2}{\pi}}\,\kappa\int_0^t f(\tau)\mathscr{F}_s^{-1}\{k\exp[-\kappa(t-\tau)k^2]\}d\tau \\ &= \frac{x}{\sqrt{4\pi\kappa}}\int_0^t f(\tau)\exp\left[-\frac{x^2}{4\kappa(t-\tau)}\right]\frac{d\tau}{(t-\tau)^{3/2}} \end{aligned}\tag{2.15.8}$$

in which $\mathscr{F}_s^{-1}\{k\exp(-t\kappa k^2)\}=\frac{x}{2\sqrt{2}}\cdot\frac{\exp(-x^2/4\kappa t)}{(\kappa t)^{3/2}}$ is used.

In particular, $f(t)=T_0=\text{constant}$, (2.15.7) reduces to

$$U_s(k,t)=\sqrt{\frac{2}{\pi}}\frac{T_0}{k}[1-\exp(-\kappa\,t\,k^2)]. \tag{2.15.9}$$

Inversion gives the solution

$$u(x,t)=\left(\frac{2T_0}{\pi}\right)\int_0^\infty\frac{\sin kx}{k}[1-\exp(-\kappa\,t\,k^2)]dk. \tag{2.15.10}$$

Making use of the integral

$$\int_0^\infty e^{-k^2a^2}\,\frac{\sin kx}{k}dk=\frac{\pi}{2}\,erf\left(\frac{x}{2a}\right), \tag{2.15.11}$$

the solution becomes

$$\begin{aligned}u(x,t) &= \frac{2T_0}{\pi}\left[\frac{\pi}{2}-\frac{\pi}{2}\,erf\left(\frac{x}{2\sqrt{\kappa t}}\right)\right] \\ &= T_0\;erfc\left(\frac{x}{2\sqrt{\kappa t}}\right), \end{aligned}\tag{2.15.12}$$

where the *error function*, *erf*$(x)$ is defined by

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\alpha^2}d\alpha, \tag{2.15.13}$$

so that

$$erf(0)=0, erf(\infty)=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\alpha^2}d\alpha=1, \text{ and } erf(-x)=-erf(x),$$

and the *complementary error function*, $erfc(x)$ is defined by

$$erfc(x)=1-erf(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\alpha^2}d\alpha, \tag{2.15.14}$$

so that

$$erfc(x)=1-erf(x), \quad erfc(0)=1, \quad erfc(\infty)=0,$$

and

$$erfc(-x)=1-erf(-x)=1+erf(x)=2-erfc(x).$$

Equation (2.15.1) with boundary condition (2.15.4) is solved by the Fourier cosine transform

$$U_c(k,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty \cos kx\, u(x,t)dx.$$

Application of this transform to (2.15.1) gives

$$\frac{dU_c}{dt}+\kappa k^2U_c=-\sqrt{\frac{2}{\pi}}\,\kappa f(t). \tag{2.15.15}$$

The solution of (2.15.15) with $U_c(k,0)=0$ is

$$U_c(k,t)=-\sqrt{\frac{2}{\pi}}\,\kappa\int_0^t f(\tau)\exp[-k^2\kappa(t-\tau)]d\tau. \tag{2.15.16}$$

Since

$$\mathscr{F}_c^{-1}\{\exp(-t\kappa k^2)\}=\frac{1}{\sqrt{2\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right), \tag{2.15.17}$$

the inverse Fourier cosine transform gives the final form of the solution

$$u(x,t)=-\sqrt{\frac{\kappa}{\pi}}\int_0^t \frac{f(\tau)}{\sqrt{t-\tau}}\exp\left[-\frac{x^2}{4\kappa(t-\tau)}\right]d\tau. \tag{2.15.18}$$

□

***Example 2.15.2***
(*The Laplace Equation in the Quarter Plane*). Solve the Laplace equation

$$u_{xx}+u_{yy}=0, \quad 0<x,\ y<\infty, \tag{2.15.19}$$

with the boundary conditions

$$u(0,y)=a, \qquad u(x,0)=0, \tag{2.15.20a}$$

$$\nabla u \to 0 \qquad \text{as } r=\sqrt{x^2+y^2}\to\infty, \tag{2.15.20b}$$

where $a$ is a constant.

We apply the Fourier sine transform with respect to $x$ to find

$$\frac{d^2U_s}{dy^2} - k^2\, U_s + \sqrt{\frac{2}{\pi}}\, ka = 0.$$

The solution of this inhomogeneous equation is

$$U_s(k,y) = Ae^{-ky} + \sqrt{\frac{2}{\pi}}\cdot\frac{a}{k},$$

where $A$ is a constant to be determined from $U_s(k,0)=0$. Consequently,

$$U_s(k,y) = \frac{a}{k}\sqrt{\frac{2}{\pi}}(1-e^{-ky}). \tag{2.15.21}$$

The inverse transformation gives the formal solution

$$u(x,y) = \frac{2a}{\pi}\int_0^\infty \frac{1}{k}(1-e^{-ky})\sin kx\, dk$$

Or,

$$\begin{aligned} u(x,y) &= \frac{2a}{\pi}\left[\int_0^\infty \frac{\sin kx}{k}dk - \int_0^\infty \frac{1}{k}e^{-ky}\sin kx\, dk\right] \\ &= a - \frac{2a}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{y}{x}\right) = \frac{2a}{\pi}\tan^{-1}\left(\frac{y}{x}\right), \end{aligned} \tag{2.15.22}$$

in which (2.13.9) is used. ▯

***Example 2.15.3***

(*The Laplace Equation in a Semi-Infinite Strip with the Dirichlet Data*). Solve the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad 0<x<\infty, \quad 0<y<b, \tag{2.15.23}$$

with the boundary conditions

$$u(0,y) = 0,\ u(x,y)\to 0 \quad \text{as} \quad x\to\infty \text{ for } 0<y<b \tag{2.15.24}$$

$$u(x,b) = 0,\ u(x,0) = f(x) \quad \text{for} \quad 0<x<\infty. \tag{2.15.25}$$

In view of the Dirichlet data, the Fourier sine transform with respect to $x$ can be used to solve this problem. Applying the Fourier sine transform to (2.15.23)–(2.15.25) gives

$$\frac{d^2 U_s}{dy^2} - k^2 U_s = 0, \tag{2.15.26}$$

$$U_s(k,b) = 0, \quad U_s(k,0) = F_s(k). \tag{2.15.27}$$

The solution of (2.15.26) with (2.15.27) is

$$U_s(k,y) = F_s(k)\,\frac{\sinh[k(b-y)]}{\sinh kb}. \tag{2.15.28}$$

The inverse Fourier sine transform gives the formal solution

$$\begin{aligned} u(x,y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \frac{\sinh[k(b-y)]}{\sinh kb} \sin kx\, dk \\ &= \frac{2}{\pi} \int_0^\infty \left[ \int_0^\infty f(l) \sin kl\, dl \right] \frac{\sinh[k(b-y)]}{\sinh kb} \sin kx\, dk. \end{aligned} \tag{2.15.29}$$

In the limit as $kb \to \infty$, $\frac{\sinh[k(b-y)]}{\sinh kb} \sim \exp(-ky)$, hence the above problem reduces to the corresponding problem in the quarter plane, $0 < x < \infty, 0 < y < \infty$. Thus, solution (2.15.29) becomes

$$\begin{aligned} u(x,y) &= \frac{2}{\pi} \int_0^\infty f(l) dl \int_0^\infty \sin kl \sin kx \exp(-ky) dk \\ &= \frac{1}{\pi} \int_0^\infty f(l) dl \int_0^\infty \{\cos k(x-l) - \cos k(x+l)\} \exp(-ky) dk \\ &= \frac{1}{\pi} \int_0^\infty f(l) \left[ \frac{y}{(x-l)^2 + y^2} - \frac{y}{(x+l)^2 + y^2} \right] dl. \end{aligned} \tag{2.15.30}$$

This is the exact integral solution of the problem. If $f(x)$ is an odd function of $x$, then solution (2.15.30) reduces to the solution (2.12.10) of the same problem in the half plane. ▯

## 2.16 Evaluation of Definite Integrals

The Fourier transform can be employed to evaluate certain definite integrals. Although the method of evaluation may not be very rigorous, it is quite simple and straightforward. The method can be illustrated by means of examples.

***Example 2.16.1***
Evaluate the integral

$$I(a,b) = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, \quad a>0, b>0. \tag{2.16.1}$$

If we write $f(x) = e^{-a|x|}$ and $g(x) = e^{-b|x|}$ then $F(k) = \sqrt{\frac{2}{\pi}} \frac{a}{(k^2+a^2)}$, $G(k) = \sqrt{\frac{2}{\pi}} \frac{b}{(k^2+b^2)}$. The Convolution Theorem 2.5.5 gives (2.5.19), that is,

$$\int_{-\infty}^{\infty} F(k)G(k)dk = \int_{-\infty}^{\infty} f(x)g(-x)dx.$$

Or, equivalently,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dk}{(k^2+a^2)(k^2+b^2)} &= \frac{\pi}{2ab} \int_{-\infty}^{\infty} e^{-|x|(a+b)} dx \\ &= \frac{\pi}{ab} \int_0^{\infty} e^{-(a+b)x} dx = \frac{\pi}{ab(a+b)}. \end{aligned} \tag{2.16.2}$$

This is the desired result.

Further

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}. \tag{2.16.3}$$

□

***Example 2.16.2***
Show that

$$\int_0^{\infty} \frac{x^{-p}dx}{(a^2+x^2)} = \frac{\pi}{2} a^{-(p+1)} \sec\left(\frac{\pi p}{2}\right). \tag{2.16.4}$$

We write

$$f(x) = e^{-ax} \quad \text{so that} \quad F_c(k) = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)}.$$

$$g(x) = x^{p-1} \quad \text{so that} \quad G_c(k) = \sqrt{\frac{2}{\pi}} k^{-p} \Gamma(p) \cos\left(\frac{\pi p}{2}\right).$$

Using Parseval's result for the Fourier cosine transform gives

$$\int_0^{\infty} F_c(k)G_c(k)dk = \int_0^{\infty} f(x)g(x)dx.$$

Or,

$$\frac{2a}{\pi}\cos\left(\frac{\pi p}{2}\right)\Gamma(p)\int_0^\infty \frac{k^{-p}dk}{k^2+a^2} = \int_0^\infty x^{p-1}e^{-ax}dx$$

$$= \frac{1}{a^p}\int_0^\infty e^{-t}t^{p-1}dt = \frac{\Gamma(p)}{a^p}, \quad (ax=t).$$

Thus,

$$\int_0^\infty \frac{k^{-p}dk}{a^2+k^2} = \frac{\pi}{2\,a^{p+1}}\sec\left(\frac{\pi p}{2}\right).$$

□

***Example 2.16.3***
If $a>0, b>0$, show that

$$\int_0^\infty \frac{x^2\,dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2(a+b)}. \tag{2.16.5}$$

We consider

$$\mathscr{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{k}{k^2+a^2} = F_s(k)$$

$$\mathscr{F}_s\{e^{-bx}\} = \sqrt{\frac{2}{\pi}}\frac{k}{k^2+b^2} = G_s(k).$$

Then the Convolution Theorem for the Fourier cosine transform gives

$$\int_0^\infty F_s(k)G_s(k)\cos kx\,dk = \frac{1}{2}\int_0^\infty g(\xi)[f(\xi+x)+f(\xi-x)]d\xi.$$

Putting $x=0$ gives

$$\int_0^\infty F_s(k)G_s(k)dk = \int_0^\infty g(\xi)f(\xi)d\xi,$$

or,

$$\int_0^\infty \frac{k^2\,dk}{(k^2+a^2)(k^2+b^2)} = \frac{\pi}{2}\int_0^\infty e^{-(a+b)\xi}d\xi = \frac{\pi}{2(a+b)}.$$

□

***Example 2.16.4***
Show that

$$\int_0^\infty \frac{x^2 dx}{(x^2+a^2)^4} = \frac{\pi}{(2a)^5}, \quad a > 0. \tag{2.16.6}$$

We write $f(x) = \frac{1}{2(x^2+a^2)}$ so that $f'(x) = -\frac{x}{(x^2+a^2)^2}$, and $\mathscr{F}\{f(x)\} = F(k) = \sqrt{\frac{\pi}{2}}\left(\frac{1}{2a}\right)\exp(-a|k|)$.

Making reference to the Parseval relation (2.4.19), we obtain

$$\int_{-\infty}^\infty |f'(x)|^2 dx = \int_{-\infty}^\infty |\mathscr{F}\{f'(x)\}|^2 dk = \int_{-\infty}^\infty |(ik)\mathscr{F}\{f(x)\}|^2 dk.$$

Thus,

$$\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)^4}\,dx = \frac{\pi}{2}\int_{-\infty}^\infty k^2 \cdot \frac{1}{(2a)^2}\exp(-2a|k|)dk$$

$$= \frac{\pi}{(2a)^2}\int_0^\infty k^2 \exp(-2ak)dk = \frac{2\pi}{(2a)^5}.$$

This gives the desired result. ▯

## 2.17 Applications of Fourier Transforms in Mathematical Statistics

In probability theory and mathematical statistics, the characteristic function of a random variable is defined by the Fourier transform or by the Fourier-Stieltjes transform of the distribution function of a random variable. Many important results in probability theory and mathematical statistics can be obtained, and their proofs can be simplified with rigor by using the methods of characteristic functions. Thus, the Fourier transforms play an important role in probability theory and mathematical statistics.

***DEFINITION 2.17.1*** *(Distribution Function). The distribution function $F(x)$ of a random variable $X$ is defined as the probability, that is, $F(x) = P(X < x)$ for every real number $x$.*

It is immediately evident from this definition that the distribution function satisfies the following properties:

(i) $F(x)$ is a non-decreasing function, that is, $F(x_1) \leq F(x_2)$ if $x_1 < x_2$.

(ii) $F(x)$ is continuous only from the left at a point $x$, that is, $F(x-0) = F(x)$, but $F(x+0) \neq F(x)$.

(iii) $F(-\infty) = 0$ and $F(+\infty) = 1$.

If $X$ is a continuous variable and if there exists a non-negative function $f(x)$ such that for every real $x$ the following relation holds:

$$F(x) = \int_{-\infty}^{x} f(x)dx, \tag{2.17.1}$$

where $F(x)$ is the distribution function of the random variable $X$, then the function $f(x)$ is called the *probability density* or simply the *density function* of the random variable $X$.

It is immediately obvious that every density function $f(x)$ satisfies the following properties:

(i)

$$F(+\infty) = \int_{-\infty}^{\infty} f(x)dx = 1. \tag{2.17.2a}$$

(ii) For every real $a$ and $b$ where $a < b$,

$$P(a \leq X \leq b) = F(b) - F(a) = \int_{a}^{b} f(x)dx. \tag{2.17.2b}$$

(iii) If $f(x)$ is continuous at some point $x$, then $F'(x) = f(x)$.

It is noted that every real function $f(x)$ which is non-negative, and integrable over the whole real line and satisfies (2.17.2ab), is the probability density function of a continuous random variable $X$. On the other hand, the function $F(x)$ defined by (2.17.1) satisfies all properties of a distribution function.

***DEFINITION 2.17.2*** *(Characteristic Function). If $X$ is a continuous random variable with the density function $f(x)$, then the characteristic function, $\phi(t)$ of the random variable $X$ or of the distribution function $F(x)$ is defined by the formula*

$$\phi(t) = E(\exp(itX)) = \int_{-\infty}^{\infty} f(x)\exp(itx)dx, \tag{2.17.3}$$

where $E[g(X)]$ is called the *expected value* of the random variable $g(X)$.

In problems of mathematical statistics, it is convenient to define the Fourier transform of $f(x)$ and its inverse in a slightly different way by

$$\mathscr{F}\{f(x)\} = \phi(t) = \int_{-\infty}^{\infty} \exp(itx) f(x) dx, \tag{2.17.4}$$

$$\mathscr{F}^{-1}\{\phi(t)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-itx) \phi(t) dt. \tag{2.17.5}$$

Evidently, the characteristic function of $F(x)$ is the Fourier transform of the density function $f(x)$. The Fourier transform of the distribution function follows from the fact that

$$\mathscr{F}\{F'(x)\} = \mathscr{F}\{f(x)\} = \phi(t),$$

or,

$$\mathscr{F}\{F(x)\} = it^{-1}\phi(t). \tag{2.17.6}$$

The *composition of two distribution functions* $F_1(x)$ and $F_2(x)$ is defined by

$$F(x) = F_1(x) * F_2(x) = \int_{-\infty}^{\infty} F_1(x-y) F_2'(y) dy. \tag{2.17.7}$$

Thus, the Fourier transform of (2.17.7) gives

$$\begin{aligned} i\,t^{-1}\phi(t) &= \mathscr{F}\left\{ \int_{-\infty}^{\infty} F_1(x-y) F_2'(y) dy \right\} \\ &= \mathscr{F}\{F_1(x)\}\mathscr{F}\{f_2(x)\} = it^{-1}\phi_1(t)\phi_2(t), \end{aligned}$$

whence an important result follows:

$$\phi(t) = \phi_1(t)\phi_2(t), \tag{2.17.8}$$

where $\phi_1(t)$ and $\phi_2(t)$ are the characteristic functions of the distribution functions $F_1(x)$ and $F_2(x)$ respectively.

The *nth moment* of a random variable $X$ is defined by

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f(x) dx, \quad n = 1, 2, 3, \ldots. \tag{2.17.9}$$

provided this integral exists. The first moment $m_1$ (or simply $m$) is called the *expectation* of $X$ and has the form

$$m = E(X) = \int_{-\infty}^{\infty} x\, f(x) dx. \tag{2.17.10}$$

Thus, the moment of any order $n$ is calculated by evaluating the integral (2.17.9). However, the evaluation of the integral is, in general, a difficult task. This difficulty can be resolved with the help of the characteristic function defined by (2.17.4). Differentiating (2.17.4) $n$ times and putting $t=0$ gives a fairly simple formula

$$m_n = \int_{-\infty}^{\infty} x^n f(x)dx = (-i)^n \phi^{(n)}(0), \tag{2.17.11}$$

where $n=1,2,3,\ldots$.

When $n=1$, the expectation of a random variable $X$ becomes

$$m_1 = E(X) = \int_{-\infty}^{\infty} x f(x)dx = (-i)\phi'(0). \tag{2.17.12}$$

Thus, the simple formula (2.17.11) involving the derivatives of the characteristic function provides for the existence and the computation of the moment of any arbitrary order.

Similarly, the variance $\sigma^2$ of a random variable is given in terms of the characteristic function as

$$\begin{aligned}\sigma^2 &= \int_{-\infty}^{\infty} (x-m)^2 f(x)dx = m_2 - m_1^2 \\ &= \{\phi'(0)\}^2 - \phi''(0). \end{aligned} \tag{2.17.13}$$

***Example 2.17.1***

Find the moments of the normal distribution defined by the density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}. \tag{2.17.14}$$

The characteristic function of the normal distribution is the Fourier transform of $f(x)$, which is

$$\phi(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] dx.$$

We substitute $x-m=y$ and use Example 2.3.1 to obtain

$$\phi(t) = \frac{\exp(itm)}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ity} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy = \exp\left(itm - \frac{1}{2}t^2\sigma^2\right). \tag{2.17.15}$$

Thus,

$$\begin{aligned} m_1 &= (-i)\phi'(0) = m, \\ m_2 &= -\phi''(0) = (m^2 + \sigma^2), \\ m_3 &= m(m^2 + 3\sigma^2). \end{aligned}$$

Finally, the variance of the normal distribution is

$$m_2 - m_1^2 = \sigma^2. \tag{2.17.16}$$

□

The above discussion reveals that characteristic functions are very useful for investigation of certain problems in mathematical statistics. We close this section by discussing more properties of characteristic functions.

***THEOREM 2.17.1***
(*Addition Theorem*). The characteristic function of the sum of a finite number of independent random variables is equal to the product of their characteristic functions.

**PROOF** Suppose $X_1, X_2, \ldots, X_n$ are $n$ independent random variables and $Z = X_1 + X_2 + \cdots + X_n$. Further, suppose $\phi_1(t), \phi_2(t), \ldots, \phi_n(t)$, and $\phi(t)$ are the characteristic functions of $X_1, X_2, \ldots, X_n$ and $Z$ respectively.

Then we have

$$\phi(t) = E[\exp(itZ)] = E\left[\exp\left\{it\left(X_1 + X_2 + \cdots + X_n\right)\right\}\right],$$

which is, by the independence of the random variables,

$$\begin{aligned} &= E(e^{itX_1})E(e^{itX_2})\cdots E(e^{itX_n}) \\ &= \phi_1(t)\,\phi_2(t)\cdots\phi_n(t). \end{aligned} \tag{2.17.17}$$

This proves the *Addition Theorem.* ∎

***Example 2.17.2***
Find the expected value and the standard deviation of the sum of $n$ independent normal random variables.

Suppose $X_1, X_2, \ldots, X_n$ are $n$ independent random variables with the normal distributions $N(m_r, \sigma_r)$, where $r = 1, 2, \ldots, n$. The respective characteristic functions of these distributions are

$$\phi_r(t) = \exp\left[itm_r - \frac{1}{2}t^2\sigma_r^2\right], \quad r = 1, 2, 3, \ldots, n. \tag{2.17.18}$$

Because of the independence of $X_1, X_2, \dots, X_n$, the random variable $Z = X_1 + X_2 + \cdots + X_n$ has the characteristic function

$$\begin{aligned}\phi(t) &= \phi_1(t)\phi_2(t)\cdots\phi_n(t)\\ &= \exp\left[it(m_1+m_2+\cdots+m_n) - \frac{1}{2}(\sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2)t^2\right]. \quad (2.17.19)\end{aligned}$$

This represents the characteristic function of the normal distribution $N(m_1 + \cdots + m_n, \sqrt{\sigma_1^2+\cdots+\sigma_n^2})$. Thus, the expected value of $Z$ is $(m_1+m_2+\cdots+m_n)$ and its standard deviation is $(\sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2)^{\frac{1}{2}}$. ∎

Finally, we state the fundamental *Central Limit Theorems* without proof.

***THEOREM 2.17.2***

(*The Lévy-Cramér Theorem*). Suppose $\{X_n\}$ is a sequence of random variables, $F_n(x)$ and $\phi_n(t)$ are respectively the distribution and characteristic functions of $X_n$. Then the sequence $\{F_n(x)\}$ is convergent to a distribution function $F(x)$ if and only if the sequence $\{\phi_n(t)\}$ is convergent at every point $t$ on the real line to a function $\phi(t)$ continuous in some neighborhood of the origin. The limit function $\phi(t)$ is then the characteristic function of the limit distribution function $F(x)$, and the convergence $\phi_n(t)\to\phi(t)$ is uniform in every finite interval on the $t$-axis.

***THEOREM 2.17.3***

(*The Central Limit Theorem in Probability*). Supose $f(x)$ is a nonnegative absolutely integrable function in $\mathbb{R}$ and has the following properties:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1, \quad \int_{-\infty}^{\infty} x\,f(x)\,dx = 1, \quad \int_{-\infty}^{\infty} x^2\,f(x)\,dx = 1.$$

If $f^n = f * f * \ldots * f$ is the convolution product of $f$ with itself $n$ times, then

$$\lim_{n\to\infty}\int_{a\sqrt{n}}^{b\sqrt{n}} f^n(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2}\,dx \qquad -\infty < a < b < \infty. \quad (2.17.20)$$

For a proof of the theorem, we refer the reader to Chandrasekharan (1989).

All these ideas developed in this section can be generalized for the multidimensional distribution functions by the use of multiple Fourier transforms. We refer interested readers to Lukacs (1960).

## 2.18 Multiple Fourier Transforms and Their Applications

***DEFINITION 2.18.1*** *Under the assumptions on $f(\mathbf{x})$ similar to those made for the one dimensional case, the multiple Fourier transform of $f(\mathbf{x})$, where $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ is the $n$-dimensional vector, is defined by*

$$\mathscr{F}\{f(\mathbf{x})\}=F(\boldsymbol{\kappa})=\frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\{-i(\boldsymbol{\kappa}\cdot\mathbf{x})\}f(\mathbf{x})d\mathbf{x}, \tag{2.18.1}$$

*where $\boldsymbol{\kappa}=(k_1,k_2,\ldots,k_n)$ is the $n$-dimensional transform vector and $\boldsymbol{\kappa}\cdot\mathbf{x}=(k_1x_1+k_2x_2+\cdots+k_nx_n)$.*

*The inverse Fourier transform is similarly defined by*

$$\mathscr{F}^{-1}\{F(\boldsymbol{\kappa})\}=f(\mathbf{x})=\frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\{i(\boldsymbol{\kappa}\cdot\mathbf{x})\}\,F(\boldsymbol{\kappa})\,d\boldsymbol{\kappa}. \tag{2.18.2}$$

*In particular, the double Fourier transform is defined by*

$$\mathscr{F}\{f(x,y)\}=F(k,\ell)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-i(\boldsymbol{\kappa}\cdot\mathbf{r})\}\,f(x,y)\,dxdy, \tag{2.18.3}$$

*where $\mathbf{r}=(x,y)$ and $\boldsymbol{\kappa}=(k,\ell)$.*

*The inverse Fourier transform is given by*

$$\mathscr{F}^{-1}\{F(k,\ell)\}=f(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{i(\boldsymbol{\kappa}\cdot\mathbf{r})\}\,F(k,\ell)\,dk\,d\ell. \tag{2.18.4}$$

*Similarly, the three-dimensional Fourier transform and its inverse are defined by the integrals*

$$\mathscr{F}\{f(x,y,z)\}=F(k,\ell,m)$$
$$=\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-i(\boldsymbol{\kappa}\cdot\mathbf{r})\}\,f(x,y,z)\,dx\,dy\,dz, \tag{2.18.5}$$

$$\mathscr{F}^{-1}\{F(k,\ell,m)\}=f(x,y,z)$$
$$\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{i(\boldsymbol{\kappa}\cdot\mathbf{r})\}\,F(k,\ell,m)\,dk\,d\ell\,dm. \tag{2.18.6}$$

The operational properties of these multiple Fourier transforms are similar to those of the one-dimensional case. In particular, results (2.4.7) and (2.4.8) relating the Fourier transforms of derivatives to the Fourier transforms of given functions are valid for the higher dimensional case as well. In higher dimensions, they are applied to the transforms of partial derivatives of $f(\mathbf{x})$ under the assumptions that $f$ and its partial derivatives vanish at infinity.

We illustrate the multiple Fourier transform method by the following examples of applications:

***Example 2.18.1***
(*The Dirichlet Problem for the Three-Dimensional Laplace Equation in the Half-Space*). The boundary value problem for $u(x,y,z)$ satisfies the following equation and boundary conditions:

$$\nabla^2 u \equiv u_{xx} + u_{yy} + u_{zz} = 0, \quad -\infty < x, y < \infty, \quad z > 0, \tag{2.18.7}$$

$$u(x,y,0) = f(x,y) \qquad -\infty < x, y < \infty \tag{2.18.8}$$

$$u(x,y,z) \to 0 \qquad \text{as } r = \sqrt{x^2+y^2+z^2} \to \infty. \tag{2.18.9}$$

We use the double Fourier transform defined by (2.18.3) to the system (2.18.7)–(2.18.9) which reduces to

$$\frac{d^2U}{dz^2} - \kappa^2 U = 0 \qquad \text{for} \qquad z > 0, \quad (\kappa^2 = k^2 + l^2)$$

$$U(k,\ell,0) = F(k,\ell).$$

Thus, the solution of this transformed problem is

$$U(k,\ell,z) = F(k,\ell)\exp(-|\boldsymbol{\kappa}|z) = F(k,\ell)G(k,\ell), \tag{2.18.10}$$

where $\boldsymbol{\kappa} = (k,\ell)$ and $G(k,\ell) = \exp(-|\boldsymbol{\kappa}|z)$ so that

$$g(x,y) = \mathscr{F}^{-1}\{\exp(-|\boldsymbol{\kappa}|z)\} = \frac{z}{(x^2+y^2+z^2)^{3/2}}. \tag{2.18.11}$$

Applying the Convolution Theorem to (2.18.10), we obtain the formal solution

$$u(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)g(x-\xi, y-\eta, z)\,d\xi\,d\eta$$

$$= \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\xi,\eta)\,d\xi\,d\eta}{[(x-\xi)^2+(y-\eta)^2+z^2]^{3/2}}. \tag{2.18.12}$$

□

***Example 2.18.2***
(*The Two-Dimensional Diffusion Equation*). We solve the two-dimensional diffusion equation

$$u_t = K\nabla^2 u, \quad -\infty < x, y < \infty, \quad t > 0, \tag{2.18.13}$$

with the initial and boundary conditions

$$u(x, y, 0) = f(x, y) \quad -\infty < x, y < \infty, \tag{2.18.14}$$
$$u(x, y, t) \to 0 \quad \text{as} \quad r = \sqrt{x^2 + y^2} \to \infty, \tag{2.18.15}$$

where $K$ is the diffusivity constant.

The double Fourier transform of $u(x, y, t)$ defined by (2.18.3) is used to reduce the system (2.18.13)–(2.18.14) into the form

$$\frac{dU}{dt} = -\kappa^2 KU, \qquad t > 0,$$
$$U(k, \ell, 0) = F(k, \ell).$$

The solution of this system is

$$U(k, \ell, t) = F(k, \ell)\exp(-tK\kappa^2) = F(k, \ell)G(k, \ell), \tag{2.18.16}$$

where

$$G(k, \ell) = \exp(-K\kappa^2 t),$$

so that

$$g(x, y) = \mathscr{F}^{-1}\{\exp(-tK\kappa^2)\} = \frac{1}{2Kt}\exp\left(-\frac{x^2 + y^2}{4Kt}\right). \tag{2.18.17}$$

Finally, the Convolution Theorem gives the formal solution

$$u(x, y, t) = \frac{1}{4\pi Kt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)\exp\left[-\frac{(x-\xi)^2 + (y-\eta)^2}{4Kt}\right] d\xi\, d\eta. \tag{2.18.18}$$

Or, equivalently,

$$u(x, y, t) = \frac{1}{4\pi Kt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r}')\exp\left\{-\frac{|\mathbf{r} - \mathbf{r}'|^2}{4Kt}\right\} d\mathbf{r}', \tag{2.18.19}$$

where $\mathbf{r}' = (\xi, \eta)$.

We make the change of variable $(\mathbf{r}' - \mathbf{r}) = \sqrt{4Kt}\,\mathbf{R}$ to reduce (2.18.19) in the form

$$u(x, y, t) = \frac{1}{\pi\sqrt{4Kt}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r} + \sqrt{4Kt}\,\mathbf{R})\exp(-R^2)\, d\mathbf{R}. \tag{2.18.20}$$

Similarly, the formal solution of the initial value problem for the three-dimensional diffusion equation

$$u_t = K(u_{xx} + u_{yy} + u_{zz}), \quad -\infty < x, y, z < \infty, \quad t > 0 \tag{2.18.21}$$

$$u(x, y, z, 0) = f(x, y, z), \quad -\infty < x, y, z < \infty \tag{2.18.22}$$

is given by

$$u(x, y, z, t) = \frac{1}{(4\pi K t)^{3/2}} \int \int_{-\infty}^{\infty} \int f(\xi, \eta, \zeta) \exp\left(-\frac{r^2}{4Kt}\right) d\xi \, d\eta \, d\zeta, \tag{2.18.23}$$

where

$$r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2.$$

Or, equivalently,

$$u(x, y, z, t) = \frac{1}{(4\pi K t)^{3/2}} \int \int_{-\infty}^{\infty} \int f(\mathbf{r}') \exp\left\{-\frac{|\mathbf{r} - \mathbf{r}'|^2}{4Kt}\right\} d\xi \, d\eta \, d\zeta, \tag{2.18.24}$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (\xi, \eta, \zeta)$.

Making the change of variable $\mathbf{r}' - \mathbf{r} = \sqrt{4tK}\mathbf{R}$, solution (2.18.24) reduces to

$$u(x, y, z, t) = \frac{1}{\pi^{3/2} 4Kt} \int \int_{-\infty}^{\infty} \int f(\mathbf{r} + \sqrt{4Kt}\,\mathbf{R}) \exp(-R^2) \, d\mathbf{R}. \tag{2.18.25}$$

This is known as the *Fourier solution.* □

***Example 2.18.3***

(*The Cauchy Problem for the Two-Dimensional Wave Equation*). The initial value problem for the wave equation in two dimensions is governed by

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad -\infty < x, y < \infty, \quad t > 0, \tag{2.18.26}$$

with the initial data

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = f(x, y), \quad -\infty < x, y < \infty, \tag{2.18.27ab}$$

where $c$ is a constant. We assume that $u$ and its first partial derivatives vanish at infinity.

We apply the two-dimensional Fourier transform defined by (2.18.3) to the system (2.18.26)–(2.18.27ab), which becomes

$$\frac{d^2U}{dt^2} + c^2\kappa^2 U = 0, \quad \kappa^2 = k^2 + \ell^2,$$

$$U(k, \ell, 0) = 0, \quad \left(\frac{dU}{dt}\right)_{t=0} = F(k, \ell).$$

The solution of this transformed system is

$$U(k,\ell,t)=F(k,\ell)\frac{\sin(c\kappa t)}{c\kappa}. \tag{2.18.28}$$

The inverse Fourier transform gives the formal solution

$$u(x,y,t) = \frac{1}{2\pi c}\int_{-\infty}^{\infty}\int \exp(i\boldsymbol{\kappa}\cdot\mathbf{r})\,\frac{\sin(c\kappa t)}{\kappa}\,F(\boldsymbol{\kappa})d\boldsymbol{\kappa} \tag{2.18.29}$$

$$= \frac{1}{4i\pi c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F(\boldsymbol{\kappa})}{\kappa}\left[\exp\left\{i\kappa\left(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}+ct\right)\right\}\right.$$
$$\left.-\exp\left\{i\kappa\left(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}-ct\right)\right\}\right]d\boldsymbol{\kappa}. \tag{2.18.30}$$

The form of this solution reveals an interesting feature of the wave equation. The exponential terms $\exp\left\{i\kappa\left(ct\pm\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}\right)\right\}$ involved in the integral solution (2.18.30) represent plane wave solutions of the wave equation (2.18.26). Thus, the solutions remain constant on the planes $\boldsymbol{\kappa}\cdot\mathbf{r}=$ constant that move parallel to themselves with velocity $c$. Evidently, solution (2.18.30) represents a superposition of the plane wave solutions traveling in all possible directions.

Similarly, the solution of the Cauchy problem for the three-dimensional wave equation

$$u_{tt}=c^2(u_{xx}+u_{yy}+u_{zz}),\ -\infty<x,y,z<\infty,\ t>0, \tag{2.18.31}$$

$$u(x,y,z,0)=0,\ u_t(x,y,z,0)=f(x,y,z),\ -\infty<x,y,z<\infty \tag{2.18.32ab}$$

is given by

$$u(\mathbf{r},t)=\frac{1}{2ic(2\pi)^{3/2}}\int\int_{-\infty}^{\infty}\int\frac{F(\boldsymbol{\kappa})}{\kappa}\left[\exp\left\{i\kappa\left(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}+ct\right)\right\}\right.$$
$$\left.-\exp\left\{i\kappa\left(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}-ct\right)\right\}\right]d\boldsymbol{\kappa}, \tag{2.18.33}$$

where $\mathbf{r}=(x,y,z)$ and $\boldsymbol{\kappa}=(k,\ell,m)$.

In particular, when $f(x,y,z)=\delta(x)\delta(y)\delta(z)$ so that $F(\boldsymbol{\kappa})=(2\pi)^{-3/2}$, solution (2.18.33) becomes

$$u(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int\int_{-\infty}^{\infty}\int\left(\frac{\sin c\kappa t}{c\kappa}\right)\ \exp(i(\boldsymbol{\kappa}\cdot\mathbf{r}))d\boldsymbol{\kappa}. \tag{2.18.34}$$

In terms of the spherical polar coordinates $(\kappa,\theta,\phi)$ where the polar axis (the $z$-axis) is taken along the $\mathbf{r}$ direction with $\boldsymbol{\kappa}\cdot\mathbf{r}=\kappa r\cos\theta$, we write (2.18.34) in the form

$$u(r,t) = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} \exp(i\kappa r\ \cos\theta)\,\frac{\sin c\kappa t}{c\kappa}\cdot\kappa^2\ \sin\theta\, d\kappa$$
$$= \frac{1}{2\pi^2 cr}\int_0^{\infty} \sin(c\kappa t)\sin(\kappa r)\ d\kappa$$
$$= \frac{1}{8\pi^2 cr}\int_{-\infty}^{\infty} [e^{i\kappa(ct-r)} - e^{i\kappa(ct+r)}]\ d\kappa.$$

Or,

$$u(r,t) = \frac{1}{4\pi cr}[\delta(ct-r) - \delta(ct+r)]. \tag{2.18.35}$$

For $t>0,\ ct+r>0$ so that $\delta(ct+r)=0$ and hence,

$$u(\mathbf{r},t) = \frac{1}{4\pi cr}\,\delta(ct-r) = \frac{1}{4\pi c^2 r}\,\delta(t-\frac{r}{c}). \tag{2.18.36}$$

▯

***Example 2.18.4***
(*The Three-Dimensional Poisson Equation*). The solution of the Poisson equation

$$-\nabla^2 u = f(\mathbf{r}), \tag{2.18.37}$$

where $\mathbf{r}=(x,y,z)$ is given by

$$u(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\mathbf{r},\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}, \tag{2.18.38}$$

where the Green's function $G(\mathbf{r},\boldsymbol{\xi})$ of the operator, $-\nabla^2$, is

$$G(\mathbf{r},\boldsymbol{\xi}) = \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\boldsymbol{\xi}|}. \tag{2.18.39}$$

To obtain the fundamental solution, we need to solve the equation

$$-\nabla^2\ G(\mathbf{r},\boldsymbol{\xi}) = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta), \qquad \mathbf{r}\neq\boldsymbol{\xi}. \tag{2.18.40}$$

Application of the three-dimensional Fourier transform defined by (2.18.5) to (2.18.40) gives

$$\kappa^2\ \hat{G}(\boldsymbol{\kappa},\boldsymbol{\xi}) = \frac{1}{(2\pi)^{3/2}}\ \exp(-i\boldsymbol{\kappa}\cdot\boldsymbol{\xi}), \tag{2.18.41}$$

where $\hat{G}(\boldsymbol{\kappa},\boldsymbol{\xi})=\mathscr{F}\{G(\mathbf{r},\boldsymbol{\xi})\}$ and $\boldsymbol{\kappa}=(k,\ell,m)$.

The inverse Fourier transform gives the formal solution

$$G(\mathbf{r},\boldsymbol{\xi}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{i\boldsymbol{\kappa}\cdot(\mathbf{r}-\boldsymbol{\xi})\}\frac{d\boldsymbol{\kappa}}{\kappa^2}$$
$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(i\boldsymbol{\kappa}\cdot\mathbf{x})\frac{d\boldsymbol{\kappa}}{\kappa^2}, \tag{2.18.42}$$

where $\mathbf{x}=|\mathbf{r}-\boldsymbol{\xi}|$.

We evaluate this integral using polar coordinates in the $\boldsymbol{\kappa}$-space with the axis along the $\mathbf{x}$-axis. In terms of spherical polar coordinates $(\kappa,\theta,\phi)$ so that $\boldsymbol{\kappa}\cdot\mathbf{x}=\kappa R\cos\theta$ where $R=|\mathbf{x}|$. Thus, (2.18.42) becomes

$$G(\mathbf{r},\xi) = \frac{1}{(2\pi)^3}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\int_0^{\infty}\exp(i\kappa R\ \cos\theta)\,\kappa^2\sin\theta\cdot\frac{d\kappa}{\kappa^2}$$
$$= \frac{1}{(2\pi)^2}\int_0^{\infty}2\,\frac{\sin\,(\kappa R)}{\kappa R}\ d\kappa = \frac{1}{4\pi R} = \frac{1}{4\pi|\mathbf{r}-\boldsymbol{\xi}|}, \tag{2.18.43}$$

provided $R>0$.

In electrodynamics, the fundamental solution (2.18.43) has a well-known interpretation. Physically, it represents the potential at point $\mathbf{r}$ generated by the unit point charge distribution at point $\boldsymbol{\xi}$. This is what can be expected because $\delta(\mathbf{r}-\boldsymbol{\xi})$ is the charge density corresponding to a unit point charge at $\boldsymbol{\xi}$.

The solution of (2.18.37) is then given by

$$u(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\mathbf{r},\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi} = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(\boldsymbol{\xi})d\boldsymbol{\xi}}{|\mathbf{r}-\boldsymbol{\xi}|}. \tag{2.18.44}$$

The integrand in (2.18.44) consists of the given charge distribution $f(\mathbf{r})$ at $\mathbf{r}=\boldsymbol{\xi}$ and Green's function $G(\mathbf{r},\boldsymbol{\xi})$. Physically, $G(\mathbf{r},\boldsymbol{\xi})\ f(\boldsymbol{\xi})$ represents the resulting potentials due to elementary point charges, and the total potential due to a given charge distribution $f(\mathbf{r})$ is then obtained by the integral superposition of the resulting potentials. This is called the *principle of superposition.* ▯

***Example 2.18.5***

(*The Two-Dimensional Helmholtz Equation*). To find the fundamental solution of the two-dimensional Helmholtz equation

$$-\nabla^2 G + \alpha^2 G = \delta(x-\xi)\delta(y-\eta), \quad -\infty < x, y < \infty. \tag{2.18.45}$$

It is convenient to make the change of variables $x-\xi=x^*$, $y-\eta=y^*$. Consequently, (2.18.45) reduces to the form, dropping the asterisks,

$$G_{xx}+G_{yy}-\alpha^2 G=-\delta(x)\delta(y). \tag{2.18.46}$$

Application of the double Fourier transform $\hat{G}(\boldsymbol{\kappa})=\mathscr{F}\{G(x,y)\}$ to (2.18.46) gives

$$\hat{G}(\boldsymbol{\kappa})=\frac{1}{2\pi}\frac{1}{(\kappa^2+\alpha^2)}, \tag{2.18.47}$$

where $\boldsymbol{\kappa}=(k,\ell)$ and $\kappa^2=k^2+\ell^2$.

The inverse Fourier transform yields the solution

$$G(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\kappa^2+\alpha^2)^{-1}\exp(i\boldsymbol{\kappa}\cdot\mathbf{x})\,dk\,d\ell. \tag{2.18.48}$$

In terms of polar coordinates $(x,y)=r(\cos\theta,\sin\theta)$, $(k,\ell)=\rho(\cos\phi,\sin\phi)$, the integral solution (2.18.48) becomes

$$G(x,y)=\frac{1}{4\pi^2}\int_0^{\infty}\frac{\rho d\rho}{(\rho^2+\alpha^2)}\int_0^{2\pi}\exp\{ir\rho\ \cos(\phi-\theta)\}d\phi,$$

which is, replacing the second integral by $2\pi J_0(r\rho)$,

$$=\frac{1}{2\pi}\int_0^{\infty}\frac{\rho\ J_0(r\rho)d\rho}{(\rho^2+\alpha^2)}. \tag{2.18.49}$$

In terms of the original coordinates, the fundamental solution of (2.18.45) is given by

$$G(\mathbf{r},\boldsymbol{\xi})=\frac{1}{2\pi}\int_0^{\infty}\frac{\rho\ J_0\left[\rho\left\{(x-\xi)^2+(y-\eta)^2\right\}^{\frac{1}{2}}\right]d\rho}{(\rho^2+\alpha^2)}. \tag{2.18.50}$$

Accordingly, the solution of the inhomogeneous equation

$$(\nabla^2-\alpha^2)u=-f(x,y) \tag{2.18.51}$$

is

$$u(x,y)=\int\int_{-\infty}^{\infty}G(\mathbf{r},\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}, \tag{2.18.52}$$

where $G(\mathbf{r},\boldsymbol{\xi})$ is given by (2.18.50).

Since the integral solution (2.18.49) does not exist for $\alpha = 0$, Green's function for the two-dimensional Poisson equation (2.18.51) cannot be derived from (2.18.49). Instead, we differentiate (2.18.49) with respect to $r$ to obtain

$$\frac{\partial G}{\partial r} = \frac{1}{2\pi} \int_0^\infty \frac{\rho^2 J_0'(r\rho)d\rho}{(\rho^2 + \alpha^2)}$$

which is, for $\alpha = 0$,

$$\frac{\partial G}{\partial r} = \frac{1}{2\pi} \int_0^\infty J_0'(r\rho)d\rho = -\frac{1}{2\pi r}.$$

Integrating this result gives

$$G(r, \theta) = -\frac{1}{2\pi} \log r.$$

In terms of the original coordinates, the Green's function becomes

$$G(\mathbf{r}, \boldsymbol{\xi}) = -\frac{1}{4\pi} \log[(x-\xi)^2 + (y-\eta)^2]. \tag{2.18.53}$$

This is Green's function for the two-dimensional Poisson equation $\nabla^2 = -f(x, y)$. Thus, the solution of the Poisson equation is

$$u(x, y) = \int_{-\infty}^\infty \int_{-\infty}^\infty G(\mathbf{r}, \boldsymbol{\xi}) f(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{2.18.54}$$

□

***Example 2.18.6***

(*Diffusion of Vorticity from a Vortex Sheet*). We solve the two-dimensional vorticity equation in the $x, y$ plane given by

$$\zeta_t = \nu \nabla^2 \zeta \tag{2.18.55}$$

with the initial condition

$$\zeta(x, y, 0) = \zeta_0(x, y), \tag{2.18.56}$$

where $\zeta = v_x - u_y$.

Application of the double Fourier transform defined by

$$\hat{\zeta}(k, \ell, t) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \exp[-i(kx + \ell y)]\zeta(x, y, t)dx\, dy$$

to (2.18.55)–(2.18.56) gives

$$\frac{d\hat{\zeta}}{dt} = -\nu(k^2+\ell^2)\hat{\zeta},$$
$$\hat{\zeta}(k,\ell,0) = \hat{\zeta}_0(k,\ell).$$

Thus, the solution of the transformed system is

$$\hat{\zeta}(k,\ell,t) = \hat{\zeta}_0(k,\ell)\exp[-\nu(k^2+\ell^2)t]. \tag{2.18.57}$$

The inversion theorem for Fourier transform gives the formal solution

$$\zeta(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\zeta}_0(k,\ell)\exp[i(\kappa\cdot\mathbf{r})-\nu\kappa^2 t]\,dk\,d\ell, \tag{2.18.58}$$

where $\boldsymbol{\kappa} = (k,\ell)$ and $\kappa^2 = k^2+\ell^2$.

In particular, if $\zeta_0(x,y) = V\delta(x)$ represents a vortex sheet of constant strength $V$ per unit width in the plane $x=0$, we find $\hat{\zeta}_0(k,\ell) = V\delta(\ell)$ and hence,

$$\begin{aligned}\zeta(x,y,t) &= \frac{V}{2\pi}\int_{-\infty}^{\infty}\exp\{ikx-\nu k^2 t\}dk \\ &= \frac{V}{2\sqrt{\pi\nu t}}\exp\left(-\frac{x^2}{4\nu t}\right).\end{aligned} \tag{2.18.59}$$

Apart from a constant, the velocity field is given by

$$u(x,t)=0, \quad v(x,t) = \frac{V}{\sqrt{\pi}}\,erf\left(\frac{x}{2\sqrt{\nu t}}\right). \tag{2.18.60}$$

$\square$

## 2.19 Exercises

1. Find the Fourier transforms of each of the following functions:

(a) $f(x)=\dfrac{1}{1+x^2}$,

(b) $f(x)=\dfrac{x}{1+x^2}$,

(c) $f(x)=\delta^{(n)}(x)$,

(d) $f(x)=x\,\exp(-a|x|),\ a>0$,

(e) $f(x)=e^x\,\exp(-e^x)$,

(f) $f(x)=x\,\exp\left(-\dfrac{ax^2}{2}\right),\ a>0$,

(g) $f(x)=x^2\,\exp\left(-\dfrac{1}{2}x^2\right)$,

(h) $f(x)=\left\{\begin{array}{ll} 1-|x|, & |x|\le 1 \\ 0, & |x|>1 \end{array}\right\}$,

(i) $f(x)=\left\{\begin{array}{ll} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{array}\right\}$,

(j) $h_n(x)=(-1)^n\,\exp\left(\dfrac{1}{2}x^2\right) \times \left(\dfrac{d}{dx}\right)^n \exp(-x^2)$,

(k) $f(x)=\chi_{[a,b]}(x)\,e^{i\alpha x}$,

(l) $f(x)=\begin{matrix} cos \\ sin \end{matrix}\ (ax^2)$.

2. Show that

(a) $\mathscr{F}\{\delta(x-ct)+\delta(x+ct)\}=\sqrt{\dfrac{2}{\pi}}\,\cos(kct)$,

(b) $\mathscr{F}\{H(ct-|x|)\}=\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin kct}{k}$,

(c) $\mathscr{F}\left\{f\left(\dfrac{x}{a}+b\right)\right\}=a\,\exp(iabk)\,F(ak)$,

(d) $\mathscr{F}\{e^{ibx}f(ax)\}=\dfrac{1}{a}\,F\left(\dfrac{k+b}{a}\right)$.

3. Show that

(a) $i\dfrac{d}{dk}F(k)=\mathscr{F}\{x\ f(x)\}$,

(b) $i^n\dfrac{d^n}{dk^n}F(k)=\mathscr{F}\{x^n\ f(x)\}$.

4. Use exercise 3(b) to find the Fourier transform of $f(x)=x^2\,\exp(-ax^2)$.

5. Prove the following:

(a) $\mathscr{F}\left\{(a^2-x^2)^{-\frac{1}{2}}H(a-|x|)\right\}=\sqrt{\dfrac{\pi}{2}}\,J_0(ak),\ a>0$.

(b) $\mathscr{F}\{P_n(x)\ H(1-|x|)\} = (-i)^n \dfrac{1}{\sqrt{k}}\ J_{n+\frac{1}{2}}(k),$

where $P_n(x)$ is the Legendre polynomial of degree $n$.

(c) If $f(x)$ has a finite discontinuity at a point $x = a$, then

$$\mathscr{F}\{f'(x)\} = (ik)\ F(k) - \frac{1}{\sqrt{2\pi}}\exp(-ika)[f]_a,$$

where $[f]_a = f(a+0) - f(a-0)$.

Generalize this result for $\mathscr{F}\{f^{(n)}(x)\}$.

6. Find the convolution $(f * g)(x)$ if

(a) $f(x) = e^{ax}, \quad g(x) = \chi_{[0,\infty]}(x),\ a \neq 0,$

(b) $f(x) = \sin bx, \quad g(x) = \exp(-a|x|),\ a > 0,$

(c) $f(x) = \chi_{[a,b]}(x), \quad g(x) = x^2,$

(d) $f(x) = \exp(-x^2), \quad g(x) = \exp(-x^2).$

7. Prove the following results for the convolution:

(a) $\delta(x) * f(x) = f(x),$ (b) $\delta'(x) * f(x) = f'(x),$

(c) $\dfrac{d}{dx}\{f(x) * g(x)\} = f'(x) * g(x) = f(x) * g'(x),$

(d) $\displaystyle\int_{-\infty}^{\infty} (f * g)(x)dx = \int_{-\infty}^{\infty} f(u)du \int_{-\infty}^{\infty} g(v)dv\ ,$

(e) $\dfrac{d^2}{dx^2}\ (f * g)\ (x) = (f' * g')\ (x) = (f'' * g)\ (x),$

(f) $(f * g)^{(n+l)}\ (x) = f^{(n)}\ (x) * g^{(l)}\ (x),$

(g) If $f$ and $g$ are both even or both odd, then $(f * g)(x)$ is even,

(h) If $f$ is even or $g$ is odd, or vice versa, then $(f * g)(x)$ is odd,

(i) If $g(x) = \dfrac{1}{2a}\ H(a-|x|)$, then $(f * g)(x)$ is the average of the function $f(x)$ in $[x-a,\ x+a]$,

(j) If $G_t(x) = \dfrac{1}{\sqrt{4\pi kt}} \displaystyle\int_{-\infty}^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4kt}\right]\ d\xi,$

then $G_t(x) * G_s(x) = G_{t+s}(x).$

8. Use the Fourier transform to solve the following ordinary differential equations in $-\infty < x < \infty$:

(a) $y''(x) - y(x) + 2f(x) = 0$, where $f(x) = 0$ when $x < -a$ and when $x > a$, and $y(x)$ and its derivatives vanish at $x = \pm\infty$,

(b) $2y''(x) + xy'(x) + y(x) = 0$, (c) $y''(x) + x\, y'(x) + y(x) = 0$,
(d) $y''(x) + x\, y'(x) + x\, y(x) = 0$, (e) $\ddot{y}(t) + 2\alpha\, \dot{y}(t) + \omega^2 y(t) = f(t)$.

9. Solve the following integral equations for an unknown function $f(x)$:

(a) $\displaystyle\int_{-\infty}^{\infty} \phi(x-t) f(t) dt = g(x)$.

(b) $\displaystyle\int_{-\infty}^{\infty} \exp(-at^2) f(x-t) dt = \exp(-bx^2),\ a > b > 0$.

(c) $\displaystyle\int_{-\infty}^{\infty} f(x-t) f(t) dt = \frac{b}{(x^2+b^2)}$.

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{f(t)dt}{(x-t^2)+a^2} = \frac{\sqrt{2\pi}}{(x^2+b^2)}$ for $b > a > 0$.

(e) $\displaystyle\frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(t)dt}{x-t} = \phi(x)$,

where the integral in (e) is treated as the Cauchy Principal value.

10. Solve the Cauchy problem for the Klein-Gordon equation

$$u_{tt} - c^2\ u_{xx} + a^2\ u = 0,\ -\infty < x < \infty,\ t > 0.$$
$$u(x,0) = f(x),\ \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) \text{ for } -\infty < x < \infty.$$

11. Solve the telegraph equation

$$u_{tt} - c^2\ u_{xx} + u_t - a u_x = 0, \quad -\infty < x < \infty,\ t > 0.$$
$$u(x,0) = f(x),\ \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) \quad \text{for} \quad -\infty < x < \infty.$$

Show that the solution is unstable when $c^2 < a^2$. If $c^2 > a^2$, show that the bounded integral solution is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \exp[-k^2(c^2-a^2)t + ik(x+at)] dk$$

where $A(k)$ is given in terms of the transformed functions of the initial data. Hence, deduce the asymptotic solution as $t \to \infty$ in the form

$$u(x,t) = A(0) \sqrt{\frac{\pi}{2(c^2-a^2)t}}\ \exp\left[-\frac{(x+at)^2}{4(c^2-a^2)t}\right].$$

12. Solve the equation

$$u_{tt} + u_{xxxx} = 0, \quad -\infty < x < \infty, \; t > 0$$
$$u(x,0) = f(x), \; u_t(x,0) = 0 \quad \text{for} \quad -\infty < x < \infty.$$

13. Find the solution of the dissipative wave equation

$$u_{tt} - c^2 \, u_{xx} + \alpha u_t = 0, \; -\infty < x < \infty, \; t > 0,$$
$$u(x,0) = f(x), \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) \quad \text{for} \quad -\infty < x < \infty,$$

where $\alpha > 0$ is the dissipation parameter.

14. Obtain the Fourier cosine transforms of the following functions:
(a) $f(x) = x \, \exp(-ax), \; a > 0,$ (b) $f(x) = e^{-ax} \, \cos \, x, \; a > 0,$
(c) $f(x) = \dfrac{1}{x},$ (d) $K_0(ax),$
where $K_0(ax)$ is the *modified Bessel function.*

15. Find the Fourier sine transform of the following functions:
(a) $f(x) = x \, \exp(-ax), \; a > 0,$ (b) $f(x) = \dfrac{1}{x} \exp(-ax), \; a > 0,$
(c) $f(x) = \dfrac{1}{x},$ (d) $f(x) = \dfrac{x}{a^2 + x^2}.$

16. (a) If $F(k) = \mathscr{F}\{\exp(-ax^2)\}, \; a > 0,$ show that $F(k)$ satisfies the differential equation

$$2a \, \frac{dF}{dk} + k \, F(k) = 0 \; \text{ with } \; F(0) = \frac{1}{\sqrt{2a}}.$$

(b) If $F_c(k) = \mathscr{F}_c\{\exp(-ax^2)\}$, show that $F_c(k)$ satisfies the equation

$$\frac{dF_c}{dk} + \left(\frac{k}{2a}\right) F_c = 0 \; \text{ with } \; F_c(0) = 1.$$

17. Prove the following for the Fourier sine transform

(a) $$\int_0^\infty F_s(k) G_c(k) \sin kx \; dk = \frac{1}{2} \int_0^\infty g(\xi)[f(\xi + x) - f(\xi - x)] d\xi,$$

(b) $$\int_0^\infty F_c(k) G_s(k) \sin kx \; dk = \frac{1}{2} \int_0^\infty f(\xi)[g(\xi + x) - g(\xi - x)] d\xi.$$

18. Solve the integral equation

$$\int_0^\infty f(x) \sin kx \; dk = \left\{ \begin{array}{cc} 1 - k, & 0 \le k < 1 \\ 0, & k > 1 \end{array} \right\}.$$

19. Solve Example 2.15.1 with the boundary data

$$u(0,t)=0, \ u(x,t)\to 0 \quad \text{as} \quad x\to\infty, \quad \text{for} \quad t>0.$$

20. Apply the Fourier cosine transform to find the solution $u(x,y)$ of the problem

$$u_{xx}+u_{yy}=0, \ 0<x<\infty, \ 0<y<\infty,$$
$$u(x,0)=H(a-x), a>x; \ u_x(0,y)=0, \ 0<x,y<\infty.$$

21. Use the Fourier cosine (or sine) transform to solve the following integral equation:

(a) $\displaystyle\int_0^\infty f(x)\cos kx \, dx=\sqrt{\frac{\pi}{2k}},$ (b) $\displaystyle\int_0^\infty f(x)\sin kx \, dx=\frac{a}{a^2+k^2},$

(c) $\displaystyle\int_0^\infty f(x)\sin kx dx=\frac{\pi}{2}\, J_0(ak),$ (d) $\displaystyle\int_0^\infty f(x)\cos kx \, dx=\frac{\sin ak}{k}.$

22. Solve the diffusion equation in the semi-infinite line

$$u_t=\kappa\, u_{xx}, \ 0\le x<\infty, \ t>0,$$

with the boundary and initial data

$$u(0,t)=0 \quad \text{for} \quad t>0,$$
$$u(x,t)\to 0 \quad \text{as} \quad x\to\infty \quad \text{for} \quad t>0,$$
$$u(x,0)=f(x) \quad \text{for} \quad 0<x<\infty.$$

23. Use the Parseval formula to evaluate the following integrals with $a>0$ and $b>0$:

(a) $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^2},$ (c) $\displaystyle\int_{-\infty}^{\infty}\frac{\sin^2 ax}{x^2}dx,$

(b) $\displaystyle\int_{-\infty}^{\infty}\frac{\sin ax}{x(x^2+b^2)}dx$ (d) $\displaystyle\int_{-\infty}^{\infty}\frac{\exp(-bx^2)dx}{(x^2+a^2)}.$

24. Show that

$$\int_0^\infty \frac{\sin ax \sin bx}{x^2}dx=\frac{\pi}{2}\min(a,b).$$

25. If $f(x)=\exp(-ax)$ and $g(x)=H(t-x)$, show that

$$\int_0^\infty \frac{\sin tx}{x(x^2+a^2)}dx = \frac{\pi}{2a^2}[1-\exp(-at)].$$

26. Use the Poisson summation formula to find the sum of each of the following series with non-zero $a$:

(a) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(1+n^2a^2)}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin an}{n}$,

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 an}{n^2}$, (d) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{a}{n^2+a^2}$.

27. The Fokker-Planck equation (Reif, 1965) is used to describe the evolution of probability distribution functions $u(x,t)$ in nonequilibrium statistical mechanics and has the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+x\right)u.$$

The fundamental solution of this equation is defined by the equation

$$\left[\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+x\right)\right] G(x,\xi;t,\tau) = \delta(x-\xi)\delta(t-\tau).$$

Show that the fundamental solution is

$$G(x,\xi;t,\tau) = [2\pi\{1-\exp[-2(t-\tau)]\}]^{-\frac{1}{2}} \exp\left[-\frac{\{x-\xi\,\exp[-(t-\tau)]\}^2}{2[1-\exp\{-2(t-\tau)\}]}\right].$$

Hence, derive

$$\lim_{t\to\infty} G(x,\xi;t,\tau) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right).$$

With the initial condition $u(x,0)=f(x)$, show that the function $u(x,t)$ tends to the normal distribution as $t\to\infty$, that is,

$$\lim_{t\to\infty} u(x,t) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)\int_{-\infty}^{\infty} f(\xi)d\xi.$$

28. The transverse vibration of an infinite elastic beam of mass $m$ per unit length and the bending stiffness $EI$ is governed by

$$u_{tt} + a^2\, u_{xxxx} = 0, \quad \left(a^2 = \frac{EI}{m}\right), \; -\infty < x < \infty, \; t > 0.$$

Solve this equation subject to the boundary and initial data

$$u(0,t)=0 \quad \text{for all} \quad t>0,$$
$$u(x,0)=\phi(x), \; u_t(x,0)=\psi''(x) \quad \text{for} \quad 0<x<\infty.$$

Show that the Fourier transform solution is

$$U(k,t)=\mathbf{\Phi}(k)\cos(atk^2)-\mathbf{\Psi}(k)\sin(atk^2).$$

Find the integral solution for $u(x,t)$.

29. Solve the Lamb (1904) problem in geophysics that satisfies the Helmholtz equation in an infinite elastic half-space

$$u_{xx}+u_{zz}+\frac{\omega^2}{{c_2}^2}u=0, \; -\infty<x<\infty, \; z>0,$$

where $\omega$ is the frequency and $c_2$ is the shear wave speed.

At the surface of the half-space ($z=0$), the boundary condition relating the surface stress to the impulsive point load distribution is

$$\mu\frac{\partial u}{\partial z}=-P\delta(x) \quad \text{at} \quad z=0,$$

where $\mu$ is one of the Lamé's constants, $P$ is a constant and

$$u(x,z)\to 0 \quad \text{as} \quad z\to\infty \quad \text{for} \quad -\infty<x<\infty.$$

Show that the solution in terms of polar coordinates is

$$\begin{aligned} u(x,z) &= \frac{P}{2i\mu}{H_0}^{(2)}\left(\frac{\omega r}{c_2}\right) \\ &\sim \frac{P}{2i\mu}\left(\frac{2c_2}{\pi\omega r}\right)^{\frac{1}{2}}\exp\left(\frac{\pi i}{4}-\frac{i\omega r}{c_2}\right) \quad \text{for} \quad \omega r>>c_2. \end{aligned}$$

30. Find the solution of the Cauchy-Poisson problem (Debnath, 1994, p. 83) in an inviscid water of infinite depth which is governed by

$$\phi_{xx}+\phi_{zz}=0, \; -\infty<x<\infty, \; -\infty<z\le 0, \; t>0,$$
$$\left.\begin{aligned} \phi_z-\eta_t &= 0 \\ \phi_t+g\eta &= 0 \end{aligned}\right\} \text{ on } z=0, \; t>0,$$
$$\phi_z\to 0 \quad \text{as} \quad z\to-\infty,$$
$$\phi(x,0,0)=0 \quad \text{and} \quad \eta(x,0)=P\delta(x),$$

where $\phi=\phi(x,z,t)$ is the velocity potential, $\eta(x,t)$ is the free surface elevation, and $P$ is a constant.

Derive the asymptotic solution for the free surface elevation in the limit as $t\to\infty$.

31. Obtain the solutions for the velocity potential $\phi(x,z,t)$ and the free surface elevation $\eta(x,t)$ involved in the two-dimensional surface waves in water of finite (or infinite) depth $h$. The governing equation, boundary, and free surface conditions and initial conditions (see Debnath 1994, p. 92) are

$$\phi_{xx} + \phi_{zz} = 0, \ -h \le z \le 0, \ -\infty < x < \infty, \ t > 0$$
$$\left.\begin{aligned} \phi_t + g\eta &= -\frac{P}{\rho} p(x) \exp(i\,\omega\, t) \\ \phi_z - \eta_t &= 0 \end{aligned}\right\} z = 0, \ t > 0$$
$$\phi(x,z,0) = 0 = \eta(x,0) \text{ for all } x \text{ and } z.$$

32. Solve the steady-state surface wave problem (Debnath, 1994, p. 47) on a running stream of infinite depth due to an external steady pressure applied to the free surface. The governing equation and the free surface conditions are

$$\phi_{xx} + \phi_{zz} = 0, \ -\infty < x < \infty, \ -\infty < z < 0, \ t > 0,$$
$$\left.\begin{aligned} \phi_x + U\phi_x + g\eta &= -\frac{P}{\rho}\,\delta(x) \exp(\epsilon t) \\ \eta_t + U\eta_x &= \phi_z \end{aligned}\right\} z = 0, \ (\epsilon > 0),$$
$$\phi_z \to 0 \quad \text{as} \quad z \to -\infty.$$

where $U$ is the stream velocity, $\phi(x,z,t)$ is the velocity potential, and $\eta(x,t)$ is the free surface elevation.

33. Use the Fourier sine transform to solve the following initial and boundary value problem for the wave equation:

$$u_{tt} = c^2 u_{xx}, \ 0 < x < \infty, \ t > 0,$$
$$u(x,0) = 0, \ u_t(x,0) = 0 \quad \text{for} \quad 0 < x < \infty,$$
$$u(0,t) = f(t) \quad \text{for} \quad t > 0,$$

where $f(t)$ is a given function.

34. Solve the following initial and boundary value problem for the wave equation using the Fourier cosine transform:

$$u_{tt} = c^2 u_{xx}, \ 0 < x < \infty, \ t > 0,$$
$$u(0,t) = f(t) \quad \text{for} \quad t > 0,$$
$$u(x,0) = 0, u_t(x,0) = 0 \quad \text{for} \quad 0 < x < \infty,$$

where $f(t)$ is a known function.

35. Apply the Fourier transform to solve the initial value problem for the dissipative wave equation

$$u_{tt} = c^2 u_{xx} + \alpha\, u_{xxt}, \;\; -\infty < x < \infty, \;\; t > 0,$$
$$u(x,0) = f(x), \;\; u_t(x,0) = \alpha f''(x) \quad \text{for} \quad -\infty < x < \infty,$$

where $\alpha$ is a positive constant.

36. Use the Fourier sine transform to solve the initial and boundary value problem for free vibrations of a semi-infinite string:

$$u_{tt} = c^2 u_{xx}, \;\; 0 < x < \infty, \;\; t > 0,$$
$$u(0,t) = 0, \;\; t \geq 0,$$
$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x) \quad \text{for} \quad 0 < x < \infty.$$

37. The static deflection $u(x,y)$ in a thin elastic disk in the form of a quadrant satisfies the boundary value problem

$$u_{xxxx} + 2\, u_{xxyy} + u_{yyyy} = 0, \;\; 0 < x < \infty, \;\; 0 < y < \infty,$$
$$u(0,y) = u_{xx}(0,y) = 0 \quad \text{for} \quad 0 < y < \infty,$$
$$u(x,0) = \frac{ax}{1+x^2}, \;\; u_{yy}(x,0) = 0 \quad \text{for} \quad 0 < x < \infty,$$

where $a$ is a constant, and $u(x,y)$ and its derivatives vanish as $x \to \infty$ and $y \to \infty$.

Use the Fourier sine transform to show that

$$u(x,y) = \frac{a}{2} \int_0^\infty (2+ky)\exp[-(1+y)k] \sin kx \; dx$$
$$= \frac{ax}{x^2+(1+y)^2} + \frac{axy(1+y)}{[x^2+(1+y)^2]^2}$$

38. In exercise 37, replace the conditions on $y=0$ with the conditions

$$u(x,0) = 0, \;\; u_{yy}(x,0) = \frac{ax}{(1+x^2)^2} \quad \text{for} \quad 0 < x < \infty.$$

Show that the solution is

$$u(x,y) = -\frac{ax}{4} \int_0^\infty \exp[-(1+y)k] \sin kx \; dk$$
$$= -\frac{1}{4} \frac{axy}{[x^2+(1+y)^2]}.$$

39. In exercise 37, solve the biharmonic equation in $0<x<\infty,\ 0<y<b$ with the boundary conditions

$$u(0,y) = a\sin y,\ u_{xx}(0,y)=0 \quad \text{for} \quad 0<y<b,$$
$$u(x,0) = u_{yy}(x,0)=u(x,b)=u_{yy}(x,b)=0 \quad \text{for} \quad 0<x<\infty,$$

and $u(x,y),\ u_x(x,y)$ vanish as $x\to\infty$.

40. Use the Fourier transform to solve the boundary value problem

$$u_{xx}+u_{yy}=-x\ \exp(-x^2),\ -\infty<x<\infty,\ 0<y<\infty,$$

$u(x,0)=0$, for $-\infty<x<\infty$, $u$ and its derivative vanish as $y\to\infty$. Show that

$$u(x,y)=\frac{1}{\sqrt{4\pi}}\int_0^\infty [1-\exp(-ky)]\frac{\sin kx}{k}\ \exp\left(-\frac{k^2}{4}\right)dk.$$

41. Using the definition of the characteristic function for the discrete random variable $X$

$$\phi(t)=E[\exp(itX)]=\sum_r p_r\ \exp(itx_r)$$

where $p_r=P(X=x_r)$, show that the characteristic function of the binomial distribution

$$p_r=\binom{n}{r}p^r(1-p)^{n-r}$$

is

$$\phi(t)=[1+p(e^{it}-1)]^n.$$

Find the moments.

42. Show that the characteristic function of the Poisson distribution

$$p_r=P(X=r)=\frac{\lambda^r}{r!}\,e^{-\lambda},\ r=0,1,2,\ldots$$

is

$$\phi(t)=\exp[\lambda(e^{it}-1)].$$

Find the moments.

43. Find the characteristic function of

(a) The gamma distribution whose density function is

$$f(x)=\frac{a^p}{\Gamma(p)}\,x^{p-1}e^{-ax}H(x),$$

(b) The beta distribution whose density function is

$$f(x) = \left\{ \begin{array}{lll} \dfrac{x^{p-1}(1-x)^{q-1}}{B(p,q)} & \text{for} & 0 < x < 1, \\ 0 & \text{for} & x < 0 \text{ and } \quad x > 1 \end{array} \right\},$$

(c) The Cauchy distribution whose density function is

$$f(x) = \frac{1}{\pi} \frac{\lambda}{[\lambda^2 + (x-\mu)^2]},$$

(d) The Laplace distribution whose density function is

$$f(x) = \frac{1}{2\lambda} \exp\left(-\frac{|x-u|}{\lambda}\right), \qquad \lambda > 0.$$

44. Find the density function of the random variable $X$ whose characteristic function is

$$\phi(t) = (1 - |t|) H(1 - |t|).$$

45. Find the characteristic function of uniform distribution whose density function is

$$f(x) = \left\{ \begin{array}{ll} 0, & x < 0 \\ 1, & 0 \le x \le a \\ 0, & x > a \end{array} \right\}.$$

46. Solve the *initial value problem* (Debnath, 1994, p. 115) for the two-dimensional surface waves at the free surface of a running stream of velocity $U$. The problem satisfies the equation, boundary, and initial conditions

$$\phi_{xx} + \phi_{zz} = 0, \;\; -\infty < x < \infty, \;\; -h \le z \le 0, \quad t > 0,$$

$$\left. \begin{array}{l} \phi_x + U\phi_x + g\eta = -\dfrac{P}{\rho}\, \delta(x) \exp(i\,\omega\, t) \\ \eta_t + U\eta_x - \phi_z = 0 \end{array} \right\} \text{ on } z = 0, \;\; t > 0,$$

$$\phi(x, z, 0) = \eta(x, 0) = 0, \quad \text{for all } x \text{ and } z.$$

47. Apply the Fourier tranform to solve the equation

$$u_{xxxx} + u_{yy} = 0, \;\; -\infty < x < \infty, \;\; y \ge 0,$$

satisfying the conditions

$$u(x, 0) = f(x), \;\; u_y(x, 0) = 0 \quad \text{for} \quad -\infty < x < \infty,$$

$u(x, y)$ and its partial derivatives vanish as $|x| \to \infty$.

48. The transverse vibration of a thin membrane of great extent satisfies the wave equation

$$c^2(u_{xx}+u_{yy})=u_{tt},\ \ -\infty<x,y<\infty,\ \ t>0,$$

with the initial and boundary conditions

$$\begin{aligned}&u(x,y,t)\to 0 \quad \text{as} \quad |x|\to\infty,\ |y|\to\infty \quad \text{for all} \quad t\geq 0,\\ &u(x,y,0)=f(x,y),\ u_t(x,y,0)=0 \quad \text{for all} \quad x,y.\end{aligned}$$

Apply the double Fourier transform method to solve this problem.

49. Solve the diffusion problem with a source $q(x,t)$

$$\begin{aligned}&u_t=\kappa\ u_{xx}+q(x,t),\ \ -\infty<x<\infty,\ \ t>0,\\ &u(x,0)=0 \quad \text{for} \quad -\infty<x<\infty.\end{aligned}$$

Show that the solution is

$$u(x,t)=\frac{1}{\sqrt{4\pi\,t\,\kappa}}\int_0^t(t-\tau)^{-\frac{1}{2}}\,d\tau\int_{-\infty}^{\infty}q(k,\tau)\exp\left[-\frac{(x-k)^2}{4\kappa(t-\tau)}\right]dk.$$

50. The function $u(x,t)$ satisfies the diffusion problem in a half-line

$$\begin{aligned}&u_t=\kappa\,u_{xx}+q(x,t),\ \ 0\leq x<\infty,\ \ t>0,\\ &u(x,0)=0,\ u(0,t)=0 \quad \text{for} \quad x\geq 0 \quad \text{and} \quad t>0.\end{aligned}$$

Show that

$$u(x,t)=\sqrt{\frac{2}{\pi}}\int_0^t d\tau\int_0^{\infty}Q_s(k,\tau)\exp[-\kappa k^2(t-\tau)]\sin kx\ dk,$$

where $Q_s(k,t)$ is the Fourier sine transform of $q(x,t)$.

51. Apply the triple Fourier transform to solve the initial value problem

$$\begin{aligned}&u_t=\kappa(u_{xx}+u_{yy}+u_{zz}),\ \ -\infty<x,\ y,\ z<\infty,\ \ t>0,\\ &u(\mathbf{x},0)=f(\mathbf{x}) \quad \text{for all} \quad x,y,z,\end{aligned}$$

where $\mathbf{x}=(x,y,z)$.

52. Use the Fourier transform with respect to $t$ and Laplace transform with respect to $x$ to solve the telegraph equation

$$\begin{aligned}&u_{tt}+a\ u_t+bu=c^2\ u_{xx}, \quad 0<x<\infty,\ -\infty<t<\infty,\\ &u(0,t)=f(t),\ u_x(0,t)=g(t), \quad \text{for} \quad -\infty<t<\infty,\end{aligned}$$

where $a,\ b,\ c$ are constants and $f(t)$ and $g(t)$ are arbitrary functions of time $t$.

53. Determine the steady-state temperature distribution in a disk occupying the semi-infinite strip $0 < x < \infty,\ 0 < y < 1$ if the edges $x = 0$ and $y = 0$ are insulated, and the edge $y = 1$ is kept at a constant temperature $T_0\ H(a - x)$. Assuming that the disk loses heat due to its surroundings according to Newton's law with proportionality constant $h$, solve the boundary value problem

$$\begin{aligned}&u_{xx} + u_{yy} - hu = 0,\ 0 < x < \infty,\ 0 < y < 1,\\ &u(x, 1) = T_0\ H(a - x), \quad \text{for} \quad 0 < x < \infty,\\ &u_x(0, y) = 0 = u_y(x, 0) \quad \text{for} \quad 0 < x < \infty,\ 0 < y < 1.\end{aligned}$$

54. Use the double Fourier transform to solve the following equations:

(a) $u_{xxxx} - u_{yy} + 2u = f(x, y)$,
(b) $u_{xx} + 2u_{yy} + 3u_x - 4u = f(x, y)$,

where $f(x, y)$ is a given function.

55. Use the Fourier transform to solve the Rossby wave problem in an inviscid $\beta$-plane ocean bounded by walls at $y = 0$ and $y = 1$ where $y$ and $x$ represent vertical and horizontal directions. The fluid is initially at rest and then, at $t = 0+$, an arbitrary disturbance localized to the vicinity of $x = 0$ is applied to generate Rossby waves. This problem satisfies the Rossby wave equation

$$\frac{\partial}{\partial t}[(\nabla^2 - \kappa^2)\psi] + \beta\psi_x = 0,\ -\infty < x < \infty,\ 0 \le y \le 1,\ t > 0,$$

with the boundary and initial conditions

$$\begin{aligned}&\psi_x(x, y) = 0 \quad \text{for} \quad 0 < x < \infty,\ y = 0 \quad \text{and} \quad y = 1,\\ &\psi(x, y, t) = \psi_0(x, y) \text{ at } t = 0 \quad \text{for all } x \text{ and } y.\end{aligned}$$

56. Find the transfer function and the corresponding impulse response function of the input and output of the $RC$ circuit governed by the equation

$$R\,\frac{dq}{dt} + \frac{1}{C}\,q\,(t) = e\,(t)\,,$$

where $R$, $C$ are constants, $q\,(t)$ is the electric charge and $e\,(t)$ is the given voltage.

57. Prove the Poisson summation formula for the Fourier cosine transform $\mathscr{F}_c\,\{f(x)\} = F_c(k)$ in the form

$$\sqrt{a}\left[\frac{1}{2}f(0) + \sum_{n=1}^{\infty} f(na)\right] = \sqrt{b}\left[\frac{1}{2}F_c(0) + \sum_{n=1}^{\infty} F_c(nb)\right],$$

where $ab = 2\pi$ and $a > 0$.

Apply this formula to the following examples:

(a) $f(x) = e^{-x}$, $\qquad F_c(k) = \sqrt{\frac{2}{\pi}}\,(1+k^2)^{-1}$,

(b) $f(x) = \exp(-\frac{1}{2}x^2)$, $\qquad F_c(k) = \exp(-\frac{1}{2}k^2)$,

(c) $f(x) = \exp(-\frac{1}{2}x^2)\cos\,\alpha\, x$, $\qquad F_c(k) = \exp\left[-\frac{1}{2}(\alpha^2+k^2)\right]\cosh(k\alpha)$,

(d) $f(x) = \begin{cases} \frac{2^{\frac{1}{2}-\nu}}{\Gamma(\nu+\frac{1}{2})}(1-x^2)^{\nu-\frac{1}{2}}, & 0 \le x < 1, \\ 0, & x \ge 1. \end{cases}$

$$F_c(k) = k^{-\nu} J_\nu(k), \quad k > 0; \quad F_c(0) = \frac{1}{2^\nu \Gamma(\nu+1)}.$$

# 3

## *Laplace Transforms and Their Basic Properties*

> "What we know is not much. What we do not know is immense."
>
> Pierre-Simon Laplace

> "The algebraic analysis soon makes us forget the main object [of our research] by focusing our attention on abstract combinations and it is only at the end that we return to the original objective. But in abandoning oneself to the operations of analysis, one is led to the generality of this method and the inestimable advantage of transforming the reasoning by mechanical procedures to results often inaccessible by geometry....No other language has the capacity for the elegance that arises from a long sequence of expressions linked one to the other and all stemming from one fundamental idea."
>
> Pierre-Simon Laplace

> "... For Laplace, on the contrary, mathematical analysis was an instrument that he bent to his purposes for the most varied applications, but always subordinating the method itself to the content of each question. Perhaps posterity will...."
>
> Simeon-Denis Poisson

### 3.1 Introduction

In this chapter, we present the formal definition of the Laplace transform and calculate the Laplace transforms of some elementary functions directly from the definition. The existence conditions for the Laplace transform are stated in Section 3.3. The basic operational properties of the Laplace transforms including convolution and its properties, and the differentiation and integration of Laplace transforms are discussed in some detail. The inverse Laplace transform is introduced in Section 3.7, and four methods of evaluation of the inverse transform are developed with examples. The Heaviside Expansion Theorem and the Tauberian theorems for the Laplace transform are discussed.

## 3.2 Definition of the Laplace Transform and Examples

We start with the *Fourier Integral Formula* (2.2.4), which expresses the representation of a function $f_1(x)$ defined on $-\infty < x < \infty$ in the form

$$f_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \int_{-\infty}^{\infty} e^{-ikt} f_1(t) dt. \tag{3.2.1}$$

We next set $f_1(x) \equiv 0$ in $-\infty < x < 0$ and write

$$f_1(x) = e^{-cx} f(x) H(x) = e^{-cx} f(x), \qquad x > 0, \tag{3.2.2}$$

where $c$ is a positive fixed number, so that (3.2.1) becomes

$$f(x) = \frac{e^{cx}}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \int_{0}^{\infty} \exp\{-t(c+ik)\} f(t) dt. \tag{3.2.3}$$

With a change of variable, $c + ik = s$, $i\,dk = ds$ we rewrite (3.2.3) as

$$f(x) = \frac{e^{cx}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp\{(s-c)x\} ds \int_{0}^{\infty} e^{-st} f(t) dt. \tag{3.2.4}$$

Thus, the *Laplace transform* of $f(t)$ is formally defined by

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \int_{0}^{\infty} e^{-st} f(t) dt, \qquad \operatorname{Re} s > 0, \tag{3.2.5}$$

where $e^{-st}$ is the *kernel* of the transform and $s$ is the *transform variable* which is a complex number. Under broad conditions on $f(t)$, its transform $\bar{f}(s)$ is analytic in $s$ in the half-plane, where $\operatorname{Re} s > a$.

Result (3.2.4) then gives the formal definition of the *inverse Laplace transform*

$$\mathscr{L}^{-1}\{\bar{f}(s)\} = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \bar{f}(s)\, ds, \qquad c > 0. \tag{3.2.6}$$

Obviously, $\mathscr{L}$ and $\mathscr{L}^{-1}$ are linear integral operators.

Using the definition (3.2.5), we can calculate the Laplace transforms of some simple and elementary functions.

***Example 3.2.1***

If $f(t) = 1$ for $t > 0$, then

$$\bar{f}(s) = \mathscr{L}\{1\} = \int_0^\infty e^{-st} dt = \frac{1}{s}. \tag{3.2.7}$$

□

***Example 3.2.2***

If $f(t) = e^{at}$, where $a$ is a constant, then

$$\mathscr{L}\{e^{at}\} = \bar{f}(s) = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \qquad s > a. \tag{3.2.8}$$

□

***Example 3.2.3***

If $f(t) = \sin at$, where $a$ is a real constant, then

$$\begin{aligned} \mathscr{L}\{\sin at\} &= \int_0^\infty e^{-st} \sin at \, dt = \frac{1}{2i} \int_0^\infty [e^{-t(s-ia)} - e^{-t(s+ia)}] \, dt \qquad (3.2.9) \\ &= \frac{1}{2i} \left[ \frac{1}{s-ia} - \frac{1}{s+ia} \right] = \frac{a}{s^2+a^2}. \end{aligned}$$

Similarly,

$$\mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2}. \tag{3.2.10}$$

□

***Example 3.2.4***

If $f(t) = \sinh at$ or $\cosh at$, where $a$ is a real constant, then

$$\mathscr{L}\{\sinh at\} = \int_0^\infty e^{-st} \sinh at \, dt = \frac{a}{s^2-a^2}, \tag{3.2.11}$$

$$\mathscr{L}\{\cosh at\} = \int_0^\infty e^{-st} \cosh at \, dt = \frac{s}{s^2-a^2}. \tag{3.2.12}$$

□

***Example 3.2.5***
If $f(t) = t^n$, where $n$ is a positive integer, then

$$\bar{f}(s) = \mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}. \tag{3.2.13}$$

We recall (3.2.7) and formally differentiate it with respect to $s$. This gives

$$\int_0^\infty t\, e^{-st}\, dt = \frac{1}{s^2}, \tag{3.2.14}$$

which means that

$$\mathscr{L}\{t\} = \frac{1}{s^2}. \tag{3.2.15}$$

Differentiating (3.2.14) with respect to $s$ gives

$$\mathscr{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\, dt = \frac{2}{s^3}. \tag{3.2.16}$$

Similarly, differentiation of (3.2.7) $n$ times yields

$$\mathscr{L}\{t^n\} = \int_0^\infty t^n e^{-st}\, dt = \frac{n!}{s^{n+1}}. \tag{3.2.17}$$

□

***Example 3.2.6***
If $a(> -1)$ is a real number, then

$$\mathscr{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}, \qquad (s > 0). \tag{3.2.18}$$

We have

$$\mathscr{L}\{t^a\} = \int_0^\infty t^a e^{-st}\, dt,$$

which is, by putting $st = x$,

$$= \frac{1}{s^{a+1}} \int_0^\infty x^a e^{-x} dx = \frac{\Gamma(a+1)}{s^{a+1}},$$

where $\Gamma(a)$ represents the *gamma function* defined by the integral

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx, \qquad a > 0. \tag{3.2.19}$$

It can be shown that the gamma function satisfies the relation

$$\Gamma(a+1) = a\Gamma(a). \tag{3.2.20}$$

Obviously, result (3.2.18) is an extension of (3.2.17). The latter is a special case of the former when $a$ is a positive integer.

In particular, when $a = -\frac{1}{2}$, result (3.2.18) gives

$$\mathscr{L}\left\{\frac{1}{\sqrt{t}}\right\} = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, \quad \text{where} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{3.2.21}$$

Similarly,

$$\mathscr{L}\left\{\sqrt{t}\right\} = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} = \frac{\sqrt{\pi}}{2}\frac{1}{s^{3/2}}, \tag{3.2.22}$$

where

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

□

***Example 3.2.7***

If $f(t) = erf\left(\frac{a}{2\sqrt{t}}\right)$, then

$$\mathscr{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s}(1 - e^{-a\sqrt{s}}), \tag{3.2.23}$$

where $erf(t)$ is the *error function* defined by (2.5.13).

To prove (3.2.23), we begin with the definition (3.2.5) so that

$$\mathscr{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\} = \int_0^\infty e^{-st}\left[\frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}} e^{-x^2} dx\right] dt,$$

which is, by putting $x = \frac{a}{2\sqrt{t}}$ or $t = \frac{a^2}{4x^2}$ and interchanging the order of integration,

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx \int_0^{a^2/4x^2} e^{-st} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \frac{1}{s} \left\{1 - \exp\left(-\frac{a^2 s}{4x^2}\right)\right\} dx$$

$$= \frac{1}{s} \cdot \frac{2}{\sqrt{\pi}} \left[\int_0^\infty e^{-x^2} dx - \int_0^\infty \exp\left\{-\left(x^2 + \frac{sa^2}{4x^2}\right)\right\} dx\right],$$

where the integral

$$\int_0^\infty \exp\left\{-\left(x^2 + \frac{\alpha^2}{x^2}\right)\right\} dx = \frac{1}{2}\left[\int_0^\infty \left(1 - \frac{\alpha}{x^2}\right) \exp\left[-\left(x + \frac{\alpha}{x}\right)^2 + 2\alpha\right] + \int_0^\infty \left(1 + \frac{\alpha}{x^2}\right) \exp\left[-\left(x - \frac{\alpha}{x}\right)^2 - 2\alpha\right]\right] dx,$$

which is, by putting $y = \left(x \pm \frac{\alpha}{x}\right)$, $dy = \left(1 \mp \frac{\alpha}{x^2}\right) dx$, and observing that the first integral vanishes,

$$= \frac{1}{2} e^{-2\alpha} \int_{-\infty}^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2} e^{-2\alpha}, \qquad \alpha = \frac{a\sqrt{s}}{2}.$$

Consequently,

$$\mathscr{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s} \frac{2}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} e^{-a\sqrt{s}}\right] = \frac{1}{s}[1 - e^{-a\sqrt{s}}].$$

We use (3.2.23) to find the Laplace transform of the complementary error function defined by (2.10.14) and obtain

$$\mathscr{L}\left\{erfc\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s} e^{-a\sqrt{s}}. \tag{3.2.24}$$

The proof of this result follows from $erfc(x) = 1 - erf(x)$ and $\mathscr{L}\{1\} = \frac{1}{s}$.

□

***Example 3.2.8***
If $f(t) = J_0(at)$ is a *Bessel function* of order zero, then

$$\mathscr{L}\{J_0(at)\} = \frac{1}{\sqrt{s^2 + a^2}}. \tag{3.2.25}$$

Using the series representation of $J_0(at)$, we obtain

$$\begin{aligned}
\mathscr{L}\{J_0(at)\} &= \mathscr{L}\left[1-\frac{a^2t^2}{2^2}+\frac{a^4t^4}{2^2\cdot 4^2}-\frac{a^6t^6}{2^2\cdot 4^2\cdot \cdot 6^2}+\cdots\right] \\
&= \frac{1}{s}-\frac{a^2}{2^2}\frac{2!}{s^3}+\frac{a^4}{2^2\cdot 4^2}\cdot\frac{4!}{s^5}-\frac{a^6}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{6!}{s^7}+\cdots \\
&= \frac{1}{s}\left[1-\frac{1}{2}\left(\frac{a^2}{s^2}\right)+\frac{1\cdot 3}{2\cdot 4}\left(\frac{a^4}{s^4}\right)-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{a^6}{s^6}\right)+\cdots\right] \\
&= \frac{1}{s}\left[\left(1+\frac{a^2}{s^2}\right)^{-\frac{1}{2}}\right]=\frac{1}{\sqrt{a^2+s^2}}.
\end{aligned}$$

▯

## 3.3 Existence Conditions for the Laplace Transform

A function $f(t)$ is said to be of *exponential order* $a(>0)$ on $0\le t<\infty$ if there exists a positive constant $K$ such that for all $t>T$

$$|f(t)|\le Ke^{at}, \tag{3.3.1}$$

and we write this symbolically as

$$f(t)=O(e^{at}) \qquad \text{as} \quad t\to\infty. \tag{3.3.2}$$

Or, equivalently,

$$\lim_{t\to\infty} e^{-bt}|f(t)|\le K\lim_{t\to\infty} e^{-(b-a)t}=0, \quad b>a. \tag{3.3.3}$$

Such a function $f(t)$ is simply called an *exponential order* as $t\to\infty$, and clearly, it does not grow faster than $Ke^{at}$ as $t\to\infty$.

***THEOREM 3.3.1***

If a function $f(t)$ is continuous or piecewise continuous in every finite interval $(0,T)$, and of exponential order $e^{at}$, then the Laplace transform of $f(t)$ exists for all $s$ provided Re $s>a$.

**PROOF** We have

$$|\bar{f}(s)|=\left|\int_0^\infty e^{-st}f(t)dt\right|\le\int_0^\infty e^{-st}|f(t)|dt \tag{3.3.4}$$

$$\le K\int_0^\infty e^{-t(s-a)}dt=\frac{K}{s-a},\ \text{for Re } s>a.$$

Thus, the proof is complete.

It is noted that the conditions as stated in Theorem 3.3.1 are sufficient rather than necessary conditions.

It also follows from (3.3.4) that $\lim_{s\to\infty}|\bar{f}(s)| = 0$, that is, $\lim_{s\to\infty}\bar{f}(s) = 0$. This result can be regarded as the limiting property of the Laplace transform. However, $\bar{f}(s) = s$ or $s^2$ is not the Laplace transform of any continuous (or piecewise continuous) function because $\bar{f}(s)$ does not tend to zero as $s \to \infty$.

Further, a function $f(t) = \exp(at^2)$, $a > 0$ cannot have a Laplace transform even though it is continuous but is *not* of the exponential order because

$$\lim_{t\to\infty} \exp(at^2 - st) = \infty.$$

∎

## 3.4 Basic Properties of Laplace Transforms

***THEOREM 3.4.1*** *(Heaviside's First Shifting Theorem).*
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathscr{L}\{e^{-at}f(t)\} = \bar{f}(s+a), \tag{3.4.1}$$

where $a$ is a real constant.

**PROOF** We have, by definition,

$$\mathscr{L}\{e^{-at}f(t)\} = \int_0^\infty e^{-(s+a)t} f(t)dt = \bar{f}(s+a).$$

∎

***Example 3.4.1***
The following results readily follow from (3.4.1)

$$\mathscr{L}\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}, \tag{3.4.2}$$

$$\mathscr{L}\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}, \tag{3.4.3}$$

$$\mathscr{L}\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2}. \tag{3.4.4}$$

***THEOREM 3.4.2***
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then the Second Shifting property holds:

$$\mathscr{L}\{f(t-a)H(t-a)\} = e^{-as}\,\bar{f}(s) = e^{-as}\,\mathscr{L}\{f(t)\}, \qquad a > 0. \tag{3.4.5}$$

Or, equivalently,

$$\mathscr{L}\{f(t)H(t-a)\} = e^{-as}\,\mathscr{L}\{f(t+a)\}. \tag{3.4.6}$$

where $H(t-a)$ is the Heaviside unit step function defined by (2.3.9).

It follows from the definition that

$$\begin{aligned}\mathscr{L}\{f(t-a)H(t-a)\} &= \int_0^\infty e^{-st} f(t-a)H(t-a)dt \\ &= \int_a^\infty e^{-st} f(t-a)dt,\end{aligned}$$

which is, by putting $t - a = \tau$,

$$= e^{-sa}\int_0^\infty e^{-s\tau} f(\tau)d\tau = e^{-sa}\,\bar{f}(s).$$

We leave it to the reader to prove (3.4.6).

In particular, if $f(t) = 1$, then

$$\mathscr{L}\{H(t-a)\} = \frac{1}{s}\exp(-sa). \tag{3.4.7}$$

$\square$

***Example 3.4.2***
Use the shifting property (3.4.5) or (3.4.6) to find the Laplace transform of

$$\text{(a)}\quad f(t) = \left\{\begin{array}{lc} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \\ 0, & t > 2 \end{array}\right\}, \qquad\qquad \text{(b)}\quad g(t) = \sin t\, H(t-\pi).$$

To find $\mathscr{L}\{f(t)\}$, we write $f(t)$ as

$$f(t) = 1 - 2\,H(t-1) + H(t-2).$$

Hence,

$$\begin{aligned}\bar{f}(s) = \mathscr{L}\{f(t)\} &= \mathscr{L}\{1\} - 2\,\mathscr{L}\{H(t-1)\} + \mathscr{L}\{H(t-2)\} \\ &= \frac{1}{s} - \frac{2\,e^{-s}}{s} + \frac{e^{-2s}}{s}.\end{aligned}$$

To obtain $\mathscr{L}\{g(t)\}$, we use (3.4.6) so that

$$\bar{g}(s) = \mathscr{L}\{\sin t\, H(t-\pi)\} = -e^{-\pi s}\,\mathscr{L}\{\cos t\} = -\frac{s\,e^{-\pi s}}{s^2+1}.$$

□

**Scaling Property:**

$$\mathscr{L}\{f(at)\} = \frac{1}{|a|}\,\bar{f}\left(\frac{s}{a}\right), \quad a \neq 0. \tag{3.4.8}$$

***Example 3.4.3***

Show that the Laplace transform of the square wave function $f(t)$ defined by

$$f(t) = H(t) - 2H(t-a) + 2H(t-2a) - 2H(t-3a) + \cdots \tag{3.4.9}$$

is

$$\bar{f}(s) = \frac{1}{s}\tanh\left(\frac{as}{2}\right). \tag{3.4.10}$$

The graph of $f(t)$ is shown in Figure 3.1.

$$\begin{aligned} f(t) &= H(t) - 2H(t-a) = 1 - 2\cdot 0 = 1, \qquad 0<t<a \\ f(t) &= H(t) - 2H(t-a) + 2H(t-2a) \\ &= 1 - 2\cdot 1 + 2\cdot 0 = -1, \qquad 0<a<t<2a. \end{aligned}$$

Thus,

$$\begin{aligned} \bar{f}(s) &= \frac{1}{s} - 2\cdot\frac{e^{-as}}{s} + 2\cdot\frac{e^{-2as}}{s} - 2\cdot\frac{e^{-3as}}{s} + \cdots \\ &= \frac{1}{s}[1 - 2r(1 - r + r^2 - \cdots)], \qquad \text{where} \quad r = e^{-as} \\ &= \frac{1}{s}\left[1 - \frac{2r}{1+r}\right] = \frac{1}{s}\left[1 - \frac{2e^{-as}}{1+e^{-as}}\right] \\ &= \frac{1}{s}\left(\frac{1-e^{-as}}{1+e^{-as}}\right) = \frac{1}{s}\left(\frac{e^{\frac{sa}{2}} - e^{-\frac{as}{2}}}{e^{\frac{sa}{2}} + e^{-\frac{as}{2}}}\right) = \frac{1}{s}\tanh\left(\frac{as}{2}\right). \end{aligned}$$

□

***Example 3.4.4***

(*The Laplace Transform of a Periodic Function*). If $f(t)$ is a periodic function of period $a$, and if $\mathscr{L}\{f(t)\}$ exists, show that

$$\mathscr{L}\{f(t)\} = [1 - \exp(-as)]^{-1}\int_0^a e^{-st} f(t)\,dt. \tag{3.4.11}$$

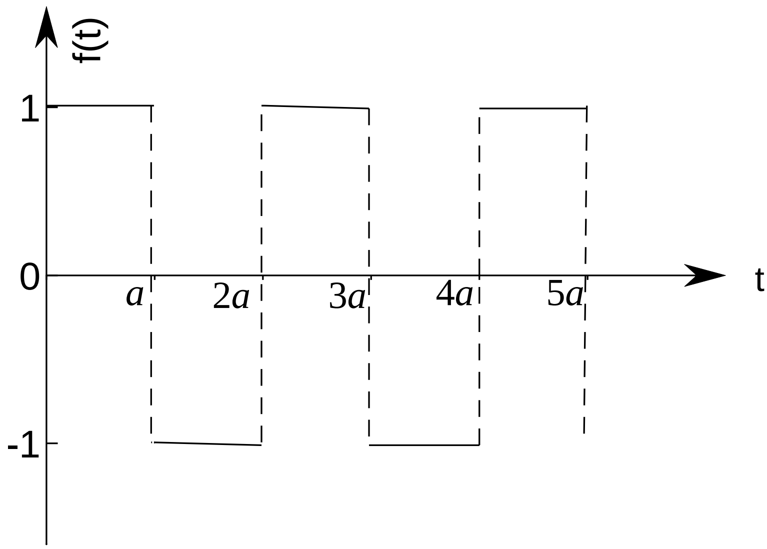


**Figure 3.1** Square wave function.

We have, by definition,

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^a e^{-st} f(t)dt + \int_a^\infty e^{-st} f(t)dt.$$

Letting $t = \tau + a$ in the second integral gives

$$\bar{f}(s) = \int_0^a e^{-st} f(t)dt + \exp(-sa)\int_0^\infty e^{-s\tau} f(\tau + a)d\tau,$$

which is, due to $f(\tau + a) = f(\tau)$ and replacing the dummy variable $\tau$ by $t$ in the second integral,

$$= \int_0^a e^{-st} f(t)dt + \exp(-sa)\int_0^\infty e^{-st} f(t)dt.$$

Finally, combining the second term with the left hand side, we obtain (3.4.11).

In particular, we calculate the Laplace transform of a rectified sine wave, that is, $f(t) = |\sin at|$. This is a periodic function with period $\frac{\pi}{a}$. We have

$$\int_0^{\frac{\pi}{a}} e^{-st} \sin at\, dt = \left[\frac{e^{-st}(-a\cos at - s\sin at)}{(s^2 + a^2)}\right]_0^{\frac{\pi}{a}} = \frac{a\left\{1 + \exp\left(-\frac{s\pi}{a}\right)\right\}}{(s^2 + a^2)}.$$

Clearly, the property (3.4.11) gives

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \frac{a}{(s^2+a^2)}\cdot\frac{1+\exp\left(-\frac{s\pi}{a}\right)}{1-\exp\left(-\frac{s\pi}{a}\right)}\\ &= \frac{a}{(s^2+a^2)}\left[\frac{\exp\left(\frac{s\pi}{2a}\right)+\exp\left(-\frac{s\pi}{2a}\right)}{\exp\left(\frac{2\pi}{2a}\right)-\exp\left(-\frac{s\pi}{2a}\right)}\right]\\ &= \frac{a}{s^2+a^2}\coth\left(\frac{\pi s}{2a}\right).\end{aligned}$$

$\square$

***THEOREM 3.4.3*** *(Laplace Transforms of Derivatives).*
If $\mathscr{L}\{f(t)\}=\bar{f}(s)$, then

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0) = s\bar{f}(s) - f(0), \tag{3.4.12}$$
$$\mathscr{L}\{f''(t)\} = s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0) = s^2\bar{f}(s) - s\,f(0) - f'(0). \tag{3.4.13}$$

More generally,

$$\mathscr{L}\{f^{(n)}(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0), \tag{3.4.14}$$

where $f^{(r)}(0)$ is the value of $f^{(r)}(t)$ at $t=0,\ r=0,1,...,(n-1)$.

**PROOF** We have, by definition,

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)dt,$$

which is, integrating by parts,

$$\begin{aligned}&= \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)dt\\ &= s\bar{f}(s) - f(0),\end{aligned}$$

in which we assumed $f(t)\,e^{-st}\to 0$ as $t\to\infty$.
Similarly,

$$\begin{aligned}\mathscr{L}\{f''(t)\} &= s\mathscr{L}\{f'(t)\} - f'(0), \qquad \text{by (3.4.12)}\\ &= s[s\ \bar{f}(s) - f(0)] - f'(0)\\ &= s^2\bar{f}(s) - sf(0) - f'(0),\end{aligned}$$

where we have assumed $e^{-st} f'(t) \to 0$ as $t \to \infty$.

A similar procedure can be used to prove the general result (3.4.14).

It may be noted that similar results hold when the Laplace transform is applied to partial derivatives of a function of two or more independent variables. For example, if $u(x,t)$ is a function of two variables $x$ and $t$, then

$$\mathscr{L}\left\{\frac{\partial u}{\partial t}\right\} = s\bar{u}(x,s) - u(x,0), \tag{3.4.15}$$

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2\bar{u}(x,s) - s\ u(x,0) - \left[\frac{\partial u}{\partial t}\right]_{t=0}, \tag{3.4.16}$$

$$\mathscr{L}\left\{\frac{\partial u}{\partial x}\right\} = \frac{d\bar{u}}{dx}, \quad \mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2\bar{u}}{dx^2}. \tag{3.4.17}$$

Results (3.4.12) to (3.4.14) imply that the Laplace transform reduces the operation of differentiation into algebraic operation. In view of this, the Laplace transform can be used effectively to solve ordinary or partial differential equations. $\blacksquare$

***Example 3.4.5***
Use (3.4.14) to find $\mathscr{L}\{t^n\}$.

Here $f(t) = t^n$, $f'(t) = nt^{n-1}, \cdots, f^{(n)}(t) = n!$ and $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$.

Thus,

$$\mathscr{L}\{n!\} = s^n \mathscr{L}\{t^n\}.$$

Or,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^n}\mathscr{L}\{1\} = \frac{n!}{s^{n+1}}.$$

$\square$

## 3.5 The Convolution Theorem and Properties of Convolution

***THEOREM 3.5.1*** *(Convolution Theorem)***.**
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$ and $\mathscr{L}\{g(t)\} = \bar{g}(s)$, then

$$\mathscr{L}\{f(t) * g(t)\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = \bar{f}(s)\bar{g}(s). \tag{3.5.1}$$

Or, equivalently,

$$\mathscr{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = f(t) * g(t), \tag{3.5.2}$$

where $f(t)*g(t)$ is called the *convolution* of $f(t)$ and $g(t)$ and is defined by the integral

$$f(t)*g(t) = \int_0^t f(t-\tau)g(\tau)d\tau. \tag{3.5.3}$$

The integral in (3.5.3) is often referred to as the *convolution integral* (or *Faltung*) and is denoted simply by $(f*g)(t)$.

**PROOF** We have, by definition,

$$\mathscr{L}\{f(t)*g(t)\} = \int_0^\infty e^{-st}dt \int_0^t f(t-\tau)g(\tau)d\tau, \tag{3.5.4}$$

where the region of integration in the $\tau - t$ plane is as shown in Figure 3.2. The integration in (3.5.4) is first performed with respect to $\tau$ from $\tau = 0$ to $\tau = t$ of the vertical strip and then from $t = 0$ to $\infty$ by moving the vertical strip from $t = 0$ outwards to cover the whole region under the line $\tau = t$.

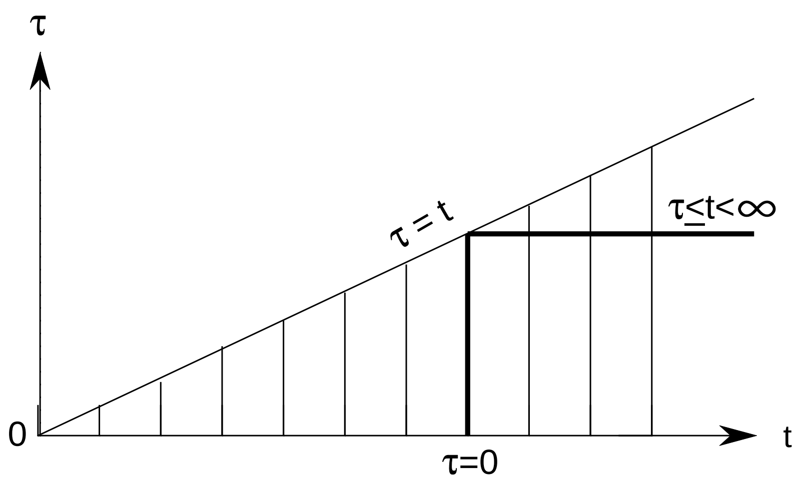


**Figure 3.2** Region of integration.

We now change the order of integration so that we integrate first along the horizontal strip from $t = \tau$ to $\infty$ and then from $\tau = 0$ to $\infty$ by moving the horizontal strip vertically from $\tau=0$ upwards. Evidently, (3.5.4) becomes

$$\mathscr{L}\{f(t)*g(t)\}=\int_0^\infty g(\tau)d\tau\int_{t=\tau}^\infty e^{-st}f(t-\tau)d\tau,$$

which is, by the change of variable $t-\tau=x$,

$$\begin{aligned}\mathscr{L}\{f(t)*g(t)\} &= \int_0^\infty g(\tau)d\tau\int_0^\infty e^{-s(x+\tau)}f(x)dx\\ &= \int_0^\infty e^{-s\tau}g(\tau)d\tau\int_0^\infty e^{-sx}f(x)dx=\bar{g}(s)\,\bar{f}(s).\end{aligned}$$

This completes the proof. ∎

**PROOF (Second Proof.)** We have, by definition,

$$\begin{aligned}\bar{f}(s)\bar{g}(s) &= \int_0^\infty e^{-s\sigma}f(\sigma)d\sigma\int_0^\infty e^{-s\mu}g(\mu)d\mu\\ &= \int_0^\infty\int_0^\infty e^{-s(\sigma+\mu)}f(\sigma)g(\mu)d\sigma\,d\mu, \qquad (3.5.5)\end{aligned}$$

where the double integral is taken over the entire first quadrant $R$ of the $\sigma-\mu$ plane bounded by $\sigma=0$ and $\mu=0$ as shown in Figure 3.3(a).

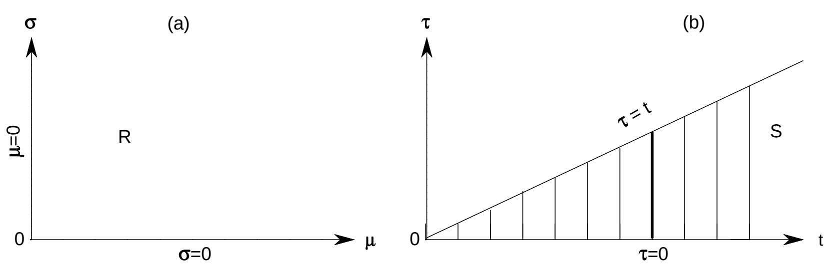


**Figure 3.3** Regions of integration.

We make the change of variables $\mu=\tau$, $\sigma=t-\mu=t-\tau$ so that the axes $\sigma=0$ and $\mu=0$ transform into the lines $\tau=0$ and $\tau=t$, respectively, as shown in

Figure 3.3(b) in the $\tau - t$ plane. Consequently, (3.5.5) becomes

$$\begin{aligned}\bar{f}(s)\bar{g}(s) &= \int_0^\infty e^{-st}dt \int_{\tau=0}^{\tau=t} f(t-\tau)g(\tau)d\tau \\ &= \mathscr{L}\left\{\int_0^t f(t-\tau)g(\tau)d\tau\right\} \\ &= \mathscr{L}\left\{f(t)*g(t)\right\}.\end{aligned}$$

This proves the theorem. ∎

***Note:*** A more rigorous proof of the convolution theorem can be found in any standard treatise (see Doetsch, 1950) on Laplace transforms. The convolution operation has the following properties:

$$f(t)*\{g(t)*h(t)\} = \{f(t)*g(t)\}*h(t), \quad \text{(Associative)}, \quad (3.5.6)$$

$$f(t)*g(t) = g(t)*f(t), \quad \text{(Commutative)}, \quad (3.5.7)$$

$$f(t)*\{ag(t) + bh(t)\} = af(t)*g(t) + bf(t)*h(t), \quad \text{(Distributive)}, \quad (3.5.8)$$

$$f(t)*\{ag(t)\} = \{af(t)\}*g(t) = a\{f(t)*g(t)\}, \quad (3.5.9)$$

$$\mathscr{L}\left\{f_1*f_2*f_3*\cdots*f_n\right\} = \bar{f}_1(s)\bar{f}_2(s)\cdots\bar{f}_n(s), \quad (3.5.10)$$

$$\mathscr{L}\left\{f^{*n}\right\} = \{\bar{f}(s)\}^n, \quad (3.5.11)$$

where $a$ and $b$ are constants. $f^{*n} = f*f*\cdots*f$ is sometimes called the *nth convolution.*

***Remark:*** By virtue of (3.5.6) and (3.5.7), it is clear that the set of all Laplace transformable functions forms a commutative semigroup with respect to the operation $*$. The set of all Laplace transformable functions does not form a group because $f*g^{-1}$ does not, in general, have a Laplace transform.

We now prove the associative property. We have

$$f(t)*\{g(t)*h(t)\} = \int_0^t f(\tau) \int_0^{t-\tau} g(t-\sigma-\tau)h(\sigma)d\sigma\ d\tau \quad (3.5.12)$$

$$\begin{aligned}&= \int_0^t h(\sigma) \int_0^{t-\sigma} g(t-\tau-\sigma)f(\tau)d\tau\ d\sigma \\ &= h(t)*\{f(t)*g(t)\} = \{f(t)*g(t)\}*h(t), \quad (3.5.13)\end{aligned}$$

where (3.5.13) is obtained from (3.5.12) by interchanging the order of integration combined with the fact that $0 \le \sigma \le t-\tau$ and $0 \le \tau \le t$ imply $0 \le \tau \le t-\sigma$ and $0 \le \sigma \le t$. Properties (3.5.10) and (3.5.11) follow immediately from the associative law of the convolution.

To prove (3.5.7), we recall the definition of the convolution and make a change of variable $t-\tau=t'$. This gives

$$f(t)*g(t)=\int_0^t f(t-\tau)g(\tau)d\tau=\int_0^t g(t-t')f(t')dt'=g(t)*f(t).$$

The proofs of (3.5.8)–(3.5.9) are very simple and hence, may be omitted.

***Example 3.5.1***

Obtain the convolutions

(a) $t*e^{at}$, (b) $(\sin at*\sin at)$, (c) $\frac{1}{\sqrt{\pi t}}*e^{at}$,

(d) $1*\frac{a}{2}\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}$, (e) $\cos t*e^{2t}$, (f) $t*t*t$.

We have

(a) $t*e^{at}=\int_0^t \tau e^{a(t-\tau)}d\tau=e^{at}\int_0^t \tau e^{-a\tau}d\tau=\frac{1}{a^2}(e^{at}-at-1).$

(b) $\sin at*\sin at=\int_0^t \sin a\tau\;\sin a(t-\tau)d\tau=\frac{1}{2a}(\sin at-at\cos at).$

(c) $\frac{1}{\sqrt{\pi t}}*e^{at}=\frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\tau}}e^{a(t-\tau)}d\tau,$

which is, by putting $\sqrt{a\tau}=x$,

$$\frac{1}{\sqrt{\pi t}}*e^{at}=\frac{2e^{at}}{\sqrt{\pi a}}\int_0^{\sqrt{at}} e^{-x^2}dx=\frac{e^{at}}{\sqrt{a}}\,erf\left(\sqrt{at}\right).$$

(d) We have

$$1*\frac{a}{2}\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}=\frac{a}{2\sqrt{\pi}}\int_0^t \frac{e^{-a^2/4\tau}}{\tau^{3/2}}d\tau,$$

which is, by letting $\frac{a}{2\sqrt{\tau}}=x$,

$$=\frac{2}{\sqrt{\pi}}\int_{\frac{a}{2\sqrt{t}}}^{\infty} e^{-x^2}dx=erfc\left(\frac{a}{2\sqrt{t}}\right).$$

$$\text{(e)}\ \cos t * e^{2t} = \int_0^t \cos(t-\tau)\, e^{2\tau}\, d\tau = \frac{1}{2}\int_0^t e^{2\tau}\left\{e^{i(t-\tau)} + e^{-i(t-\tau)}\right\} d\tau$$

$$= \left[\frac{e^{i(t-\tau)+2\tau}}{2(2-i)} + \frac{e^{-i(t-\tau)+2\tau}}{2(2+i)}\right] = \frac{2}{5}\, e^{2t} + \frac{1}{5}\,(\sin t - 2\cos t)\,.$$

$$\text{(f)}\ (t*t)*t = \left[\int_0^t (t-\tau)\,\tau\, d\tau\right] * t = \frac{1}{6}t^3 * t$$

$$= \frac{1}{6}\int_0^t (t-\tau)\,\tau^3\, d\tau = \frac{t^5}{5!}.$$

□

***Example 3.5.2***

Using the Convolution Theorem 3.5.1, prove that

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \tag{3.5.14}$$

where $\Gamma(m)$ is the gamma function, and $B(m,n)$ is the beta function defined by

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad (m>0, n>0). \tag{3.5.15}$$

To prove (3.5.14), we consider

$$f(t) = t^{m-1}\ (m>0) \quad \text{and} \quad g(t) = t^{n-1},\ (n>0).$$

Evidently, $\bar{f}(s) = \dfrac{\Gamma(m)}{s^m}$ and $\bar{g}(s) = \dfrac{\Gamma(n)}{s^n}$.

We have

$$f*g = \int_0^t \tau^{m-1}(t-\tau)^{n-1}d\tau = \mathscr{L}^{-1}\{\bar{f}(s)\bar{g}(s)\}$$

$$= \Gamma(m)\Gamma(n)\mathscr{L}^{-1}\{s^{-(m+n)}\}$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\, t^{m+n-1}.$$

Letting $t=1$, we derive the result

$$\int_0^1 \tau^{m-1}(1-\tau)^{n-1}d\tau = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$$

which proves the result (3.5.14). □

## 3.6 Differentiation and Integration of Laplace Transforms

***THEOREM 3.6.1***
If $f(t) = O(e^{at})$ as $t \to \infty$, then the *Laplace integral*

$$\int_0^\infty e^{-st} f(t)dt, \tag{3.6.1}$$

is uniformly convergent with respect to $s$ provided $s \geq a_1$ where $a_1 > a$.

**PROOF** Since

$$|e^{-st} f(t)| \leq K e^{-t(s-a)} \leq K e^{-t(a_1-a)} \quad \text{for all } s \geq a_1$$

and $\int_0^\infty e^{-t(a_1-a)} dt$ exists for $a_1 > a$, by Weierstrass' test, the Laplace integral is uniformly convergent for all $s > a_1$ where $a_1 > a$. This completes the proof.

▮

In view of the uniform convergence of (3.6.1), differentiation of (3.2.5) with respect to $s$ within the integral sign is permissible. Hence,

$$\begin{aligned} \frac{d}{ds}\bar{f}(s) &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)dt \\ &= -\int_0^\infty t f(t) e^{-st} dt = -\mathscr{L}\{t f(t)\}. \end{aligned} \tag{3.6.2}$$

Similarly, we obtain

$$\frac{d^2}{ds^2}\bar{f}(s) = (-1)^2 \mathscr{L}\{t^2 f(t)\}, \tag{3.6.3}$$

$$\frac{d^3}{ds^3}\bar{f}(s) = (-1)^3 \mathscr{L}\{t^3 f(t)\}. \tag{3.6.4}$$

More generally,

$$\frac{d^n}{ds^n}\bar{f}(s) = (-1)^n \mathscr{L}\{t^n f(t)\}. \tag{3.6.5}$$

Results (3.6.5) can be stated in the following theorem:

**THEOREM 3.6.2** *(Derivatives of the Laplace Transform).*
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \bar{f}(s), \tag{3.6.6}$$

where $n = 0, 1, 2, 3, \ldots$.

***Example 3.6.1***
Show that

(a) $\mathscr{L}\{t^n e^{-at}\} = \dfrac{n!}{(s+a)^{n+1}}$, (b) $\mathscr{L}\{t \cos at\} = \dfrac{s^2 - a^2}{(s^2+a^2)^2}$,

(c) $\mathscr{L}\{t \sin at\} = \dfrac{2as}{(s^2+a^2)^2}$, (d) $\mathscr{L}\{t\, f'(t)\} = -\left\{ s \dfrac{d}{ds} \bar{f}(s) + \bar{f}(s) \right\}$.

(a) Application of Theorem 3.6.2 gives

$$\mathscr{L}\{t^n e^{-at}\} = (-1)^n \frac{d^n}{ds^n} \cdot \frac{1}{(s+a)} = (-1)^{2n} \frac{n!}{(s+a)^{n+1}}.$$

(b)
$$\mathscr{L}\{t \cos at\} = (-1) \frac{d}{ds} \left( \frac{s}{s^2+a^2} \right) = \frac{s^2 - a^2}{(s^2+a^2)^2}.$$

Results (c) and (d) can be proved similarly. ▯

**THEOREM 3.6.3** *(Integral of the Laplace Transform).*
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathscr{L}\left\{ \frac{f(t)}{t} \right\} = \int_s^\infty \bar{f}(s) ds. \tag{3.6.7}$$

**PROOF** In view of the uniform convergence of (3.6.1), $\bar{f}(s)$ can be integrated with respect to $s$ in $(s,\infty)$ so that

$$\begin{aligned}\int_s^\infty \bar{f}(s)ds &= \int_s^\infty ds \int_0^\infty e^{-st} f(t)dt\\ &= \int_0^\infty f(t)dt \int_s^\infty e^{-st}ds\\ &= \int_0^\infty \frac{f(t)}{t} e^{-st}dt = \mathscr{L}\left\{\frac{f(t)}{t}\right\}.\end{aligned}$$

This proves the theorem. ∎

***Example 3.6.2***
Show that

(a) $\mathscr{L}\left\{\dfrac{\sin at}{t}\right\} = \tan^{-1}\left(\dfrac{a}{s}\right)$, (b) $\mathscr{L}\left\{\dfrac{e^{-a^2/4t}}{\sqrt{\pi t^3}}\right\} = \dfrac{2}{a}\exp(-a\sqrt{s})$.

(a) Using (3.6.7), we obtain

$$\mathscr{L}\left\{\frac{\sin at}{t}\right\} = a\int_s^\infty \frac{ds}{s^2+a^2} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right).$$

(b) $\mathscr{L}\left\{\dfrac{1}{t}\cdot\dfrac{e^{-a^2/4t}}{\sqrt{\pi t}}\right\} = \displaystyle\int_s^\infty \bar{f}(s)ds = \int_s^\infty \frac{e^{-a\sqrt{s}}}{\sqrt{s}}ds$, by Table B-4 of Laplace transforms,

which is, by putting $a\sqrt{s} = x$,

$$= \frac{2}{a}\int_{a\sqrt{s}}^\infty e^{-x}dx = \frac{2}{a}\exp(-a\sqrt{s}).$$

□

***THEOREM 3.6.4*** *(The Laplace Transform of an Integral).*
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{\bar{f}(s)}{s}. \tag{3.6.8}$$

**PROOF** We write

$$g(t) = \int_0^t f(\tau)d\tau$$

so that $g(0) = 0$ and $g'(t) = f(t)$. Then it follows from (3.4.10) that

$$\bar{f}(s) = \mathscr{L}\{f(t)\} = \mathscr{L}\{g'(t)\} = s\,\bar{g}(s) = s\,\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\}.$$

Dividing both sides by $s$, we obtain (3.6.8).

It is noted that the Laplace transform of an integral corresponds to the division of the transform of its integrand by $s$. Result (3.6.8) can be used for evaluation of the inverse Laplace transform. ∎

***Example 3.6.3***
Use result (3.6.8) to find

(a) $\mathscr{L}\left\{\int_0^t \tau^n e^{-a\tau} d\tau\right\}$, (b) $\mathscr{L}\{Si(at)\} = \mathscr{L}\left\{\int_0^t \frac{\sin a\tau}{\tau} d\tau\right\}$.

(a) We know

$$\mathscr{L}\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}.$$

It follows from (3.6.8) that

$$\mathscr{L}\left\{\int_0^t \tau^n e^{-a\tau} d\tau\right\} = \frac{n!}{s(s+a)^{n+1}}.$$

(b) Using (3.6.8) and Example 3.6.2(a), we obtain

$$\mathscr{L}\left\{\int_0^t \frac{\sin a\tau}{\tau} d\tau\right\} = \frac{1}{s}\tan^{-1}\left(\frac{a}{s}\right).$$

□

## 3.7 The Inverse Laplace Transform and Examples

It has already been demonstrated that the Laplace transform $\bar{f}(s)$ of a given function $f(t)$ can be calculated by direct integration. We now look at the inverse problem. Given a Laplace transform $\bar{f}(s)$ of an unknown function $f(t)$, how can we find $f(t)$? This is essentially concerned with the solution of the integral equation

$$\int_0^\infty e^{-st} f(t)dt = \bar{f}(s). \tag{3.7.1}$$

At this stage, it is rather difficult to handle the problem as it is. However, in simple cases, we can find the inverse transform from Table B-4 of Laplace transforms. For example

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad \mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos\ at.$$

In general, the inverse Laplace transform can be determined by using four methods: (i) Partial Fraction Decomposition, (ii) the Convolution Theorem, (iii) Contour Integration of the Laplace Inversion Integral, and (iv) Heaviside's Expansion Theorem.

(i) *Partial Fraction Decomposition Method*
If

$$\bar{f}(s) = \frac{\bar{p}(s)}{\bar{q}(s)}, \tag{3.7.2}$$

where $\bar{p}(s)$ and $\bar{q}(s)$ are polynomials in $s$, and the degree of $\bar{p}(s)$ is less than that of $\bar{q}(s)$, the method of partial fractions may be used to express $\bar{f}(s)$ as the sum of terms which can be inverted by using a table of Laplace transforms. We illustrate the method by means of simple examples.

***Example 3.7.1***
To find

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)}\right\},$$

where $a$ *is* a constant, we write

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)}\right\} &= \mathscr{L}^{-1}\left[\frac{1}{a}\left\{\frac{1}{s-a} - \frac{1}{s}\right\}\right] \\ &= \frac{1}{a}\left[\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}\right] \\ &= \frac{1}{a}(e^{at}-1).\end{aligned}$$

□

***Example 3.7.2***
Show that

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{b^2-a^2}\left(\frac{\sin at}{a} - \frac{\sin bt}{b}\right).$$

We write

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{b^2-a^2}\left[\mathscr{L}^{-1}\left\{\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right\}\right]$$
$$= \frac{1}{(b^2-a^2)}\left(\frac{\sin at}{a} - \frac{\sin bt}{b}\right).$$

$\square$

***Example 3.7.3***

Find

$$\mathscr{L}^{-1}\left\{\frac{s+7}{s^2+2s+5}\right\}.$$

We have

$$\mathscr{L}^{-1}\left\{\frac{s+7}{(s+1)^2+4}\right\} = \mathscr{L}^{-1}\left\{\frac{s+1+6}{(s+1)^2+2^2}\right\}$$
$$= \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + 3\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$
$$= e^{-t}\cos 2t + 3e^{-t}\sin 2t.$$

$\square$

***Example 3.7.4***

Evaluate the following inverse Laplace transform

$$\mathscr{L}^{-1}\left\{\frac{2s^2+5s+7}{(s-2)(s^2+4s+13)}\right\}.$$

We have

$$\mathscr{L}^{-1}\left\{\frac{2s^2+5s+7}{(s-2)(s^2+4s+13)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s-2} + \frac{s+2}{(s+2)^2+3^2} + \frac{1}{(s+2)^2+3^2}\right\}$$
$$= \mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\}$$
$$+\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\}$$
$$= e^{2t} + e^{-2t}\cos 3t + \frac{1}{3}\,e^{-2t}\sin 3t.$$

$\square$

(ii) *Convolution Theorem*

We shall apply the convolution theorem for calculation of inverse Laplace transforms.

***Example 3.7.5***

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)}\right\} = 1 * e^{at} = \int_0^t e^{a\tau} d\tau = \frac{(e^{at}-1)}{a}. \qquad \square$$

***Example 3.7.6***

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} &= t * \frac{\sin at}{a} \\ &= \frac{1}{a}\int_0^t (t-\tau)\sin a\tau \; d\tau \\ &= \frac{t}{a}\int_0^t \sin a\tau \, d\tau - \frac{1}{a}\int_0^t \tau \sin a\tau \, d\tau \\ &= \frac{1}{a^2}\left(t - \frac{1}{a}\sin at\right).\end{aligned}$$

$\square$

***Example 3.7.7***

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} &= \frac{\sin at}{a} * \frac{\sin at}{a} \\ &= \frac{1}{a^2}\int_0^t \sin a\tau \sin a(t-\tau) d\tau \\ &= \frac{1}{2a^3}(\sin at - at\cos at).\end{aligned}$$

$\square$

***Example 3.7.8***

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-a)}\right\} &= \frac{1}{\sqrt{\pi t}} * e^{at}, (a>0)\\ &= \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\tau}}\, e^{a(t-\tau)}d\tau\\ &= \frac{2\,e^{at}}{\sqrt{\pi a}}\int_0^{\sqrt{at}} e^{-x^2}dx, \quad \left(\text{putting } \sqrt{a\tau}=x\right)\\ &= \frac{e^{at}}{\sqrt{a}}\, erf(\sqrt{at}). \end{aligned} \tag{3.7.3}$$

□

***Example 3.7.9***
Show that

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\, e^{-a\sqrt{s}}\right\} = erfc\left(\frac{a}{2\sqrt{t}}\right). \tag{3.7.4}$$

In view of Example 3.6.2(b), and the Convolution Theorem 3.5.1, we obtain

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s}\, e^{-a\sqrt{s}}\right\} &= 1*\frac{a}{2}\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}\\ &= \frac{a}{2\sqrt{\pi}}\int_0^t \frac{e^{-a^2/4\tau}}{\tau^{3/2}}d\tau,\end{aligned}$$

which is, by putting $\dfrac{a}{2\sqrt{\tau}} = x$,

$$= \frac{2}{\sqrt{\pi}}\int_{\frac{a}{2\sqrt{t}}}^{\infty} e^{-x^2}dx = erfc\left(\frac{a}{2\sqrt{t}}\right).$$

□

***Example 3.7.10***
Show that

$$\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}+a}\right\} = \frac{1}{\sqrt{\pi t}} - a\exp(ta^2)\, erfc(a\sqrt{t}). \tag{3.7.5}$$

We have

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}+a}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}}-\frac{a}{\sqrt{s}(\sqrt{s}+a)}\right\}\\
&= \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} - a\mathscr{L}^{-1}\left\{\frac{\sqrt{s}-a}{\sqrt{s}(s-a^2)}\right\}\\
&= \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} - a\mathscr{L}^{-1}\left\{\frac{1}{s-a^2}\right\} + a^2\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-a^2)}\right\}\\
&= \frac{1}{\sqrt{\pi t}} - a\exp(a^2 t) + a\exp(a^2 t)\, erf(a\sqrt{t}), \qquad \text{by (3.7.3)}\\
&= \frac{1}{\sqrt{\pi t}} - a\exp(a^2 t)\, erfc(a\sqrt{t}).
\end{aligned}$$

□

***Example 3.7.11***
If $f(t)=\mathscr{L}^{-1}\{\bar{f}(s)\}$, then

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\,\bar{f}(s)\right\} = \int_0^t f(x)dx. \tag{3.7.6}$$

We have, by the Convolution Theorem with $g(t)=1$ so that $\bar{g}(s)=\frac{1}{s}$,

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\,\bar{f}(s)\right\} = \int_0^t f(t-\tau)d\tau,$$

which is, by putting $t-\tau=x$,

$$= \int_0^t f(x)dx.$$

(iii) *Contour Integration of the Laplace Inversion Integral*
In Section 3.2, in inverse Laplace transform is defined by the complex integral formula

$$\mathscr{L}^{-1}\{\bar{f}(s)\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)ds, \tag{3.7.7}$$

where $c$ is a suitable real constant and $\bar{f}(s)$ is an analytic function of the complex variable $s$ in the right half-plane Re $s>a$.

The details of evaluation of (3.7.7) depend on the nature of the singularities of $\bar{f}(s)$. Usually, $\bar{f}(s)$ is a single valued function with a finite or enumerably infinite number of polar singularities. Often it has branch points. The path of integration is the straight line $L$ (see Figure 3.4(a)) in the complex $s$-plane with equation $s = c + iR$, $-\infty < R < \infty$, Re $s = c$ being chosen so that all the singularities of the integrand of (3.7.7) lie to the left of the line $L$. This line is called by *Bromwich Contour*. In practice, the Bromwich Contour is closed by an arc of a circle of radius $R$ as shown in Figure 3.4(a), and then the limit as $R \to \infty$ is taken to expand the contour of integration to infinity so that all the singularities of $\bar{f}(s)$ lie inside the contour of integration.

When $\bar{f}(s)$ has a branch point at the origin, we draw the modified contour of integration by making a cut along the negative real axis and a small semicircle $\gamma$ surrounding the origin as shown in Figure 3.4(b).

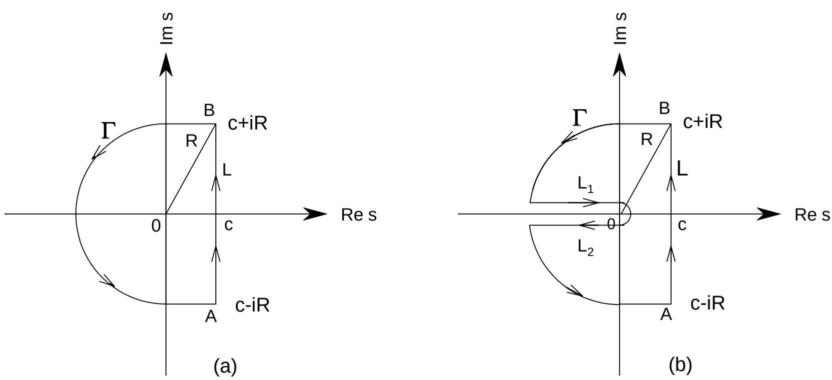


**Figure 3.4** The Bromwich contour and the contour of integration.

In either case, the Cauchy Residue Theorem is used to evaluate the integral

$$\int_L e^{st}\bar{f}(s)ds + \int_\Gamma e^{st}\bar{f}(s)ds = \int_C e^{st}\bar{f}(s)ds$$
$$= 2\pi i \times [\text{sum of the residues of } e^{st}\bar{f}(s) \text{ at the poles inside } C]. \quad (3.7.8)$$

Letting $R \to \infty$, the integral over $\Gamma$ tends to zero, and this is true in most problems of interest. Consequently, result (3.7.7) reduces to the form

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{c-iR}^{c+iR} e^{st}\bar{f}(s)ds = \text{sum of the residues of } e^{st}\bar{f}(s) \text{ at the poles of } \bar{f}(s). \quad (3.7.9)$$

We illustrate the above method of evaluation by simple examples. ∎

***Example 3.7.12***
If $\bar{f}(s)=\frac{s}{s^2+a^2}$, show that

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)ds=\cos at.$$

Clearly, the integrand has two simple poles at $s=\pm ia$ and the residues at these poles are

$$\begin{aligned}R_1 &= \text{Residue of } e^{st}\bar{f}(s) \text{ at } s=ia\\ &= \lim_{s\to ia}(s-ia)\frac{s\,e^{st}}{(s^2+a^2)}=\frac{1}{2}e^{iat}.\end{aligned}$$

$$\begin{aligned}R_2 &= \text{Residue of } e^{st}\bar{f}(s) \text{ at } s=-ia\\ &= \lim_{s\to -ia}(s+ia)\frac{s\,e^{st}}{(s^2+a^2)}=\frac{1}{2}e^{-iat}.\end{aligned}$$

Hence,

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)ds=R_1+R_2=\frac{1}{2}(e^{iat}+e^{-iat})=\cos at,$$

as obtained earlier.

If $\bar{g}(s)=e^{st}\bar{f}(s)$ has a pole of order $n$ at $s=z$, then the residue $R_1$ of $\bar{g}(s)$ at this pole is given by the formula

$$R_1=\lim_{s\to z}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-z)^n\bar{g}(s)]. \tag{3.7.10}$$

This is obviously true for a simple pole $(n=1)$ and for a double pole $(n=2)$. ▯

***Example 3.7.13***
Evaluate

$$\mathscr{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}.$$

Clearly

$$\bar{g}(s)=e^{st}\bar{f}(s)=\frac{s\,e^{st}}{(s^2+a^2)^2}$$

has double poles at $s=\pm ia$. The residue formula (3.7.10) for double poles gives

$$\begin{aligned}R_1 &= \lim_{s\to ia}\frac{d}{ds}\left[(s-ia)^2\frac{s\,e^{st}}{(s^2+a^2)^2}\right]\\ &= \lim_{s\to ia}\frac{d}{ds}\left[\frac{s\,e^{st}}{(s+ia)^2}\right]=\frac{t\,e^{iat}}{4ia}.\end{aligned}$$

Similarly, the residue at the double pole at $s=-ia$ is $(-t\,e^{-iat})/4ia$.

Thus,

$$f(t)=\text{Sum of the residues}=\frac{t}{4ia}(e^{iat}-e^{-iat})=\frac{t}{2a}\sin at, \tag{3.7.11}$$

as given in Table B-4 of Laplace transforms. ▯

***Example 3.7.14***

Evaluate

$$\mathscr{L}^{-1}\left\{\frac{\cosh(\alpha x)}{s\cosh(\alpha\ell)}\right\}, \qquad \alpha=\sqrt{\frac{s}{a}}.$$

We have

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\frac{\cosh(\alpha x)}{\cosh(\alpha\ell)}\,\frac{ds}{s}.$$

Clearly, the integrand has simple poles at $s=0$ and $s=s_n=-(2n+1)^2\dfrac{a\pi^2}{4\ell^2}$, where $n=$ 0,1,2,....

$R_1=$ Residue at the pole $s=0$ is 1, and $R_n=$ Residue at the pole $s=s_n$ is

$$\frac{\exp(-s_n t)\cosh\left\{i(2n+1)\dfrac{\pi x}{2\ell}\right\}}{\left[s\dfrac{d}{ds}\left\{\cosh l\sqrt{\dfrac{s}{a}}\right\}\right]_{s=s_n}}$$
$$=\frac{4(-1)^{n+1}}{(2n+1)\pi}\exp\left[-\left\{\frac{(2n+1)\pi}{2\ell}\right\}^2 at\right]\cos\left\{(2n+1)\frac{\pi x}{2\ell}\right\}.$$

Thus,

$$\begin{aligned}f(t)&=\text{Sum of the residues at the poles}\\&=1+\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)}\exp\left[-(2n+1)^2\frac{\pi^2 at}{4\ell^2}\right]\\&\qquad\times\cos\left\{(2n+1)\frac{\pi x}{2\ell}\right\},\end{aligned} \tag{3.7.12}$$

as given later by the Heaviside Expansion Theorem. ▯

***Example 3.7.15***

Show that

$$f(t)=\mathscr{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s}\exp(st-a\sqrt{s})ds$$
$$=erfc\left(\frac{a}{2\sqrt{t}}\right). \tag{3.7.13}$$

The integrand has a branch point at $s=0$. We use the contour of integration as shown in Figure 3.4(b) which excludes the branch point at $s=0$. Thus, the Cauchy Fundamental Theorem gives

$$\frac{1}{2\pi i}\left[\int_L+\int_\Gamma+\int_{L_1}+\int_{L_2}+\int_\gamma\right]\exp(st-a\sqrt{s})\,\frac{ds}{s}=0. \tag{3.7.14}$$

It is shown that the integral on $\Gamma$ tends to zero as $R\to\infty$, and that on $L$ gives the Bromwich integral. We now evaluate the remaining three integrals in (3.7.14). On $L_1$, we have $s=re^{i\pi}=-r$ and

$$\int_{L_1}\exp(st-a\sqrt{s})\,\frac{ds}{s}=\int_{-\infty}^{0}\exp(st-a\sqrt{s})\,\frac{ds}{s}=-\int_0^\infty\exp\{-(rt+ia\sqrt{r})\}\,\frac{dr}{r}.$$

On $L_2$, $s=re^{-i\pi}=-r$ and

$$\int_{L_2}\exp(st-a\sqrt{s})\frac{ds}{s}=\int_0^{-\infty}\exp(st-a\sqrt{s})\,\frac{ds}{s}=\int_0^\infty\exp\{-rt+ia\sqrt{r}\}\,\frac{dr}{r}.$$

Thus, the integrals along $L_1$ and $L_2$ combined yield

$$-2i\int_0^\infty e^{-rt}\sin(a\sqrt{r})\,\frac{dr}{r}=-4i\int_0^{-\infty}e^{-x^2t}\,\frac{\sin ax}{x}dx,\qquad(\sqrt{r}=x). \tag{3.7.15}$$

Integrating the following standard integral with respect to $\beta$

$$\int_0^\infty e^{-x^2\alpha^2}\cos(2\beta x)dx=\frac{\sqrt{\pi}}{2\alpha}\exp\left(-\frac{\beta^2}{\alpha^2}\right), \tag{3.7.16}$$

we obtain

$$\frac{1}{2}\int_0^\infty e^{-x^2\alpha^2}\,\frac{\sin 2\beta x}{x}dx = \frac{\sqrt{\pi}}{2\alpha}\int_0^\beta \exp\left(-\frac{\beta^2}{\alpha^2}\right)d\beta$$

$$= \frac{\sqrt{\pi}}{2}\int_0^{\beta/\alpha} e^{-u^2}\,du, \qquad (\beta=\alpha u)$$

$$= \frac{\pi}{4}\,erf\left(\frac{\beta}{\alpha}\right). \tag{3.7.17}$$

In view of (3.7.17), result (3.7.15) becomes

$$-4i\int_0^\infty \exp(-tx^2)\,\frac{\sin ax}{x}\,dx = -2\pi i\; erf\left(\frac{a}{2\sqrt{t}}\right). \tag{3.7.18}$$

Finally, on $\gamma$, we have $s=re^{i\theta}$, $ds=ire^{i\theta}d\theta$, and

$$\int_\gamma |\exp(st-a\sqrt{s})|\frac{ds}{s} = i\int_\pi^{-\pi} \exp\left(rt\cos\theta - a\sqrt{r}\cos\frac{\theta}{2}\right)d\theta$$

$$= i\int_{-\pi}^{\pi} d\theta = 2\pi i, \tag{3.7.19}$$

in which the limit as $r\to 0$ is used and integration from $\pi$ to $-\pi$ is interchanged to make $\gamma$ in the counterclockwise direction.

Thus, the final result follows from (3.7.14), (3.7.18), and (3.7.19) in the form

$$\mathscr{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(st-a\sqrt{s})\,\frac{ds}{s}$$

$$= \left[1 - erf\left(\frac{a}{2\sqrt{t}}\right)\right] = erfc\left(\frac{a}{2\sqrt{t}}\right).$$

(iv) *Heaviside's Expansion Theorem*

Suppose $\bar{f}(s)$ is the Laplace transform of $f(t)$, which has a Maclaurin power series expansion in the form

$$f(t) = \sum_{r=0}^\infty a_r\,\frac{t^r}{r!}. \tag{3.7.20}$$

Taking the Laplace transform, it is possible to write formally

$$\bar{f}(s) = \sum_{r=0}^\infty \frac{a_r}{s^{r+1}}. \tag{3.7.21}$$

Conversely, we can derive (3.7.20) from a given expansion (3.7.21). This kind of expansion is useful for determining the behavior of the solution for small time. Further, it provides an alternating way to prove the Tauberian theorems. ▯

***THEOREM 3.7.1***

(*Heaviside's Expansion Theorem*). If $\bar{f}(s) = \dfrac{\bar{p}(s)}{\bar{q}(s)}$, where $\bar{p}(s)$ and $\bar{q}(s)$ are polynomials in $s$ and the degree of $\bar{q}(s)$ is higher than that of $\bar{p}(s)$, then

$$\mathscr{L}^{-1}\left\{\frac{\bar{p}(s)}{\bar{q}(s)}\right\} = \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} \exp(t\alpha_k), \tag{3.7.22}$$

where $\alpha_k$ are the distinct roots of the equation $\bar{q}(s) = 0$.

**PROOF** Without loss of generality, we can assume that the leading coefficient of $\bar{q}(s)$ is unity and write distinct factors of $\bar{q}(s)$ so that

$$\bar{q}(s) = (s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_k)\cdots(s-\alpha_n). \tag{3.7.23}$$

Using the rules of partial fraction decomposition, we can write

$$\bar{f}(s) = \frac{\bar{p}(s)}{\bar{q}(s)} = \sum_{k=1}^{n} \frac{A_k}{(s-\alpha_k)}, \tag{3.7.24}$$

where $A_k$ are arbitrary constants to be determined. In view of (3.7.23), we find

$$\bar{p}(s) = \sum_{k=1}^{n} A_k (s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_{k-1})(s-\alpha_{k+1})\cdots(s-\alpha_n).$$

Substitution of $s = \alpha_k$ gives

$$\bar{p}(\alpha_k) = A_k(\alpha_k-\alpha_1)(\alpha_k-\alpha_2)\cdots(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n), \tag{3.7.25}$$

where $k = 1, 2, 3, \ldots, n$.

Differentiation of (3.7.23) yields

$$\bar{q}'(s) = \sum_{k=1}^{n} (s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_{k-1})(s-\alpha_{k+1})\cdots(s-\alpha_n),$$

whence it follows that

$$\bar{q}'(\alpha_k) = (\alpha_k-\alpha_1)(\alpha_k-\alpha_2)\cdots(\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n). \tag{3.7.26}$$

From (3.7.25) and (3.7.26), we find

$$A_k = \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)},$$

and hence,

$$\frac{\bar{p}(s)}{\bar{q}(s)} = \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} \frac{1}{(s-\alpha_k)}. \tag{3.7.27}$$

Inversion gives immediately

$$\mathscr{L}^{-1}\left\{\frac{\bar{p}(s)}{\bar{q}(s)}\right\} = \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} \exp(t\alpha_k).$$

This proves the theorem. We give some examples of this theorem. ∎

***Example 3.7.16***
We consider

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2-3s+2}\right\}.$$

Here $\bar{p}(s) = s$, and $\bar{q}(s) = s^2 - 3s + 2 = (s-1)(s-2)$. Hence,

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2-3s+2}\right\} = \frac{\bar{p}(2)}{\bar{q}'(2)}e^{2t} + \frac{\bar{p}(1)}{\bar{q}'(1)}e^{t} = 2\,e^{2t} - e^{t}.$$

□

***Example 3.7.17***
Use Heaviside's power series expansion to evaluate

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}}\right\}, \qquad 0 < x < 1, \qquad s > 0.$$

We have

$$\begin{aligned}
\frac{1}{s}\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}} &= \frac{1}{s}\left(\frac{e^{x\sqrt{s}} - e^{-x\sqrt{s}}}{e^{\sqrt{s}} - e^{-\sqrt{s}}}\right) \\
&= \frac{1}{s}\frac{e^{-(1-x)\sqrt{s}} - e^{-(1+x)\sqrt{s}}}{1 - e^{-2\sqrt{s}}} \\
&= \frac{1}{s}\left[e^{-(1-x)\sqrt{s}} - e^{-(1+x)\sqrt{s}}\right]\left(1 - e^{-2\sqrt{s}}\right)^{-1} \\
&= \frac{1}{s}\left[e^{-(1-x)\sqrt{s}} - e^{-(1+x)\sqrt{s}}\right]\sum_{n=0}^{\infty}\exp(-2n\sqrt{s}) \\
&= \frac{1}{s}\sum_{n=0}^{\infty}\left[\exp\{-(1-x+2n)\sqrt{s}\} - \exp\{-(1+x+2n)\sqrt{s}\}\right].
\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{L}^{-1} & \left\{\frac{1}{s}\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}}\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{1}{s}\sum_{n=0}^{\infty}\left[\exp\{-(1-x+2n)\sqrt{s}\} - \exp\{-(1+x+2n)\sqrt{s}\}\right]\right\} \\ &= \sum_{n=0}^{\infty}\left[erfc\left(\frac{1-x+2n}{2\sqrt{t}}\right) - erfc\left(\frac{1+x+2n}{2\sqrt{t}}\right)\right].\end{aligned}$$

▯

***Example 3.7.18***

If $\alpha = \sqrt{\dfrac{s}{a}}$, show that

$$\mathscr{L}^{-1}\left[\frac{\cosh\alpha x}{s\cosh\alpha\ell}\right] = 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k\cos\left\{\left(k+\frac{1}{2}\right)\frac{\pi x}{\ell}\right\}\exp\left[-(2k+1)^2\frac{a\pi^2 t}{4\ell^2}\right]}{(2k+1)}. \tag{3.7.28}$$

In this case, we write

$$\mathscr{L}^{-1}\{\bar{f}(s)\} = \mathscr{L}^{-1}\left\{\frac{\bar{p}(s)}{\bar{q}(s)}\right\} = \mathscr{L}^{-1}\left\{\frac{\cosh\alpha x}{s\cosh\alpha\ell}\right\}.$$

Clearly, the zeros of $\bar{f}(s)$ are at $s=0$ and at the roots of $\cosh\alpha\ell = 0$, that is, at $s = s_k = a\left(k+\frac{1}{2}\right)^2\left(\frac{\pi i}{\ell}\right)^2$, $k = 0, 1, 2, \ldots$. Thus,

$$\alpha_k = \sqrt{\frac{s_k}{a}} = \left(k+\frac{1}{2}\right)\frac{\pi i}{\ell}, \qquad k = 0, 1, 2, \ldots.$$

Here $\bar{p}(s) = \cosh(\alpha x)$, $\bar{q}(s) = s\cosh(\alpha\ell)$. In order to apply the Heaviside Expansion Theorem, we need

$$\bar{q}'(s) = \frac{d}{ds}(s\cosh\alpha\ell) = \cosh(\alpha\ell) + \frac{1}{2}\alpha\ell\sinh(\alpha\ell).$$

For the zero $s = 0$, $\bar{q}'(0) = 1$, and for the zeros at $s = s_k$,

$$\begin{aligned}\bar{q}'(s_k) &= \frac{1}{2}\left(k+\frac{1}{2}\right)\pi i\cdot\sinh\left[\left(k+\frac{1}{2}\right)\pi i\right] \\ &= (2k+1)\frac{\pi i}{4}\cdot i\sin\left[\left(k+\frac{1}{2}\right)\pi\right] \\ &= -(2k+1)\frac{\pi}{4}\cdot\cos k\pi = (-1)^{k+1}(2k+1)\frac{\pi}{4}.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{\cosh \alpha x}{s\cosh \alpha\ell}\right\} &= 1+\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)}\cosh\left[(2k+1)\frac{\pi i x}{2\ell}\right]\exp(ts_k)\\ &= 1-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2k+1)}\cos\left[(2k+1)\frac{\pi x}{2\ell}\right]\\ &\qquad\times\exp\left[-\left(k+\frac{1}{2}\right)^2\frac{\pi^2 a t}{\ell^2}\right].\end{aligned}$$

▯

## 3.8 Tauberian Theorems and Watson's Lemma

These theorems give the behavior of object functions in terms of the behavior of transform functions. Particularly, they determine the value of the object functions $f(t)$ for large and small values of time $t$. Tauberian theorems are extremely useful and have frequent applications.

***THEOREM 3.8.1*** **(***The Initial Value Theorem***).**
If $\mathscr{L}\{f(t)\}=\bar{f}(s)$ exists, then

$$\lim_{s\to\infty}\bar{f}(s)=0. \tag{3.8.1}$$

In addition, if $f(t)$ and its derivatives exist as $t\to 0$, we obtain the *Initial Value Theorem*:

$$(i)\quad \lim_{s\to\infty}[s\bar{f}(s)]=\lim_{t\to 0}f(t)=f(0) \tag{3.8.2}$$

$$(ii)\quad \lim_{s\to\infty}[s^2\bar{f}(s)-sf(0)]=\lim_{t\to 0}f'(t)=f'(0), \quad \text{and} \tag{3.8.3}$$

$$(iii)\quad \lim_{s\to\infty}[s^{n+1}\bar{f}(s)-s^n\bar{f}(s)-\cdots-sf^{(n-1)}(0)]=f^{(n)}(0). \tag{3.8.4}$$

Results (3.8.2)–(3.8.4), which are true under fairly general conditions, determine the initial values $f(0), f'(0), \ldots, f^{(n)}(0)$ of the function $f(t)$ and its derivatives from the Laplace transform $\bar{f}(s)$.

**PROOF** To prove (3.8.1), we use the fact that the Laplace integral (3.2.5) is uniformly convergent with respect to the parameter $s$. Hence, it is permissible to take the limit $s \to \infty$ under the sign of integration so that

$$\lim_{s\to\infty} \bar{f}(s) = \int_0^\infty (\lim_{s\to\infty} e^{-st}) f(t) dt = 0.$$

Next, we use the same argument to obtain

$$\lim_{s\to\infty} \mathscr{L}\{f'(t)\} = \int_0^\infty (\lim_{s\to\infty} e^{-st}) f'(t) dt = 0.$$

Then it follows from result (3.4.10) that

$$\lim_{s\to\infty} [s\bar{f}(s) - f(0)] = 0,$$

and hence, we obtain (3.8.2), that is,

$$\lim_{s\to\infty} [s\bar{f}(s)] = f(0) = \lim_{t\to 0} f(t).$$

A similar argument combined with Theorem 3.4.2 leads to (3.8.3) and (3.8.4). ∎

***Example 3.8.1***
Verify the truth of Theorem 3.8.1 for $\bar{f}(s) = (n+1)!\, s^{-(n+1)}$ where $n$ is a positive integer. Clearly, $f(t) = t^n$. Thus, we have

$$\begin{aligned} \lim_{s\to\infty} \bar{f}(s) &= \lim_{s\to\infty} \frac{(n+1)!}{s^{n+1}} = 0, \\ \lim_{s\to\infty} s\bar{f}(s) &= 0 = f(0). \end{aligned}$$

□

***Example 3.8.2***
Find $f(0)$ and $f'(0)$ when

(a) $\bar{f}(s) = \dfrac{1}{s(s^2+a^2)}$, (b) $\bar{f}(s) = \dfrac{2s}{s^2 - 2s + 5}$.

(a) It follows from (3.8.2) and (3.8.3) that

$$\begin{aligned} f(0) &= \lim_{s\to\infty} [s\bar{f}(s)] = \lim_{s\to\infty} \frac{1}{s^2+a^2} = 0. \\ f'(0) &= \lim_{s\to\infty} [s^2\bar{f}(s) - sf(0)] = \lim_{s\to\infty} \frac{s}{s^2+a^2} = 0. \end{aligned}$$

(b)
$$f(0) = \lim_{s\to\infty} \frac{2s^2}{s^2-2s+5} = 2.$$
$$f'(0) = \lim_{s\to\infty} [s^2 \bar{f}(s) - sf(0)] = \lim_{s\to\infty} \left[\frac{2s^3}{s^2-2s+5} - 2s\right] = 4.$$

▯

***THEOREM 3.8.2*** **(***The Final Value Theorem***).**
If $\bar{f}(s) = \dfrac{\bar{p}(s)}{\bar{q}(s)}$, where $\bar{p}(s)$ and $\bar{q}(s)$ are polynomials in $s$, and the degree of $\bar{p}(s)$ is less than that of $\bar{q}(s)$, and if all roots of $\bar{q}(s) = 0$ have negative real parts with the possible exception of one root which may be at $s = 0$, then

$$\text{(i)} \quad \lim_{s\to 0} \bar{f}(s) = \int_0^\infty f(t)dt, \qquad \text{and} \tag{3.8.5}$$

$$\text{(ii)} \quad \lim_{s\to 0} [s\bar{f}(s)] = \lim_{t\to\infty} f(t), \tag{3.8.6}$$

provided the limits exist.

Result (3.8.6) is true under more general conditions, and known as the *Final Value Theorem.* This theorem determines the final value of $f(t)$ at infinity from its Laplace transform at $s = 0$. However, if $\bar{f}(s)$ is more general than the rational function as stated above, a statement of a more general theorem is needed with appropriate conditions under which it is valid.

**PROOF** To prove (i), we use the same argument as employed in Theorem 3.8.1 and find

$$\lim_{s\to 0} \bar{f}(s) = \int_0^\infty \left(\lim_{s\to 0} \exp(-st)\right) f(t)dt = \int_0^\infty f(t)\,dt.$$

As before, we can use result (3.4.12) to obtain

$$\lim_{s\to 0} \mathscr{L}\{f'(t)\} = \lim_{s\to 0} [s\bar{f}(s) - f(0)] = \int_0^\infty \left(\lim_{s\to 0} \exp(-st)\right) f'(t)dt$$
$$= \int_0^\infty f'(t)dt = f(\infty) - f(0) = \lim_{t\to\infty} [f(t) - f(0)].$$

Thus, it follows immediately that

$$\lim_{s\to 0} [s\bar{f}(s)] = \lim_{t\to\infty} f(t) = f(\infty).$$

▮

***Example 3.8.3***

Find $f(\infty)$, if it exists, from the following functions:

(a) $\bar{f}(s) = \dfrac{1}{s(s^2+2s+2)}$, (b) $\bar{f}(s) = \dfrac{1}{s-a}$,

(c) $\bar{f}(s) = \dfrac{s+a}{s^2+b^2}, (b \neq 0)$, (d) $\bar{f}(s) = \dfrac{s}{s-2}$.

(a) Clearly, $\bar{q}(s) = 0$ has roots at $s = 0$ and $s = -1 \pm i$, and the conditions of Theorem 3.8.2 are satisfied. Thus,

$$\lim_{s \to 0}[s\bar{f}(s)] = \lim_{s \to 0} \frac{1}{s^2+2s+2} = \frac{1}{2} = f(\infty).$$

(b) Here $\bar{q}(s) = 0$ has a real positive root at $s = a$ if $a > 0$, and a real negative root if $a < 0$. Thus, when $a < 0$

$$\lim_{s \to 0}[s\bar{f}(s)] = \lim_{s \to 0} \frac{s}{s-a} = 0 = f(\infty).$$

If $a > 0$, the Final Value Theorem does not apply. In fact,

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \to \infty \qquad \text{as } t \to \infty.$$

(c) Here $\bar{q}(s) = 0$ has purely imaginary roots at $s = \pm ib$ which do not have negative real parts. The Final Value Theorem does not apply. In fact, $f(t) = \cos bt + \frac{a}{b}\sin bt$ and $\lim_{t \to \infty} f(t)$ does not exist. However, $f(t)$ is bounded and oscillatory for all $t > 0$.

(d) The Final Value Theorem does not apply as $\bar{q}(s) = 0$ has a positive root at $s = 2$.

□

***Watson's Lemma.*** If (i) $f(t) = O(e^{at})$ as $t \to \infty$, that is, $|f(t)| \leq K\ \exp(at)$ for $t > T$ where $K$ and $T$ are constants, and (ii) $f(t)$ has the expansion

$$f(t) = t^{\alpha}\left[\sum_{r=0}^{n} a_r t^r + R_{n+1}(t)\right] \text{ for } 0 < t < T \text{ and } \alpha > -1, \tag{3.8.7}$$

where $|R_{n+1}(t)| < A\, t^{n+1}$ for $0 < t < T$ and $A$ is a constant, then the Laplace transform $\bar{f}(s)$ has the *asymptotic expansion*

$$\bar{f}(s) \sim \sum_{r=0}^{n} a_r \frac{\Gamma(\alpha+r+1)}{s^{\alpha+r+1}} + O\left(\frac{1}{s^{\alpha+n+2}}\right) \qquad \text{as } s \to \infty. \tag{3.8.8}$$

**PROOF** We have, for $s > a$,

$$\begin{aligned}\bar{f}(s) &= \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt \\ &= \int_0^T e^{-st}\, t^\alpha \left(\sum_{r=0}^n a_r t^r\right) dt + \int_0^T e^{-st}\, t^\alpha R_{n+1}(t)dt \\ &\qquad + \int_T^\infty e^{-st} f(t)dt. \end{aligned} \tag{3.8.9}$$

The general term of the first integral in (3.8.9) can be written as

$$\begin{aligned}\int_0^T a_r\, e^{-st} t^{\alpha+r} dt &= \int_0^\infty a_r\, e^{-st} t^{\alpha+r} dt - \int_T^\infty a_r\, e^{-st} t^{\alpha+r} dt \\ &= a_r \frac{\Gamma(\alpha+r+1)}{s^{\alpha+r+1}} + O(e^{-Ts}). \end{aligned} \tag{3.8.10}$$

As $s \to \infty$, the second integral in(3.8.9) is less in magnitude than

$$A \int_0^T e^{-st} t^{\alpha+n+1} dt = O\left(\frac{1}{s^{\alpha+n+2}}\right), \tag{3.8.11}$$

and the magnitude of the third integral in (3.8.9) is

$$\left|\int_T^\infty e^{-st} f(t)dt\right| \le K \int_T^\infty e^{-(s-a)t} dt = K \exp[-(s-a)T], \tag{3.8.12}$$

which is exponentially small as $s \to \infty$.

Finally, combining (3.8.10), (3.8.11), and (3.8.12), we obtain

$$\bar{f}(s) \sim \sum_{r=0}^n a_r \frac{\Gamma(\alpha+r+1)}{s^{\alpha+r+1}} + O\left(\frac{1}{s^{\alpha+n+2}}\right) \qquad \text{as } s \to \infty.$$

This completes the proof of Watson's lemma. ∎

This lemma is one of the most widely used methods for finding asymptotic expansions. In order to further expand its applicability, this lemma has subsequently been generalized and its converse has also been proved. The reader is referred to Erdélyi (1956), Copson (1965), Wyman (1964), Watson (1981), Ursell (1990), and Wong (1989).

***Example 3.8.4***
Find the asymptotic expansion of the *parabolic cylinder function* $D_\nu(s)$, which is valid for $\mathrm{Re}(\nu) < 0$, given by

$$D_\nu(s) = \frac{\exp\left(-\frac{s^2}{4}\right)}{\Gamma(-\nu)} \int_0^\infty \exp\left[-\left(st + \frac{t^2}{2}\right)\right] \frac{dt}{t^{\nu+1}}. \tag{3.8.13}$$

To find the asymptotic behavior of $D_\nu(s)$ as $s \to \infty$, we expand $\exp\left(-\frac{1}{2}t^2\right)$ as a power series in $t$ in the form

$$\exp\left(-\frac{1}{2}t^2\right) = \sum_{n=0}^\infty (-1)^n \frac{t^{2n}}{2^n\, n!}. \tag{3.8.14}$$

According to Watson's lemma, as $s \to \infty$,

$$\begin{aligned} D_\nu(s) &\sim \frac{\exp\left(-\frac{s^2}{4}\right)}{\Gamma(-\nu)} \sum_{n=0}^\infty \frac{(-1)^n}{2^n n!} \int_0^\infty t^{2n-\nu-1} e^{-st}\, dt \\ &= \frac{\exp\left(-\frac{s^2}{4}\right)}{\Gamma(-\nu)} \sum_{n=0}^\infty \frac{(-1)^n}{2^n n!} \frac{\Gamma(2n-\nu)}{s^{2n-\nu}}. \end{aligned} \tag{3.8.15}$$

This result is also valid for $\mathrm{Re}(\nu) \geq 0$. ▯

## 3.9 Exercises

1. Find the Laplace transforms of the following functions:

   (a) $2t + a \sin at$,
   (b) $(1-2t)\exp(-2t)$,
   (c) $t \cos at$,
   (d) $t^{3/2}$,
   (e) $H(t-3)\exp(t-3)$,
   (f) $H(t-a)\sinh(t-a)$,
   (g) $(t-3)^2 H(t-3)$,
   (h) $t\, H(t-a)$,
   (i) $(1+2\,at)\, t^{-\frac{1}{2}} \exp(at)$,
   (j) $a\, \cos^2 \omega t$.

2. If $n$ is a positive integer, show that $\mathscr{L}\{t^{-n}\}$ does not exist.

3. Use result (3.4.12) to find (a) $\mathscr{L}\{\cos at\}$ and (b) $\mathscr{L}\{\sin at\}$.

4. Use the Maclaurin series for $\sin at$ and $\cos at$ to find the Laplace transforms of these functions.

5. Show that $\mathscr{L}\left[\frac{1}{t}\{\exp(-at)-\exp(-bt)\}\right]=\log\left(\frac{s+b}{s+a}\right)$.

6. Show that $\mathscr{L}\left\{\int_0^t \frac{s(u)}{u}du\right\}=\frac{1}{s}\int_s^\infty \bar{f}(x)dx.$

7. Obtain the inverse Laplace transforms of the following functions:

(a) $\frac{s}{(s^2+a^2)(s^2+b^2)}$, (b) $\frac{1}{s^2(s^2+c^2)}$, (c) $\frac{1}{s^2}\exp(-as)$,

(d) $\frac{1}{(s-1)^2(s-2)}$, (e) $\frac{1}{s^2+2s+5}$, (f) $\frac{1}{s^2(s+1)(s+2)}$,

(g) $\frac{1}{s(s-a)^2}$, (h) $\frac{1}{s^2(s-a)^2}$, (i) $\frac{1}{s^2(s-a)}$.

8. Use the Convolution Theorem to find the inverse Laplace transforms of the following functions:

(a) $\frac{s^2}{(s^2+a^2)^2}$, (b) $\frac{1}{s\sqrt{s+4}}$, (c) $\frac{\bar{f}(s)}{s}$,

(d) $\frac{s}{(s^2+a^2)^2}$, (e) $\left(\frac{\omega}{s^2+\omega^2}\right)\bar{f}(s)$, (f) $\frac{1}{(s^2+a^2)^2}$,

(g) $\frac{s}{(s-a)(s^2+b^2)}$, (h) $\frac{1}{(s+1)^2}$, (i) $\frac{1}{s}\exp(-a\sqrt{s})$,

(j) $\frac{1}{s^2(s^2+a^2)}$, (k) $\frac{(s^2-a^2)}{(s^2+a^2)^2}$, (l) $\frac{1}{2}\ln(1+\frac{a^2}{s^2})$.

9. Show that

(a) $\mathscr{L}\{\exp(-t^2)\}=\frac{\sqrt{\pi}}{2}\exp\left(\frac{s^2}{4}\right)\left(1-erf\,\frac{s}{2}\right)$,

(b) $\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}-\sqrt{a}}\right\}=\sqrt{a}\exp(at)+\frac{1}{\sqrt{\pi t}}+\sqrt{a}\exp(at)erf(\sqrt{at})$,

(c) $$\mathscr{L}^{-1}\left\{\frac{\sinh\left(\frac{sx}{a}\right)}{s^2\cosh\left(\frac{sb}{2a}\right)}\right\}=\frac{x}{a}+\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{4b}{a\pi^2}\right)(2n+1)^{-2}$$
$$\times\left[\sin\left\{(2n+1)\frac{\pi x}{b}\right\}\cos\left\{(2n+1)\frac{\pi at}{b}\right\}\right],$$

(d) $\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s^2+a^2}}\right\}=\frac{1}{\pi}\int_{-1}^{1}\frac{e^{iatx}}{\sqrt{1-x^2}}dx.$

10. Show that

(a) $\mathscr{L}\left\{\frac{1}{t}(\sin at - at\cos at)\right\} = \tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2},$

(b) $\mathscr{L}\left\{\int_0^t \frac{1}{\tau}(\sin a\tau - a\tau\cos a\tau)d\tau\right\} = \frac{1}{s}\left[\tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2}\right].$

11. Using the Heaviside power series expansion, evaluate the inverse Laplace transforms of the following functions:

(a) $\frac{1}{\sqrt{s^2+a^2}},$ (b) $\tan^{-1}\left(\frac{a}{s}\right),$ (c) $\sinh^{-1}\left(\frac{1}{s}\right),$

(d) $\frac{1}{s}\,\mathrm{cosech}\,(x\sqrt{s}),$ (e) $\frac{1}{s}\exp\left(-\frac{1}{s}\right),$ (f) $\sin^{-1}\left(\frac{a}{s}\right).$

12. If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, show that

(i) $\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(\tau)d\tau,$

(ii) $\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \left\{\int_0^{t_1} f(\tau)d\tau\right\} dt_1 = \int_0^t (t-\tau)f(\tau)d\tau,$

(iii) $\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^3}\right\} = \int_0^t\int_0^{t_1}\int_0^{t_2} f(\tau)d\tau dt_1\, dt_2 = \int_0^t \frac{1}{2}(t-\tau)^2 f(\tau)d\tau,$

and in general

(iv) $$\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^n}\right\} = \int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{n-1}} f(\tau)d\tau\, dt_1\cdots dt_{n-1}$$
$$= \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau)d\tau.$$

13. The staircase function $f(t) = [t]$ represents the greatest integer less than or equal to $t$. Find its Laplace transform.

14. Use the convolution theorem to prove the identity

$$\int_0^t J_0(\tau)J_0(t-\tau)d\tau = \sin t.$$

15. Show that

(a) $\mathscr{L}\{t\,H(t-a)\} = \left(\frac{1}{s^2} + \frac{a}{s}\right)\exp(-sa)$,

(b) $\mathscr{L}\{t^n \exp(at)\} = n!(s-a)^{-(n+1)}$.

16. If $\mathscr{L}\{f(t)\} = \bar{f}(s)$ and $f(t)$ has a finite discontinuity at $t=a$, show that

$$\mathscr{L}\{f'(t)\} = s\bar{f}(s) - f(0) - \exp(-sa)[f]_a,$$

where $[f]_a = f(a+0) - f(a-0)$.

17. If $f(t) = H\left(t - \frac{\pi}{2}\right)\sin t$, find its Laplace transforms.

18. Establish the following results:

(a) $\mathscr{L}\{\sin^2 at\} = \frac{2a^2}{s(s^2+4a^2)}$,

(b) $\mathscr{L}\{I_0(x)\} = \frac{1}{\sqrt{s^2+a^2}}$,

(c) $\mathscr{L}\{|\sin at|\} = \frac{a}{s^2+a^2}\coth\left(\frac{\pi s}{2a}\right)$, $s>0$,

(d) $\mathscr{L}\left\{\int_0^t \frac{\sin ax}{x}dx\right\} = \frac{1}{s}\tan^{-1}\left(\frac{a}{s}\right)$,

(e) $\mathscr{L}\left\{\frac{d}{dt}(f*g)\right\} = g(0)\bar{f}(s) + \mathscr{L}\{f*g'\} = s\bar{f}(s)\bar{g}(s)$
$= \mathscr{L}\{f'*g\} + f(0)\bar{g}(s)$.

19. Establish the following results:

(a) $\mathscr{L}\{t^2 f''(t)\} = s^2\frac{d^2}{ds^2}\bar{f}(s) + 4s\frac{d}{ds}\bar{f}(s) + 2\bar{f}(s)$,

(b) $\mathscr{L}\{t^m f^{(n)}(t)\} = (-1)^m \frac{d^m}{ds^m}\left[s^n\bar{f}(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)\right]$.

20. (a) Show that $f(t) = \sin(a\sqrt{t})$ satisfies the differential equation

$$4t\ f''(t) + 2f'(t) + a^2 f(t) = 0.$$

Use this differential equation to show that

(b) $\mathscr{L}\{\sin\sqrt{t}\} = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)s^{-3/2}\exp\left(-\frac{1}{4s}\right)$, $s>0$,

(c) $\mathscr{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \Gamma\left(\frac{1}{2}\right)\frac{1}{\sqrt{s}}\exp\left(-\frac{1}{4s}\right)$, $s>0$.

21. Establish the following results:

(a) $\mathscr{L}\left\{\int_t^\infty \frac{f(x)}{x}dx\right\} = \frac{1}{s}\int_0^s \bar{f}(x)dx,$

(b) $\mathscr{L}\left\{\int_0^\infty \frac{f(x)}{x}dx\right\} = \frac{1}{s}\int_0^\infty \bar{f}(x)dx.$

22. Use exercise 21(a) to find the Laplace transform of

(a) the *cosine integral* defined by

$$Ci(t) = \int_\infty^t \frac{\cos x}{x}dx, \quad t > 0,$$

(b) the *exponential integral* defined by

$$\mathrm{E}i(t) = \int_t^\infty \frac{e^{-x}}{x}dx, \; t > 0.$$

23. Show that

(a) $\mathscr{L}\{t\,e^{-bt}\cos at\} = \frac{(s+b)^2 - a^2}{[(s+b)^2 + a^2]^2},$

(b) $\mathscr{L}\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right),$

(c) $\mathscr{L}\{L_n(t)\} = \frac{1}{s}\left(\frac{s-1}{s}\right)^n$, where $L_n(t)$ are the Laguerre polynomials of degree $n$.

24. If $\mathscr{L}\{f(t)\} = \bar{f}(s)$ and $\mathscr{L}\{g(x,t)\} = \bar{h}(s)\exp\{-x\bar{h}(s)\}$, prove that

(a) $\mathscr{L}\left\{\int_0^\infty g(x,t)f(x)dx\right\} = \bar{h}(s)\bar{f}\{\bar{h}(s)\}.$

(b) $\mathscr{L}\left\{\int_0^\infty J_0(2\sqrt{xt})f(x)dx\right\} = \frac{1}{s}\,\bar{f}\left(\frac{1}{s}\right)$, when $g(x,t) = J_0(2\sqrt{xt})$.

25. Use Exercise 24(b) to show that

(a) $\int_0^\infty J_0(2\sqrt{xt})\sin\left(\frac{x}{a}\right)dx = a\cos at, \qquad (a \neq 0),$

(b) $\int_0^\infty J_0(2\sqrt{xt})e^{-x}x^n\,dx = n!\,e^{-t}L_n(t)$.

26. Find the Laplace transform of the *triangular wave* function defined over $(0, 2a)$ by

$$f(t) = \begin{Bmatrix} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{Bmatrix}.$$

27. Use the Initial Value Theorem to find $f(0)$, and $f'(0)$ from the following functions:

(a) $\bar{f}(s) = \dfrac{s}{s^2 - 5s + 12}$, (b) $\bar{f}(s) = \dfrac{1}{s(s^2 + a^2)}$,

(c) $\bar{f}(s) = \dfrac{\exp(-sa)}{s^2 + 3s + 5}$, $a > 0$, (d) $\bar{f}(s) = \dfrac{s^2 - 1}{(s^2 + 1)}$.

28. Use the Final Value Theorem to find $f(\infty)$, if it exists, from the following functions:

(a) $\bar{f}(s) = \dfrac{1}{s(s^2 + as + b)}$, (b) $\bar{f}(s) = \dfrac{s + 2}{s^2 + 4}$,

(c) $\bar{f}(s) = \dfrac{1}{1 + as}$, (d) $\bar{f}(s) = \dfrac{3}{(s^2 + 4)^2}$.

29. If $\mathscr{L}\{f(t)\} = \bar{f}(s)$ and $\mathscr{L}\{g(t)\} = \bar{g}(s)$, establish Duhamel's integrals:

$$\mathscr{L}^{-1}\{s\bar{f}(s)\,\bar{g}(s)\} = \begin{Bmatrix} f(0)g(t) + \int_0^t f'(\tau)g(t - \tau)d\tau \\ \\ g(0)f(t) + \int_0^t g'(\tau)f(t - \tau)d\tau \end{Bmatrix}.$$

30. Using Watson's lemma, find the asymptotic expansion of

(a) $\bar{f}(s) = \int_0^\infty (1 + t^2)^{-1}\exp(-st)dt$, as $s \to \infty$,

(b) $K_0(s) = \int_0^\infty (t^2 - 1)^{-\frac{1}{2}}\exp(-st)dt$, as $s \to \infty$,

where $K_0(s)$ is the *modified Bessel* function.

31. Find the asymptotic expansion of $\bar{f}(s)$ as $s \to \infty$ when $f(t)$ is given by

(a) $(1+t)^{-1}$, (b) $\sin 2\sqrt{t}$,

(c) $\log(1+t)$, (d) $J_0(at)$.

32. Use the shifting property (3.4.5) or (3.4.6) to obtain the Laplace transform of the following functions:

(a) $f(t)=(t-a)^n\, H(t-a)$, (b) $f(t)=t^2\, H(t-a)$,

(c) $f(t)=\begin{Bmatrix} t, & 0\le t\le a \\ 0, & t\ge a \end{Bmatrix}$, (d) $f(x)=\begin{Bmatrix} w_0\left(1-\frac{2x}{l}\right), & 0<x<\frac{l}{2} \\ 0, & \frac{l}{2}<x<l \end{Bmatrix}$,

(e) $f(t)=\cos 2t\, H(t-\pi)$, (f) $f(t)=\begin{Bmatrix} 2, & 0\le t\le a \\ -2, & t\ge a \end{Bmatrix}$.

33. For the square wave function $f(t)$ given by $f(t)=a\,H(t)-a\,H(t-a)$, show that

$$\bar{f}(s)=\frac{a}{s\,(1+e^{-as})}.$$

34. If $f(t)=a\,H(t)-2a\,H(t-1)+a\,H(t-2)$, show that

$$\bar{f}(s)=\frac{a}{s}\left(1-2\,e^{-s}+e^{-2s}\right).$$

35. If $f(t)=\begin{Bmatrix} \frac{\sin t}{t}, & t\ne 0 \\ 1, & t=0 \end{Bmatrix}$, show that $\bar{f}(s)=\tan^{-1}\left(\frac{1}{s}\right)$

36. If $f_p(t)=t^{p-1}\,e^{-t}\,H(t)$, show that $(f_p * f_q)(t)$ exists if and only if $p$ and $q$ are both positive.

Hence, derive the following results

(a) $(f_p * f_q)(t)=B(p,\,q)\,f_{p+q}(t)$.
(b) $f_p'(t)=(p-1)\,f_{p-1}(t)-f_p(t)$.
(c) $(f_p * f_q)'(t)=(p-1)\,B(p-1,\,q)\,f_{p+q-1}(t)-B(p,\,q)\,f_{p+q}(t)$.
(d) $(f_p * f_q)'(t)=B(p,\,q)\left[(p+q-1)f_{p+q-1}(t)-f_{p+q}(t)\right]$.

37. A family $\{h_p(t): p>0\}$ of functions on $\mathbb{R}$ is called a *convolution semi-group* if $h_p * h_q = h_{p+q}$ for all $p,\,q>0$. Show that $h_p(t)=\frac{f_p(t)}{\Gamma(p)}$ defines a convolution semi-group where $f_p(t)$ is defined in Exercise 36.

38. Using the change of variables, $s=c+i\omega$, show that the inverse Laplace transformation is a Fourier transformation, that is,

(i) $f(t)=\mathscr{L}^{-1}\left\{\bar{f}(s)\right\}=\frac{e^{ct}}{2\pi}\int\limits_{-\infty}^{\infty}\bar{f}(c+i\omega)e^{i\omega t}d\omega.$

(ii) $f(t) = \frac{1}{\pi}\, e^{ct}\, \mathrm{Re} \int_0^\infty \bar{\mathrm{f}}(\mathrm{c} + \mathrm{i}\omega)\, \mathrm{e}^{\mathrm{i}\omega \mathrm{t}} \mathrm{d}\omega.$

Hence, for real $f(t)$, show that

(iii) $\mathscr{F}_c \left\{e^{ct} f(t)\right\} = 2\ \mathrm{Re}\left[\bar{\mathrm{f}}(\mathrm{c} + \mathrm{i}\omega)\right],$

(iv) $\mathscr{F}_s \left\{e^{ct} f(t)\right\} = 2\ \mathrm{Im}\left[\bar{\mathrm{f}}(\mathrm{c} + \mathrm{i}\omega)\right].$

# 4

# *Applications of Laplace Transforms*

"Mathematical sciences have attracted special attention since great antiquity, they are attracting still more attention today because of their influence on industry and the arts. The agreement of theory and practice brings most beneficial results, and it is not exclusively the practical side which gains; science is advancing under its influence as it discovers new objects of study and new aspects of the mathematical sciences...."

P. L. Chebyshev

"... partial differential equations are the basis of all physical theorems. In the theory of sound in gases, liquids and solids, in the investigations of elasticity, in optics, everywhere partial differential equations formulate basic laws of nature which can be checked against experiments."

Bernhard Riemann

## 4.1 Introduction

Many problems of physical interest are described by ordinary or partial differential equations with appropriate initial or boundary conditions. These problems are usually formulated as *initial value problems, boundary value problems,* or *initial-boundary value problems* that seem to be mathematically more rigorous and physically realistic in applied and engineering sciences. The Laplace transform method is particularly useful for finding solutions of these problems. The method is very effective for the solution of the response of a linear system governed by an ordinary differential equation to the *initial data* and/or to an *external disturbance* (or *external input function*). More precisely, we seek the solution of a linear system for its state at subsequent time $t > 0$ due to the initial state at $t = 0$ and/or to the disturbance applied for $t > 0$.

This chapter deals with the solutions of ordinary and partial differential equations that arise in mathematical, physical, and engineering sciences. The applications of Laplace transforms to the solutions of certain integral equations and boundary value problems are also discussed in this chapter. It is shown by examples that the Laplace transform can also be used effectively for evaluating certain definite integrals. We also give a few examples of solutions of difference and differential equations using the Laplace transform technique. The effective use of the joint Laplace and Fourier transform is illustrated by solving several initial-boundary value problems. Application of Laplace transforms to the problem of summation of infinite series in closed form is presented with examples. Finally, it is noted that the examples given in this chapter are only representative of a wide variety of problems which can be solved by the use of the Laplace transform method.

## 4.2 Solutions of Ordinary Differential Equations

As stated in the introduction of this chapter, the Laplace transform can be used as an effective tool for analyzing the basic characteristics of a linear system governed by the differential equation in response to initial data and/or to an external disturbance. The following examples illustrate the use of the Laplace transform in solving certain initial value problems described by ordinary differential equations.

***Example 4.2.1***
(*Initial Value Problem*). We consider the first-order ordinary differential equation

$$\frac{dx}{dt} + px = f(t), \qquad t > 0, \tag{4.2.1}$$

with the initial condition

$$x(t=0) = a, \tag{4.2.2}$$

where $p$ and $a$ are constants and $f(t)$ *is* an external input function so that its Laplace transform exists.

Application of the Laplace transform $\bar{x}(s)$ of the function $x(t)$ gives

$$s\,\bar{x}(s) - x(0) + p\,\bar{x}(s) = \bar{f}(s),$$

or

$$\bar{x}(s) = \frac{a}{s+p} + \frac{\bar{f}(s)}{s+p}. \tag{4.2.3}$$

The inverse Laplace transform together with the Convolution Theorem leads to the solution

$$x(t) = ae^{-pt} + \int_0^t f(t-\tau)e^{-p\tau}\,d\tau. \tag{4.2.4}$$

Thus, the solution naturally splits into two terms—the first term corresponds to the response of the initial condition and the second term is entirely due to the external input function $f(t)$.

In particular, if $f(t) = q =$ constant, then the solution (4.2.4) becomes

$$x(t) = \frac{q}{p} + \left(a - \frac{q}{p}\right) e^{-pt}. \tag{4.2.5}$$

The first term of this solution is independent of time $t$ and is usually called the *steady-state solution*. The second term depends on time $t$ and is called the *transient solution*. In the limit as $t \to \infty$, the transient solution decays to zero if $p > 0$ and the steady-state solution is attained. On the other hand, when $p < 0$, the transient solution grows exponentially as $t \to \infty$, and the solution becomes unstable.

Equation (4.2.1) describes the law of natural growth or decay process with an external forcing function $f(t)$ according as $p > 0$ or $< 0$. In particular, if $f(t) = 0$ and $p > 0$, the resulting equation (4.2.1) occurs very frequently in chemical kinetics. Such an equation describes the rate of chemical reactions. □

***Example 4.2.2***

(*Second Order Ordinary Differential Equation*). The second order linear ordinary differential equation has the general form

$$\frac{d^2x}{dt^2} + 2p\frac{dx}{dt} + qx = f(t), \qquad t > 0. \tag{4.2.6}$$

The initial conditions are

$$x(t) = a, \quad \frac{dx}{dt} = \dot{x}(t) = b \qquad \text{at } t = 0, \tag{4.2.7ab}$$

where $p, q, a$ and $b$ are constants.

Application of the Laplace transform to this general initial value problem gives

$$s^2\bar{x}(s) - s\,x(0) - \dot{x}(0) + 2p\{s\,\bar{x}(s) - x(0)\} + q\bar{x}(s) = \bar{f}(s).$$

The use of (4.2.7ab) leads to the solution for $\bar{x}(s)$ as

$$\bar{x}(s) = \frac{(s+p)a + (b+pa) + \bar{f}(s)}{(s+p)^2 + n^2}, \qquad n^2 = q - p^2. \tag{4.2.8}$$

The inverse transform gives the solution in three distinct forms depending on

$q >=< p^2$, and they are

$$x(t) = ae^{-pt}\cos nt + \frac{1}{n}(b+pa)e^{-pt}\sin nt + \frac{1}{n}\int_0^t f(t-\tau)e^{-p\tau}\sin n\tau d\tau, \qquad \text{when } n^2 = q - p^2 > 0, \tag{4.2.9}$$

$$x(t) = ae^{-pt} + (b+pa)\,t\,e^{-pt} + \int_0^t f(t-\tau)\,\tau\,e^{-p\tau}d\tau, \qquad \text{when } n^2 = q - p^2 = 0, \tag{4.2.10}$$

$$x(t) = ae^{-pt}\cosh mt + \frac{1}{m}(b+pa)\,e^{-pt}\sinh mt + \frac{1}{m}\int_0^t f(t-\tau)\,e^{-p\tau}\sinh m\tau d\tau, \quad \text{when } m^2 = p^2 - q > 0. \tag{4.2.11}$$

□

***Example 4.2.3***

(*Higher Order Ordinary Differential Equations*). We solve the linear equation of order $n$ with constant coefficients as

$$f(D)\{x(t)\} \equiv D^n x + a_1 D^{n-1}x + a_2 D^{n-2}x + \cdots + a_n x = \phi(t), \quad t > 0, \tag{4.2.12}$$

with the initial conditions

$$x(t) = x_0, \quad Dx(t) = x_1, \quad D^2x(t) = x_2, \ldots, D^{n-1}x(t) = x_{n-1}, \quad \text{at } t = 0, \tag{4.2.13}$$

where $D = \frac{d}{dt}$ is the differential operator and $x_0, x_1, \ldots, x_{n-1}$ are constants. We take the Laplace transform of (4.2.12) to get

$$\begin{aligned}(s^n\,\bar{x} - s^{n-1}\,x_0 - s^{n-2}\,x_1 - \cdots - s\,x_{n-2} - x_{n-1}) \\ + a_1\left(s^{n-1}\,\bar{x} - s^{n-2}\,x_0 - s^{n-3}\,x_1 - \cdots - x_{n-2}\right) \\ + a_2\left(x^{n-2}\,\bar{x} - s^{n-3}\,x_0 - \cdots - x_{n-3}\right) \\ + \cdots + a_{n-1}(s\,\bar{x} - x_0) + a_n\,\bar{x} = \bar{\phi}(s).\end{aligned} \tag{4.2.14}$$

Or,

$$
\begin{aligned}
(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n)\,\bar{x}(s) \\
&= \bar{\phi}(s) + (s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1})x_0 \\
&\quad + (s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-2})x_1 + \cdots + (s + a_1)x_{n-2} + x_{n-1} \\
&= \bar{\phi}(s) + \bar{\psi}(s),
\end{aligned} \tag{4.2.15}
$$

where $\bar{\psi}(s)$ is made up of all terms on the right hand side of (4.2.15) except $\bar{\phi}(s)$, and is a polynomial in $s$ of degree $(n-1)$.

Hence,

$$\bar{f}(s)\,\bar{x}(s) = \bar{\phi}(s) + \bar{\psi}(s),$$

where

$$\bar{f}(s) = s^n + a_1 s^{n-1} + \cdots + a_n.$$

Thus, the Laplace transform solution, $\bar{x}(s)$ is

$$\bar{x}(s) = \frac{\bar{\phi}(s) + \bar{\psi}(s)}{\bar{f}(s)}. \tag{4.2.16}$$

Inversion of (4.2.16) yields

$$x(t) = \mathscr{L}^{-1}\left\{\frac{\bar{\phi}(s)}{\bar{f}(s)}\right\} + \mathscr{L}^{-1}\left\{\frac{\bar{\psi}(s)}{\bar{f}(s)}\right\}. \tag{4.2.17}$$

The inverse operation on the right can be carried out by partial fraction decomposition, by the Heaviside Expansion Theorem, or by contour integration.
$\square$

**_Example 4.2.4_**
(*Third Order Ordinary Differential Equations*). We solve

$$(D^3 + D^2 - 6D)x(t) = 0, \qquad D \equiv \frac{d}{dt}, \qquad t > 0, \tag{4.2.18}$$

with the initial data

$$x(0) = 1,\ \dot{x}(0) = 0,\ \text{and } \ddot{x}(0) = 5. \tag{4.2.19}$$

The Laplace transform of equation (4.2.18) gives

$$[s^3\bar{x} - s^2 x(0) - s\,\dot{x}(0) - \ddot{x}(0)] + [s^2\bar{x} - s\,x(0) - \dot{x}(0)] - 6[s\,\bar{x} - x(0)] = 0.$$

In view of the initial conditions, we find

$$\bar{x}(s) = \frac{s^2 + s - 1}{s(s^2 + s - 6)} = \frac{s^2 + s - 1}{s(s+3)(s-2)}.$$

Or,

$$\bar{x}(s) = \frac{1}{6} \cdot \frac{1}{s} + \frac{1}{3} \cdot \frac{1}{s+3} + \frac{1}{2} \cdot \frac{1}{s-2}.$$

Inverting gives the solution

$$x(t) = \frac{1}{6} + \frac{1}{3} e^{-3t} + \frac{1}{2} e^{2t}. \tag{4.2.20}$$

▯

**Example 4.2.5**

(*System of First Order Ordinary Differential Equations*). Consider the system

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + b_1(t) \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + b_2(t) \end{aligned}\right\} \tag{4.2.21ab}$$

with the initial data

$$x_1(0) = x_{10} \quad \text{and} \quad x_2(0) = x_{20}; \tag{4.2.22ab}$$

where $a_{11}$, $a_{12}$, $a_{21}$, $a_{22}$ are constants.

Introducing the matrices

$$x \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \frac{dx}{dt} \equiv \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix}, \qquad A \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$b(t) \equiv \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} \quad \text{and} \quad x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix},$$

we can write the above system in a matrix differential system as

$$\frac{dx}{dt} = A\,x + b(t), \qquad x(0) = x_0. \tag{4.2.23ab}$$

We take the Laplace transform of the system with the initial conditions to get

$$\begin{aligned} (s - a_{11})\bar{x}_1 - a_{12}\bar{x}_2 &= x_{10} + \bar{b}_1(s), \\ -a_{21}\bar{x}_1 + (s - a_{22})\bar{x}_2 &= x_{20} + b_2(s). \end{aligned}$$

The solutions of this algebraic system are

$$\bar{x}_1(s) = \frac{\begin{vmatrix} x_{10} + \bar{b}_1(s) & -a_{12} \\ x_{20} + \bar{b}_2(s) & s - a_{22} \end{vmatrix}}{\begin{vmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{vmatrix}}, \quad \bar{x}_2(s) = \frac{\begin{vmatrix} s - a_{11} & x_{10} + \bar{b}_1(s) \\ -a_{21} & x_{20} + \bar{b}_2(s) \end{vmatrix}}{\begin{vmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{vmatrix}}. \tag{4.2.24ab}$$

Expanding these determinants, results for $\bar{x}_1(s)$ and $\bar{x}_2(s)$ can readily be inverted, and the solutions for $x_1(t)$ and $x_2(t)$ can be found in closed forms. □

***Example 4.2.6***
Solve the matrix differential system

$$\frac{dx}{dt} = Ax, \qquad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.2.25}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}.$$

This system is equivalent to

$$\begin{aligned} \frac{dx_1}{dt} - x_2 &= 0, \\ \frac{dx_2}{dt} + 2x_1 - 3x_2 &= 0, \end{aligned}$$

with

$$x_1(0) = 0 \quad \text{and} \quad x_2(0) = 1.$$

Taking the Laplace transform of the coupled system with the given initial data, we find

$$\begin{aligned} s\,\bar{x}_1 - \bar{x}_2 &= 0, \\ 2\bar{x}_1 + (s-3)\bar{x}_2 &= 1. \end{aligned}$$

This system has the solutions

$$\begin{aligned} \bar{x}_1(s) &= \frac{1}{s^2 - 3s + 2} = \frac{1}{s-2} - \frac{1}{s-1}, \\ \bar{x}_2(s) &= \frac{s}{s^2 - 3s + 2} = \frac{2}{s-2} - \frac{1}{s-1}. \end{aligned}$$

Inverting these results, we obtain

$$x_1(t) = e^{2t} - e^t, \qquad x_2(t) = 2\,e^{2t} - e^t.$$

In matrix notation, the solution is

$$x(t) = \begin{pmatrix} e^{2t} - e^t \\ 2\,e^{2t} - e^t \end{pmatrix}. \tag{4.2.26}$$

□

**Example 4.2.7**
(*Second Order Coupled Differential System*). Solve the system

$$\left.\begin{aligned} \frac{d^2x_1}{dt^2} - 3x_1 - 4x_2 = 0 \\ \frac{d^2x_2}{dt^2} + x_1 + x_2 = 0 \end{aligned}\right\} \qquad t > 0, \tag{4.2.27}$$

with the initial conditions

$$x_1(t) = x_2(t) = 0; \qquad \frac{dx_1}{dt} = 2 \quad \text{and} \quad \frac{dx_2}{dt} = 0 \text{ at } t = 0. \tag{4.2.28}$$

The use of the Laplace transform to (4.2.27) with (4.2.28) gives

$$\begin{aligned} (s^2 - 3)\bar{x}_1 - 4\bar{x}_2 = 2 \\ \bar{x}_1 + (s^2 + 1)\bar{x}_2 = 0. \end{aligned}$$

Then

$$\bar{x}_1(s) = \frac{2(s^2+1)}{(s^2-1)^2} = \frac{(s+1)^2 + (s-1)^2}{(s^2-1)^2} = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}.$$

Hence, the inversion yields

$$x_1(t) = t(e^t + e^{-t}). \tag{4.2.29}$$

$$\bar{x}_2(s) = \frac{-2}{(s^2-1)^2} = \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{(s-1)^2} - \frac{1}{(s+1)^2}\right],$$

which can be readily inverted to find

$$x_2(t) = \frac{1}{2}(e^t - e^{-t} - t\,e^t - t\,e^{-t}). \tag{4.2.30}$$

□

**Example 4.2.8**
(*The Harmonic Oscillator in a Non-Resisting Medium*). The differential equation of the oscillator in the presence of an external driving force $F\,f(t)$ is

$$\frac{d^2x}{dt^2} + \omega^2 x = F\,f(t), \tag{4.2.31}$$

where $\omega$ is the frequency and $F$ is a constant.

The initial conditions are

$$x(t) = a, \quad \dot{x}(t) = U \qquad \text{at } t = 0, \tag{4.2.32}$$

where $a$ and $U$ are constants.

Taking the Laplace transform of (4.2.31) with the initial conditions, we obtain

$$(s^2+\omega^2)\bar{x}(s) = sa + U + F\,\bar{f}(s).$$

Or,

$$\bar{x}(s) = \frac{as}{s^2+\omega^2} + \frac{U}{s^2+\omega^2} + \frac{F\,\bar{f}(s)}{s^2+\omega^2}. \tag{4.2.33}$$

Inversion together with the convolution theorem yields

$$x(t) = a\cos\omega t + \frac{U}{\omega}\sin\omega t + \frac{F}{\omega}\int_0^t f(t-\tau)\sin\,\omega\tau d\tau \tag{4.2.34}$$

$$= A\ \cos(\omega t-\phi) + \frac{F}{\omega}\int_0^t f(t-\tau)\ \sin\,\omega\tau d\tau, \tag{4.2.35}$$

where $A = \left(a^2+\dfrac{U^2}{\omega^2}\right)^{1/2}$ and $\phi = \tan^{-1}\left(\dfrac{U}{\omega a}\right)$.

The solution (4.2.35) consists of two terms. The first term represents the response to the initial data, and it describes *free oscillations* with amplitude $A$, phase $\phi$, and frequency $\omega$, which is called the *natural frequency* of the oscillator. The second term arises in response to the external force, and hence, it represents the *forced oscillations.* In order to investigate some interesting features of solution (4.2.35), we select the following cases of interest:

*(i) Zero Forcing Function.*
In this case, solution (4.2.35) reduces to

$$x(t) = A\ \cos(\omega t-\phi). \tag{4.2.36}$$

This represents simple harmonic motion with amplitude $A$, frequency $\omega$ and phase $\phi$. Evidently, the motion is oscillatory.

*(ii) Steady Forcing Function, that is,* $f(t)=1$.
In this case, solution (4.2.35) becomes

$$x - \frac{F}{\omega^2} = A\cos(\omega t-\phi) - \frac{F}{\omega^2}\cos\omega t. \tag{4.2.37}$$

In particular, when the particle is released from rest, $U=0$, (4.2.37) takes the form

$$x - \frac{F}{\omega^2} = \left(a - \frac{F}{\omega^2}\right)\cos\omega t. \tag{4.2.38}$$

This corresponds to free oscillations with the natural frequency $\omega$ and displays a shift in the equilibrium position from the origin to the point $\dfrac{F}{\omega^2}$.

*(iii) Periodic Forcing Function, that is,* $f(t) = \cos\omega_0 t$.
The transform solution can readily be found form (4.2.33) in the form

$$\bar{x}(s) = \frac{as}{s^2+\omega^2} + \frac{U}{s^2+\omega^2} + \frac{Fs}{(s^2+\omega_0^2)(s^2+\omega^2)}$$
$$= \frac{as}{s^2+\omega^2} + \frac{U}{s^2+\omega^2} + \frac{Fs}{(\omega_0^2-\omega^2)}\left(\frac{1}{s^2+\omega^2} - \frac{1}{s^2+\omega_0^2}\right). \quad (4.2.39)$$

Inversion yields the solution

$$x(t) = a\cos\omega t + \frac{U}{\omega}\sin\omega t + \frac{F}{(\omega_0^2-\omega^2)}(\cos\omega t - \cos\omega_0 t) \quad (4.2.40)$$
$$= A\cos(\omega t - \phi) + \frac{F}{(\omega_0^2-\omega^2)}\cos\omega_0 t, \quad (4.2.41)$$

where $A = \left\{\left(a + \frac{F}{\omega_0^2-\omega^2}\right)^2 + \frac{U^2}{\omega^2}\right\}^{1/2}$ and $\tan\phi = \frac{U}{\omega} \div \left(a + \frac{F}{\omega_0^2-\omega^2}\right)$.

It is noted that solution (4.2.41) consists of free oscillations of period $\left(\frac{2\pi}{\omega}\right)$ and forced oscillations of period $\left(\frac{2\pi}{\omega_0}\right)$, which is the same as that of the external periodic force. If $\omega_0 < \omega$, the phase of the forced oscillations is the same as that of the external periodic force. If $\omega_0 > \omega$, the forced term suffers from a phase change by an amount $\pi$. In other words, the forced motion is in phase or 180° out of phase with the external force according as $\omega >$ or $< \omega_0$.

When $\omega = \omega_0$, result (4.2.40) can be written as

$$x(t) = a\cos\omega t + \frac{U}{\omega}\sin\ \omega t + \frac{Ft}{(\omega_0+\omega)}\left[\frac{\sin\left\{\frac{1}{2}(\omega-\omega_0)t\right\}\sin\left\{\frac{1}{2}(\omega+\omega_0)t\right\}}{\frac{1}{2}(\omega_0-\omega)t}\right]$$
$$= a\cos\omega t + \frac{U}{\omega}\sin\omega t + \frac{Ft}{2\omega}\sin\omega t = A\cos(\omega t - \phi) + \frac{Ft}{2\omega}\sin\omega t, \quad (4.2.42)$$

where

$$A^2 = \left(a^2 + \frac{U^2}{\omega^2}\right) \quad \text{and } \tan\phi = \frac{U}{a\omega}.$$

This solution clearly shows that the amplitude of the forced motion increases with $t$. Thus, if the natural frequency is equal to the forcing frequency, the oscillations become unbounded, which is physically undesirable. This phenomenon is usually called *resonance,* and the corresponding frequency $\omega = \omega_0$ is referred to as the *resonant frequency* of the system. It may be emphasized that at the resonant frequency, the solution of the problem becomes mathematically invalid for large times, and hence, it is physically unrealistic. In most dynamical systems, this kind of situation is resolved by including dissipating and/or nonlinear effects. □

**Example 4.2.9**

(*Harmonic Oscillator in a Resisting Medium*). The differential equation of the oscillator in a resisting medium where the resistance is proportional to velocity is given by

$$\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+\omega^2 x = F\,f(t), \tag{4.2.43}$$

where $k\,(>0)$ is a constant of proportionality and the right hand side represents the external driving force. The initial state of the system is

$$x(t)=a, \quad \frac{dx}{dt}=U \qquad \text{at } t=0. \tag{4.2.44}$$

In view of the initial conditions, the Laplace transform solution of equation (4.2.43) is obtained as

$$\begin{aligned}\bar{x}(s) &= \frac{a(s+2k)+U+F\,\bar{f}(s)}{(s^2+2ks+\omega^2)}\\ &= \frac{a(s+k)+(U+ak)+\bar{F}(s)}{(s+k)^2+n^2},\end{aligned} \tag{4.2.45}$$

where $n^2=\omega^2-k^2$.

Three possible cases deserve attention:

*(i)* $k<\omega$ *(small damping).*
In this case, $n^2=\omega^2-k^2>0$ and the inversion of (4.2.45) along with the Convolution Theorem yields

$$x(t)=a\,e^{-kt}\cos nt+\frac{(U+ak)}{n}e^{-kt}\sin nt+\frac{F}{n}\int_0^t f(t-\tau)e^{-k\tau}\sin n\tau d\tau. \tag{4.2.46}$$

This is the most general solution of the problem for an arbitrary form of the external driving force.

*(ii)* $k=\omega$ *(critical damping) so that* $n^2=0$.
The solution for this case can readily be obtained from (4.2.45) by inversion and has the form

$$x(t)=a\,e^{-kt}+(U+ak)\,t\,e^{-kt}+F\int_0^t f(t-\tau)\,\tau e^{-kt}d\tau. \tag{4.2.47}$$

*(iii)* $k>\omega$ *(large damping).*
Set $n^2=-(k^2-\omega^2)=-m^2$ so that $m^2=k^2-\omega^2>0$.

The transformed solution (4.2.45) assumes the form

$$\bar{x}(s) = \frac{a(s+k) + (U+ak) + F\ \bar{f}(s)}{(s+k)^2 - m^2}. \tag{4.2.48}$$

After inversion, it turns out that

$$x(t) = a\,e^{-kt}\cosh mt + \left(\frac{U+ak}{m}\right)e^{-kt}\sinh mt + \frac{F}{m}\int_0^t f(t-\tau)e^{-k\tau}\sinh m\tau d\tau. \tag{4.2.49}$$

In order to examine the characteristic features of the problem, it is necessary to specify the nature and functional form of $f(t)$ involved in the external force term. Suppose the external driving force is zero. The solution can readily be written down in all three cases.

For $0 < k < \omega$, the solution is

$$x(t) = e^{-kt}\left(a\,\cos nt + \frac{U+ak}{n}\sin nt\right) = A\,e^{-kt}\ \cos(nt-\phi), \tag{4.2.50}$$

where $A = \left\{a^2 + \frac{(U+ak)^2}{n^2}\right\}^{1/2}$ and $\phi = \tan^{-1}\left(\frac{U+ak}{an}\right)$.

Like the harmonic oscillator in a vacuum, the motion is oscillatory with the time-dependent amplitude $Ae^{-kt}$ and the modified frequency

$$n = (\omega^2 - k^2)^{1/2} = \omega\left(1 - \frac{1}{2}\frac{k^2}{\omega^2} + \cdots\right), \qquad 0 < k < \omega.$$

This means that, when the resistance is small, the modified frequency (or the undamped natural frequency) is obviously smaller than the natural frequency, $\omega$. Although the small resistance produces an insignificant effect on the frequency, the amplitude is radically modified. It should also be noted that the amplitude decays exponentially to zero as time $t \to \infty$. The phase of the motion is also changed by the small resistance. Thus, the motion is called the *damped oscillatory motion,* and depicted by Figure 4.1.

At the critical case, $\omega = k$, and hence, $n = 0$. The solution can readily be found from (4.2.47) with $F = 0$, and has the form

$$x(t) = a\,e^{-kt} + (ak+U)\,t\,e^{-kt}. \tag{4.2.51}$$

The motion ceases to be oscillatory and decays very rapidly as $t \to \infty$.

If damping is large with no external force, solution (4.2.49) reduces to

$$x(t) = a\,e^{-kt}\ \cosh mt + \left(\frac{ak+U}{m}\right)e^{-kt}\ \sinh mt. \tag{4.2.52}$$

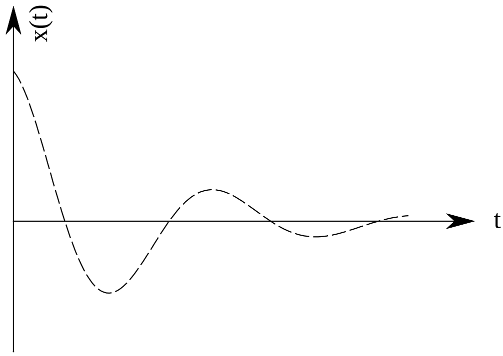


**Figure 4.1** Damped oscillatory motion.

Using $\begin{smallmatrix}\cosh\\ \sinh\end{smallmatrix}\, mt = \frac{1}{2}(e^{mt} \pm e^{-mt})$, we can write the solution as

$$x(t) = A\, e^{-(k-m)t} + B\, e^{-(k+m)t}, \tag{4.2.53}$$

where $A = \frac{1}{2}\left(a + \frac{ak+U}{m}\right)$ and $B = \dfrac{1}{2}\left(a - \dfrac{ak+U}{m}\right)$.

The above solution suggests that the motion is no longer oscillatory and in fact, it decays very rapidly as $t \to \infty$. ▯

***Example 4.2.10***

(*Harmonic Oscillator in a Resisting Medium with an External Periodic Force*), The motion is governed by the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = F\cos\omega_0 t, \qquad k > 0 \tag{4.2.54}$$

with the initial data

$$x(0) = a \quad \text{and} \quad \dot{x}(0) = U.$$

The transformed solution for the case of small damping $(k < \omega)$ is

$$\begin{aligned}\bar{x}(s) &= \frac{a(s+k) + (U+ak)}{(s+k)^2 + n^2} + \frac{Fs}{\{(s+k)^2 + n^2\}(s^2 + \omega_0^2)}\\ &= \frac{a(s+k) + (U+ak)}{(s+k)^2 + n^2} + F\left[\frac{As - B}{(s+k)^2 + n^2} - \frac{As - C}{s^2 + \omega_0^2}\right],\end{aligned} \tag{4.2.55}$$

where

$$A = \frac{\omega_0^2 - \omega^2}{(\omega^2 - \omega_0^2)^2 + 4k^2\omega_0^2}, \quad B = \frac{2k\omega^2}{(\omega^2 - \omega_0^2)^2 + 4k^2\omega_0^2},$$

and

$$C=\frac{2\,k\,\omega_0^2}{(\omega^2-\omega_0^2)^2+4k^2\omega_0^2}\quad\text{with } \omega^2=n^2+k^2.$$

The expression for $\bar{x}(s)$ can be inverted to obtain the solution

$$x(t)=(a+FA)\,e^{-kt}\,\cos nt+\frac{1}{n}(U+ak-FAk-FB)\,e^{-kt}\,\sin nt$$
$$-AF\,\cos\omega_0 t+\frac{CF}{\omega_0}\sin\omega_0 t. \tag{4.2.56}$$

It is convenient to write it in the form

$$x(t)=A_1\,\cos(\omega_0 t-\phi_1)+A_2\,e^{-kt}\,\cos(nt-\phi_2), \tag{4.2.57}$$

where

$$A_1^2=F^2\left(A^2+\frac{C^2}{\omega_0^2}\right)=\frac{F^2}{(\omega^2-\omega_0^2)^2+4k^2\omega_0^2}, \tag{4.2.58}$$

$$\tan\,\phi_1=-\frac{C}{A\omega_0}=\frac{2k\omega_0}{\omega^2-\omega_0^2}, \tag{4.2.59}$$

$$A_2^2=(a+FA)^2+\frac{1}{n^2}(U+ak-kFa-FB)^2, \tag{4.2.60}$$

and

$$\tan\,\phi_2=\frac{U+ak-kFA-FB}{n(a+FB)}. \tag{4.2.61}$$

This form of solution (4.2.57) lends itself to some interesting physical interpretations. First, the displacement field $x(t)$ essentially consists of the steady state and the transient terms, which are independently modified by the damping and driving forces involved in the equation of motion. In the limit as $t\to\infty$, the latter decays exponentially to zero. Consequently, the ultimate steady state is attained in the limit, and represented by the first term of (4.2.57). In fact, the steady-state solution is denoted by $x_{st}(t)$ and given by

$$x_{st}(t)=A_1\,\cos(\omega_0 t-\phi_1), \tag{4.2.62}$$

where $A_1$ is the amplitude, $\omega_0$ is the frequency, and $\phi_1$ represents the phase lag given by

$$\begin{aligned}\phi_1&=\tan^{-1}\left\{\frac{2k\omega_0}{(\omega^2-\omega_0^2)}\right\} &&\text{when } \omega_0<\omega,\\ &=\pi-\tan^{-1}\left\{\frac{2k\omega_0}{(\omega_0^2-\omega^2)}\right\} &&\text{when } \omega_0>\omega,\\ &=\frac{\pi}{2} &&\text{as } \omega_0\to\omega.\end{aligned}$$

It should be noted that the frequency of the steady-state solution is the same as that of the external driving force, but the amplitude and the phase are modified by the parameters $\omega,\ k$ and $\omega_0$. It is of interest to examine the nature of the amplitude and the phase with respect to the forcing frequency $\omega_0$. For a low frequency $(\omega_0 \to 0)$, $A_1 = \dfrac{F}{\omega^2}$ and $\phi_1 = 0$. As $\omega_0 \to \omega$, the amplitude of the motion is still bounded and equal to $\left(\dfrac{F}{2k\omega}\right)$ if $k \neq 0$. The displacement suffers from a phase lag of $\pi/2$. Further, we note that

$$\frac{dA_1}{d\omega_0} = \frac{2\,\omega_0 F\left(\omega^2 - \omega_0^2 - 2k^2\right)}{\left\{(\omega^2 - \omega_0^2)^2 + 4k^2\omega_0^2\right\}^{3/2}}. \tag{4.2.63}$$

It follows that $A_1$ has a minimum at $\omega_0 = 0$ with minimum value $\dfrac{F}{\omega^2}$, and a maximum at $\omega_0 = (\omega^2 - 2k^2)^{1/2}$ with maximum value $\dfrac{F}{2k(\omega^2 - 2k^2)^{1/2}}$ provided $2k^2 < \omega^2$. If $2k^2 > \omega^2$, $A_1$ has no maximum and gradually decreases. The non-dimensional amplitude $A^* = \left(\dfrac{2A_1\omega^2}{F}\right)$ is plotted against the non-dimensional frequency $\omega^* = \dfrac{\omega_0}{\omega}$ for a given value of $\frac{k}{\omega}\,(< 1)$ in Figure 4.2.

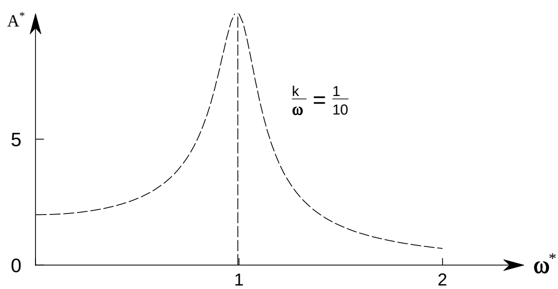


**Figure 4.2** Amplitude versus frequency with damping.

In the absence of the damping term, the amplitude $A_1$ becomes

$$A_1 = \frac{F}{\left|\omega^2 - \omega_0^2\right|},$$

which is unbounded at $\omega_0 = \omega$ and shown in Figure 4.3.

This situation has already been encountered earlier, and the frequency $\omega_0 = \omega$ was defined as the *resonant frequency*. The difficulty for the resonant case has been resolved by the inclusion of small damping effect.

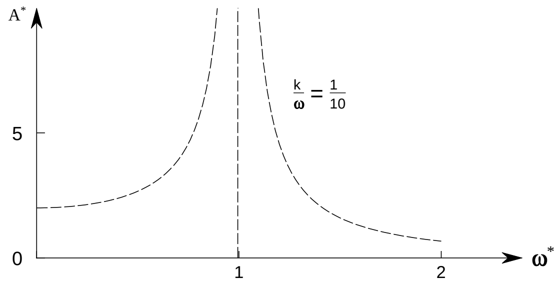


**Figure 4.3** Amplitude versus frequency without damping.

At the critical case ($k^2 = \omega^2$), the solution is found from (4.2.55) by inversion and has the form

$$x(t) = A_1 \cos(\omega_0 t - \phi) + (a + FA)\, e^{-kt} \\ + t(U + ak - FAk - FB)\, e^{-kt}. \qquad (4.2.64)$$

The transient term of this solution decays as $t \to \infty$ and the steady state is attained.

The solution for the case of high damping ($k^2 > \omega^2$) is obtained from (4.2.55) as

$$x(t) = (a + FA)\, e^{-kt} \cosh mt + \frac{1}{m}(U + ak - FAk - FB)\, e^{-kt} \sinh mt \\ - AF \cos \omega_0 t + \frac{CF}{\omega_0} \sin \omega_0 t \qquad (4.2.65)$$

where $m^2 = -n^2 = k^2 - \omega^2 > 0$. This result is somewhat similar to that of (4.2.56) or (4.2.57) with the exception that the transient term decays very rapidly as $t \to \infty$. Like previous cases, the steady state is reached in the limit. $\square$

***Example 4.2.11***

Obtain the solution of the Bessel equation

$$t\,\frac{d^2x}{dt^2} + \frac{dx}{dt} + a^2 t\, x(t) = 0, \qquad x(0) = 1. \qquad (4.2.66)$$

Application of the Laplace transform gives

$$\mathscr{L}\left\{t\frac{d^2x}{dt^2}\right\} + \mathscr{L}\left\{\frac{dx}{dt}\right\} + a^2\, \mathscr{L}\left\{t\, x(t)\right\} = 0.$$

Or,

$$-\frac{d}{ds}\left[\mathscr{L}\left\{\frac{d^2x}{dt^2}\right\}\right] + s\,\bar{x}(s) - x(0) - a^2\,\frac{d\bar{x}}{ds} = 0.$$

Or,

$$-\frac{d}{ds}[s^2\,\bar{x} - s\,x(0) - \dot{x}(0)] + s\,\bar{x}(s) - 1 - a^2\frac{d\bar{x}}{ds} = 0.$$

Thus,

$$(s^2 + a^2)\frac{d\bar{x}}{ds} + s\,\bar{x} = 0.$$

Or,

$$\frac{d\bar{x}}{\bar{x}} = -\frac{s\,ds}{s^2 + a^2}.$$

Integration gives the solution for $\bar{x}(s)$

$$\bar{x}(s) = \frac{A}{\sqrt{s^2 + a^2}},$$

where $A$ is an integrating constant. By the inverse transformation, we obtain the solution

$$x(t) = A\,J_0(at).$$

□

***Example 4.2.12***
Find the solution of the initial value problem

$$\frac{d^2x}{dt^2} + t\,\frac{dx}{dt} - 2x = 2, \qquad x(0) = \dot{x}(0) = 0.$$

Taking the Laplace transform yields

$$\mathscr{L}\left\{\frac{d^2x}{dt^2}\right\} + \mathscr{L}\left\{t\,\frac{dx}{dt}\right\} - 2\,\bar{x}(s) = \frac{2}{s}.$$

Or,

$$s^2\bar{x} - \frac{d}{ds}\{s\,\bar{x}(s)\} - 2\bar{x} = \frac{2}{s}$$
$$\frac{d\bar{x}}{ds} + \left(\frac{3}{s} - s\right)\bar{x} = -\frac{2}{s^2}.$$

This is a first order linear equation, which can be solved by the method of the integrating factor. The integrating factor is $s^3\exp\left(-\frac{1}{2}s^2\right)$. Multiplying the equation by the integrating factor and integrating, it turns out that

$$\bar{x}(s) = \frac{2}{s^3} + \frac{A}{s^3}\,\exp\left(\frac{s^2}{2}\right),$$

where $A$ is an integrating constant. As $\bar{x}(s) \to \infty$ as $s \to \infty$, we must have $A \equiv 0$. Thus, $\bar{x}(s) = \dfrac{2}{s^3}$. Inverting, we get the solution

$$x(t) = t^2.$$

□

**Example 4.2.13**

(*Current and Charge in a Simple Electric Circuit*). The current in a circuit (see Figure 4.4) containing inductance $L$, resistance $R$, and capacitance $C$ with an applied voltage $E(t)$ is governed by the equation

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I\,dt = E(t), \tag{4.2.67}$$

where $L$, $R$, and $C$ are constants and $I(t)$ is the current that is related to the accumulated charge $Q$ on the condenser at time $t$ by

$$Q(t) = \int_0^t I(t)\,dt \qquad \text{so} \quad \text{that} \qquad \frac{dQ}{dt} = I(t). \tag{4.2.68}$$

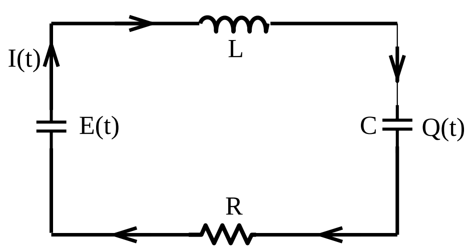


**Figure 4.4** Simple electric circuit.

If the circuit is without a condenser ($C \to \infty$), equation (4.2.67) reduces to

$$L\frac{dI}{dt} + RI = E(t), \qquad t > 0. \tag{4.2.69}$$

This can easily be solved with the initial condition $I(t=0) = I_0$. However, we solve the system (4.2.67)–(4.2.68) with the initial data

$$I(t=0) = 0, \qquad Q(t=0) = 0. \tag{4.2.70}$$

Then, in the limit $C \to \infty$, the solution of the system reduces to that of (4.2.69).

Application of the Laplace transform to (4.2.67) with (6.2.70) gives

$$\bar{I}(s) = \frac{1}{L}\frac{s\bar{E}(s)}{\left(s^2 + \frac{R}{L}s + \frac{1}{CL}\right)} = \frac{1}{L}\cdot\frac{(s+k-k)\bar{E}(s)}{(s+k)^2+n^2}, \tag{4.2.71}$$

where $k = \dfrac{R}{2L}$, $\omega^2 = \dfrac{1}{LC}$ and $n^2 = \omega^2 - k^2$.

Inversion of (4.2.71) gives the current field for three cases:

$$I(t) = \frac{1}{L}\int_0^t E(t-\tau)\left(\cos n\tau - \frac{k}{n}\sin n\tau\right)e^{-k\tau}\,d\tau, \qquad \text{if } \omega^2 > k^2 \tag{4.2.72}$$

$$= \frac{1}{L}\int_0^t E(t-\tau)(1-k\tau)e^{-k\tau}\,d\tau, \qquad \text{if } \omega^2 = k^2 \tag{4.2.73}$$

$$= \frac{1}{L}\int_0^t E(t-\tau)\left(\cosh m\tau - \frac{k}{m}\sinh m\tau\right)e^{-k\tau}\,d\tau, \text{ if } k^2 > \omega^2 \tag{4.2.74}$$

where $m^2 = -n^2$.

In particular, if $E(t) =$ constant $= E_0$, then the solution can be obtained directly from (4.2.71) by inversion as

$$I(t) = \frac{E_0}{nL}\exp\left(-\frac{Rt}{2L}\right)\sin nt, \qquad \text{if } n^2 = \frac{1}{CL} - \left(\frac{R}{2L}\right)^2 > 0, \tag{4.2.75}$$

$$= \frac{E_0}{L}\,t\,\exp\left(-\frac{Rt}{2L}\right), \qquad \text{if } \left(\frac{R}{2L}\right)^2 = \frac{1}{CL}, \tag{4.2.76}$$

$$= \frac{E_0}{mL}\exp\left(-\frac{Rt}{2L}\right)\sinh mt, \quad \text{if } m^2 = \left(\frac{R}{2L}\right)^2 - \frac{1}{CL} > 0. \tag{4.2.77}$$

It may be observed that the solution for the case of low resistance ($R^2C < 4L$), or small damping, describes a damped sinusoidal current with slowly decaying amplitude. In fact, the rate of damping is proportional to $\dfrac{R}{L}$, and when this quantity is large, the attenuation of the current is very rapid. The frequency of the oscillating current field is

$$n = \left(\frac{1}{CL} - \frac{R^2}{4L^2}\right)^{1/2},$$

which is called the *natural frequency* of the current field. If $\dfrac{R^2}{4L^2} << \dfrac{1}{CL}$, the frequency $n$ is approximately equal to

$$n \sim \frac{1}{\sqrt{CL}}.$$

The case, $\dfrac{R^2}{4L^2} = \dfrac{1}{CL}$, corresponds to *critical damping*, and the solution for this case decays exponentially with time.

The last case, $R^2C > 4L$, corresponds to high resistance or high damping. The current related to this case has the form

$$I(t) = \frac{E_0}{2mL}\left[e^{-\left(\frac{R}{2L}-m\right)t} - e^{-\left(\frac{R}{2L}+m\right)t}\right]. \tag{4.2.78}$$

It may be recognized that the solution is no longer oscillatory and decays exponentially to zero as $t \to \infty$. This is really expected in an electrical circuit with a very high resistance. If $C \to \infty$, the circuit is free from a condenser and $m \to \dfrac{R}{2L}$. Consequently, solution (4.2.77) reduces to

$$I(t) = \frac{E_0}{R}\left[1 - \exp\left(-\frac{Rt}{L}\right)\right]. \tag{4.2.79}$$

This is identical with the solution of equation (4.2.69).

We consider another special case where the alternating voltage is applied to the circuit so that

$$E(t) = E_0 \, \sin \omega_0 t. \tag{4.2.80}$$

The transformed solution for $\bar{I}(s)$ follows from (4.2.71) as

$$\bar{I}(s) = \left(\frac{E_0\omega_0}{L}\right)\frac{s}{\{(s+k)^2+n^2\}\left(s^2+\omega_0^2\right)}. \tag{4.2.81}$$

Using the rules of partial fractions, it turns out that

$$\bar{I}(s) = \left(\frac{E_0\omega_0}{L}\right)\left[\frac{As - B}{(s+k)^2+n^2} - \frac{As - C}{s^2+\omega_0^2}\right], \tag{4.2.82}$$

where $(A, B, C) \equiv \dfrac{(\omega_0^2 - \omega^2,\, 2k\omega^2,\, 2k\omega_0^2)}{(\omega^2-\omega_0^2)^2 + 4k^2\omega_0^2}$.

The inversion of (4.2.82) can be completed by Table B-4 of Laplace transforms, and the solution for $I(t)$ assumes three distinct forms according to $\omega^2 > = < k^2$.

The solution for the case of low resistance $(\omega^2 > k^2)$ is

$$I(t) = \left(\frac{E_0\omega_0}{L}\right)\left[A\,e^{-kt}\,\cos nt - \frac{1}{n}(Ak+B)e^{-kt}\,\sin nt \right.$$
$$\left. - A\cos\omega_0 t + \frac{C}{\omega_0}\sin\omega_0 t\right], \tag{4.2.83}$$

which has the equivalent form

$$I(t) = A_1\sin(\omega_0 t - \phi_1) + A_2\,e^{-kt}\cos(nt - \phi_2), \tag{4.2.84}$$

where

$$A_1^2 = \frac{E_0^2}{L^2}\left(A^2\omega_0^2 + C^2\right) = \frac{E_0^2\omega_0^2}{L^2\left\{\left(\omega^2-\omega_0^2\right)^2 + 4k^2\omega_0^2\right\}},\ \tan\phi_1 = \frac{A\omega_0}{C}, \quad (4.2.85)$$

$$A_2^2 = \left(\frac{E_0^2\omega_0^2}{L^2}\right)\left[A^2 + \frac{1}{n^2}(Ak+B)^2\right] \text{ and } \tan\phi_2 = -\frac{(Ak+B)}{An}. \quad (4.2.86)$$

The current field consists of the steady-state and transient components. The latter decays exponentially in a time scale of the order $\frac{L}{R}$. Consequently, the steady current field is established in the electric circuit and describes the sinusoidal current with constant amplitude and phase lagging by an angle $\phi_1$. The frequency of the steady oscillating current is the same as that of the applied voltage.

In the critical situation ($\omega^2 = k^2$), the current field is derived from (4.2.82) by inversion and has the form

$$I(t) = A_1\sin(\omega_0 t - \phi_1) + \left(\frac{E_0\omega_0}{L}\right)\left[Ae^{-kt} - (Ak+B)te^{-kt}\right]. \quad (4.2.87)$$

This result suggests that the transient component of the current dies out exponentially in the limit as $t\to\infty$. Eventually, the steady oscillating current is set up in the circuit and described by the first term of (4.2.87). Finally, the solution related to the case of high resistance ($\omega^2 < k^2$) can be found by direct inversion of (4.2.82) and is given by

$$\begin{aligned} I(t) &= A_1\,\sin(\omega_0 t - \phi_1) \\ &\quad + \left(\frac{E_0\omega_0}{L}\right)\left[A\cosh mt - \frac{1}{m}(Ak+B)\sinh mt\right]e^{-kt}. \end{aligned} \quad (4.2.88)$$

This solution is somewhat similar to (4.2.84) with the exception of the form of the transient term which, of course, decays very rapidly as $t\to\infty$. Consequently, the steady current field is estalished in the circuit and has the same value as in (4.2.84).

Finally, we close this example by suggesting a similarity between this electric circuit system and the mechanical system as described in Example 4.2.9. Differentiation of (4.2.67) with respect to $t$ gives a second order equation for the current field as

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dE}{dt}. \quad (4.2.89)$$

Also, an equation for the charge field $Q(t)$ can be found from (4.2.67) and (4.2.68) as

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t). \quad (4.2.90)$$

Writing $2k = \frac{R}{L}$ and $\omega^2 = \frac{1}{LC}$, the above equation can be put into the form

$$\left(\frac{d^2}{dt^2} + 2k\,\frac{d}{dt} + \omega^2\right)\begin{pmatrix} I \\ Q \end{pmatrix} = \frac{1}{L}\begin{pmatrix} \frac{dE}{dt} \\ E \end{pmatrix}. \tag{4.2.91ab}$$

These equations are very similar to equation (4.2.43) for a harmonic oscillator. □

***Example 4.2.14***

(*Current and Charge in an Electrical Network*). An electrical network is a combination of several interrelated simple electric circuits. Consider a more general network consisting of two electric circuits coupled by the mutual inductance $M$ with resistances $R_1$ and $R_2$, capacitances $C_1$ and $C_2$, and self-inductances $L_1$ and $L_2$ as shown in Figure 4.5. A time-dependent voltage $E(t)$ is applied to the first circuit at time $t = 0$, when charges and currents are zero.

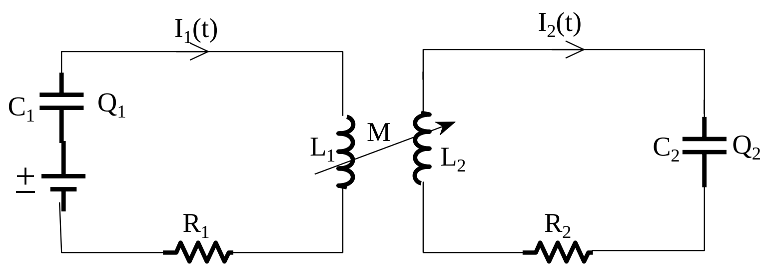


**Figure 4.5** Two coupled electric circuits.

The charge and current fields in the network are governed by the system of ordinary differential equations

$$L_1\,\frac{dI_1}{dt} + R_1 I_1 + M\frac{dI_2}{dt} + \frac{Q_1}{C_1} = E(t), \qquad t > 0 \tag{4.2.92}$$

$$M\,\frac{dI_1}{dt} + L_2\frac{dI_2}{dt} + R_2 I_2 + \frac{Q_2}{C_2} = 0, \qquad t > 0 \tag{4.2.93}$$

with

$$\frac{dQ_1}{dt} = I_1 \quad \text{and} \quad \frac{dQ_2}{dt} = I_2.$$

The initial conditions are

$$I_1 = 0,\ Q_1 = 0,\ I_2 = 0,\ Q_2 = 0 \qquad \text{at}\ t = 0. \tag{4.2.94}$$

Eliminating the currents from (4.2.92) and (4.2.93), we obtain

$$\left(L_1D^2 + R_1D + \frac{1}{C_1}\right)Q_1 + MD^2Q_2 = E(t), \tag{4.2.95}$$

$$MD^2Q_1 + \left(L_2D^2 + R_2D + \frac{1}{C_2}\right)Q_2 = 0, \tag{4.2.96}$$

where $D \equiv \dfrac{d}{dt}$.

The Laplace transform can be used to solve this system for $Q_1$ and $Q_2$. Similarly, we can find solutions for the current fields $I_1$ and $I_2$ independently or from the charge fields. We leave it as an exercise for the reader.

In the absence of the external voltage ($E = 0$) with $R_1 = R_2 = 0$, $L_1 = L_2 = L$ and $C_1 = C_2 = C$, addition and subtraction of (4.2.95) and (4.2.96) give

$$\ddot{Q}_+ + \alpha^2 Q_+ = 0, \quad \ddot{Q}_- + \beta^2 Q_- = 0, \tag{4.2.97ab}$$

where

$$Q_+ = Q_1 + Q_2, \quad Q_- = Q_1 - Q_2,$$

$$\alpha^2 = [C(L+M)]^{-1}, \quad \text{and} \quad \beta^2 = [\mathrm{C(L-M)}]^{-1}.$$

Clearly, the system executes uncoupled simple harmonic oscillations with frequencies $\alpha$ and $\beta$. Hence, the normal modes can be generated in this freely oscillatory electrical system.

Finally, in the absence of capacitances ($C_1 \to \infty$, $C_2 \to \infty$), the above network reduces to a simple one that consists of two electric circuits coupled by the mutual inductance $M$ with inductances $L_1$ and $L_2$, and resistances $R_1$ and $R_2$. As shown in Figure 4.6, an external voltage is applied to the first circuit at time $t = 0$.

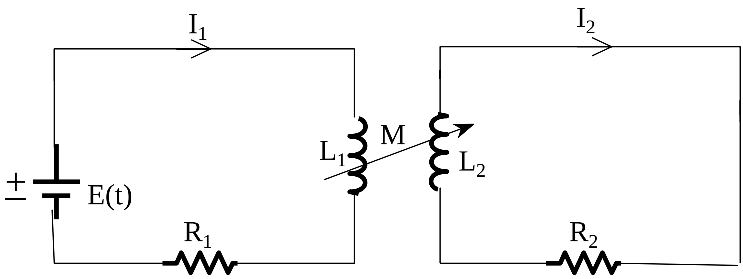


**Figure 4.6** Two coupled electric circuits without capacitances.

The current fields in the network are governed by a pair of coupled ordinary differential equations

$$L_1 \frac{dI_1}{dt} + R_1 I_1 + M \frac{dI_2}{dt} = E(t), \qquad t > 0, \tag{4.2.98}$$

$$M \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt} + R_2 I_2 = 0, \qquad t > 0, \tag{4.2.99}$$

where $I_1(t)$ and $I_2(t)$ are the currents in the first and the second circuits, respectively. The initial conditions are

$$I_1(0) = I_2(0) = 0. \tag{4.2.100}$$

We shall not pursue the problem further because the transform method of solution is a simple exercise. ▯

**Example 4.2.15**

(*Linear Dynamical Systems and Signals.* In physical and engineering sciences, a large number of linear dynamical systems with a time dependent *input signal* $f(t)$ that generates an *output signal* $x(t)$ can be described by the ordinary differential equation with constant coefficients

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_0)\, x(t) = (D^m + b_{m-1}D^{m-1} + \cdots + b_0)\, f(t), \tag{4.2.101}$$

where $D \equiv \frac{d}{dt}$ is the differential operator, $a_r$ and $b_r$ are constants.

We apply the Laplace transform to find the output $x(t)$ so that (4.2.101) becomes

$$\bar{p}_n(s)\,\bar{x}(s) - \bar{R}_{n-1} = \bar{q}_m(s)\,\bar{f}(s) - \bar{S}_{m-1}, \tag{4.2.102}$$

where

$$\bar{p}_n(s) = s^n + a_{n-1}\, s^{n-1} + \cdots + a_0, \quad \bar{q}_m(s) = s^m + a_{m-1}\, s^{m-1} + \cdots + b_0,$$

$$\bar{R}_{n-1}(s) = \sum_{r=0}^{n-1} s^{n-r-1}\, x^{(r)}(0), \qquad \bar{S}_{m-1}(s) = \sum_{r=0}^{m-1} s^{m-r-1}\, f^{(r)}(0).$$

It is convenient to express (4.2.102) in the form

$$\bar{x}(s) = \bar{h}(s)\,\bar{f}(s) + \bar{g}(s), \tag{4.2.103}$$

where

$$\bar{h}(s) = \frac{\bar{q}_m(s)}{\bar{p}_n(s)} \quad \text{and} \quad \bar{g}(s) = \frac{\bar{R}_{n-1}(s) - \bar{S}_{m-1}(s)}{\bar{p}_n(s)}, \tag{4.2.104ab}$$

and $\bar{h}(s)$ is usually called the *transfer function.*

The inverse Laplace transform combined with the Convolution Theorem leads to the formal solution

$$x(t) = \int_0^t f(t-\tau)\, h(\tau) d\tau \;+\; g(t). \tag{4.2.105}$$

With zero initial data, $\bar{g}(s) = 0$, the transfer function takes the simple form

$$\bar{h}(s) = \frac{\bar{x}(s)}{\bar{f}(s)}. \tag{4.2.106}$$

If $f(t) = \delta(t)$ so that $\bar{f}(s) = 1$, then the output function is

$$x(t) = \int_0^t \delta(t-\tau)\, h(\tau) d\tau = h(t), \tag{4.2.107}$$

and $h(t)$ is known as the *impulse response*. ▯

***Example 4.2.16***

(*Delay Differential Equations*). In many problems, the derivatives of the unknown function $x(t)$ are related to its value at different times $t - \tau$. This leads us to consider differential equations of the form

$$\frac{dx}{dt} + a\, x(t-\tau) = f(t), \tag{4.2.108}$$

where $a$ is a constant and $f(t)$ is a given function. Equations of this type are called *delay differential equations*. In general, initial value problems for these equations involve the specification of $x(t)$ in the interval $t_0 - \tau \le t < t_0$, and this information combined with the equation itself is sufficient to determine $x(t)$ for $t > t_0$.

We show how equation (4.2.108) can be solved by the Laplace transform when $t_0 = 0$ and $x(t) = x_0$ for $t \le 0$. In view of the initial condition, we can write

$$x(t-\tau) = x(t-\tau)\, H(t-\tau)$$

so equation (4.2.108) is equivalent to

$$\frac{dx}{dt} + a\, x(t-\tau)\, H(t-\tau) = f(t). \tag{4.2.109}$$

Application of the Laplace transform to (4.2.109) gives

$$s\, \bar{x}(s) - x_0 + a \exp(-\tau s)\, \bar{x}(s) = \bar{f}(s).$$

Or,

$$\bar{x}(s) = \frac{x_0 + \bar{f}(s)}{\{s + a\exp(-\tau s)\}} \tag{4.2.110}$$

$$= \frac{1}{s}\{x_0 + \bar{f}(s)\}\left[1 + \frac{a}{s}\exp(-\tau s)\right]^{-1}$$

$$= \frac{1}{s}\{x_0 + \bar{f}(s)\}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{s}\right)^n\exp(-n\tau s). \tag{4.2.111}$$

The inverse Laplace transform gives the formal solution

$$x(t) = \mathscr{L}^{-1}\left[\frac{1}{s}\{x_0 + \bar{f}(s)\}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{s}\right)^n\exp(-n\tau s)\right]. \tag{4.2.112}$$

In order to write an explicit solution, we choose $x_0 = 0$ and $f(t) = t$, and hence, (4.2.112) becomes

$$x(t) = \mathscr{L}^{-1}\left[\frac{1}{s^3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{s}\right)^n\exp(-n\tau s)\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n a^n\,\frac{(t - n\tau)^{n+2}}{(n+2)!}\,H(t - n\tau), \quad t > 0. \tag{4.2.113}$$

□

***Example 4.2.17***

(*The Renewal Equation in Statistics*). The random function $X(t)$ of time $t$ represents the number of times some event has occurred between time 0 and time $t$, and is usually referred to as a *counting process*. A random variable $X_n$ that records the time it assumes for $X$ to get the value $n$ from the $n-1$ is referred to as an *inter-arrival time*. If the random variables $X_1, X_2, X_3, \ldots$ are independent and identically distributed, then the counting process $X(t)$ is called a *renewal process*. We represent their common probability distribution function by $F(t)$ and the density function by $f(t)$ so that $F'(t) = f(t)$. The *renewal function* is defined by the expected number of times the event being counted occurs by time $t$ and is denoted by $r(t)$ so that

$$r(t) = E\{X(t)\} = \int_0^{\infty} E\{X(t)|X_1 = x\}f(x)dx, \tag{4.2.114}$$

where $E\{X(t)|X_1 = x\}$ is the conditional expected value of $X(t)$ under the condition that $X_1 = x$ and has the value

$$E\{X(t)|X_1 = x\} = [1 + r(t - x)]\,H(t - x). \tag{4.2.115}$$

Thus,

$$r(t) = \int_0^t \{1 + r(t-x)\} f(x) dx.$$

Or,

$$r(t) = F(t) + \int_0^t r(t-x)\, f(x) dx. \tag{4.2.116}$$

This is called the *renewal equation* in mathematical statistics. We solve the equation by taking the Laplace transform with respect to $t$, and the Laplace transformed equation is

$$\bar{r}(s) = \overline{F}(s) + \bar{r}(s)\, \bar{f}(s).$$

Or,

$$\bar{r}(s) = \frac{\overline{F}(s)}{1 - \bar{f}(s)}. \tag{4.2.117}$$

The inverse transform gives the formal solution of the renewal function

$$r(t) = \mathscr{L}^{-t} \left\{ \frac{\overline{F}(s)}{1 - \bar{f}(s)} \right\}. \tag{4.2.118}$$

□

## 4.3 Partial Differential Equations, Initial and Boundary Value Problems

The Laplace transform method is very useful in solving a variety of partial differential equations with assigned initial and boundary conditions. The following examples illustrate the use of the Laplace transform method.

***Example 4.3.1***
(*First-Order Initial-Boundary Value Problem*). Solve the equation

$$u_t + x u_x = x, \qquad x > 0, \quad t > 0, \tag{4.3.1}$$

with the initial and boundary conditions

$$u(x,0) = 0 \qquad \text{for} \quad x > 0, \tag{4.3.2}$$
$$u(0,t) = 0 \qquad \text{for} \quad t > 0. \tag{4.3.3}$$

We apply the Laplace transform of $u(x,t)$ with respect to $t$ to obtain

$$s\,\bar{u}(x,s) + x\,\frac{d\bar{u}}{dx} = \frac{x}{s}, \qquad \bar{u}(0,s) = 0.$$

Using the integrating factor $x^s$, the solution of this transformed equation is

$$\bar{u}(x,s) = A\,x^{-s} + \frac{x}{s(s+1)},$$

where $A$ is a constant of integration. Since $\bar{u}(0,s) = 0$, $A = 0$ for a bounded solution. Consequently,

$$\bar{u}(x,s) = \frac{x}{s(s+1)} = x\left(\frac{1}{s} - \frac{1}{s+1}\right).$$

The inverse Laplace transform gives the solution

$$u(x,t) = x(1 - e^{-t}). \tag{4.3.4}$$

$\square$

***Example 4.3.2***
Find the solution of the equation

$$x\,u_t + u_x = x, \qquad x > 0, \quad t > 0 \tag{4.3.5}$$

with the same initial and boundary conditions (4.3.2) and (4.3.3).

Application of the Laplace transform with respect to $t$ to (4.3.5) with the initial conditon gives

$$\frac{d\bar{u}}{dx} + x\,s\,\bar{u} = \frac{x}{s}.$$

Using the integrating factor $\exp\left(\frac{1}{2}\,x^2 s\right)$ gives the solution

$$\bar{u}(x,s) = \frac{1}{s^2} + A\exp\left(-\frac{1}{2}\,s\,x^2\right),$$

where $A$ is an integrating constant. Since $\bar{u}(0,s) = 0$, $A = -\dfrac{1}{s^2}$ and hence, the solution is

$$\bar{u}(x,s) = \frac{1}{s^2}\left[1 - \exp\left(-\frac{1}{2}\,x^2\,s\right)\right]. \tag{4.3.6}$$

Finally, we obtain the solution by inversion

$$u(x,t) = t - \left(t - \frac{1}{2}\,x^2\right) H\left(t - \frac{x^2}{2}\right). \tag{4.3.7}$$

Or, equivalently,

$$u(x,t) = \left\{ \begin{array}{ll} t, & 2t < x^2 \\ \frac{1}{2}x^2, & 2t > x^2 \end{array} \right\}. \tag{4.3.8}$$

□

**Example 4.3.3**
(*The Heat Conduction Equation in a Semi-Infinite Medium*). Solve the equation

$$u_t = \kappa\, u_{xx}, \quad x > 0, \quad t > 0 \tag{4.3.9}$$

with the initial and boundary conditions

$$u(x,0) = 0, \qquad x > 0 \tag{4.3.10}$$

$$u(0,t) = f(t), \qquad t > 0 \tag{4.3.11}$$

$$u(x,t) \to 0 \qquad \text{as } x \to \infty, \quad t > 0. \tag{4.3.12}$$

Application of the Laplace transform with respect to $t$ to (4.3.9) gives

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{\kappa}\,\bar{u} = 0. \tag{4.3.13}$$

The general solution of this equation is

$$\bar{u}(x,s) = A \exp\left(-x\sqrt{\frac{s}{\kappa}}\right) + B \exp\left(x\sqrt{\frac{s}{\kappa}}\right). \tag{4.3.14}$$

where $A$ and $B$ are integrating constants. For a bounded solution, $B \equiv 0$, and using $\bar{u}(0,s) = \bar{f}(s)$, we obtain the solution

$$\bar{u}(x,s) = \bar{f}(s) \exp\left(-x\sqrt{\frac{s}{\kappa}}\right). \tag{4.3.15}$$

The inversion theorem gives the solution

$$u(x,t) = \frac{x}{2\sqrt{\pi\kappa}} \int_0^t f(t-\tau)\tau^{-3/2} \exp\left(-\frac{x^2}{4\kappa\tau}\right) d\tau, \tag{4.3.16}$$

which is, by putting $\lambda = \frac{x}{2\sqrt{\kappa\tau}}$, or, $d\lambda = -\frac{x}{4\sqrt{\kappa}}\,\tau^{-3/2} d\tau$,

$$= \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{\kappa t}}}^{\infty} f\left(t - \frac{x^2}{4\kappa\lambda^2}\right) e^{-\lambda^2}\, d\lambda. \tag{4.3.17}$$

This is the formal solution of the problem.

In particular, if $f(t) = T_0 =$ constant, solution (4.3.17) becomes

$$u(x,t) = \frac{2T_0}{\sqrt{\pi}} \int_{\frac{x}{\kappa t}}^{\infty} e^{-\lambda^2} d\lambda = T_0 \, erfc\left(\frac{x}{2\sqrt{\kappa t}}\right). \tag{4.3.18}$$

Clearly, the temperature distribution tends asymptotically to the constant value $T_0$ as $t \to \infty$.

We consider another physical problem that is concerned with the determination of the temperature distribution in a semi-infinite solid when the rate of flow of heat is prescribed at the end $x = 0$. Thus, the problem is to solve diffusion equation (4.3.9) subject to conditions (4.3.10) and (4.3.12)

$$-k\left(\frac{\partial u}{\partial x}\right) = g(t) \quad \text{at}\, x = 0, \quad t > 0, \tag{4.3.19}$$

where $k$ is a constant that is called *thermal conductivity.*

Application of the Laplace transform gives the solution of the transformed problem

$$\bar{u}(x,s) = \frac{1}{k}\sqrt{\frac{\kappa}{s}}\; \bar{g}(s) \exp\left(-x\sqrt{\frac{s}{\kappa}}\right). \tag{4.3.20}$$

The inverse Laplace transform yields the solution

$$u(x,t) = \frac{1}{k}\sqrt{\frac{\kappa}{\pi}} \int_0^t g(t-\tau)\, \tau^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4\kappa t}\right) d\tau, \tag{4.3.21}$$

which is, by the change of variable $\lambda = \dfrac{x}{2\sqrt{\kappa\tau}}$,

$$= \frac{x}{k\sqrt{\pi}} \int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty} g\left(t - \frac{x^2}{4\kappa\lambda^2}\right) \lambda^{-2} e^{-\lambda^2}\, d\lambda. \tag{4.3.22}$$

In particular, if $g(t) = T_0 =$ constant, the solution becomes

$$u(x,t) = \left(\frac{T_0 x}{k\sqrt{\pi}}\right) \int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty} \lambda^{-2} e^{-\lambda^2}\, d\lambda.$$

Integrating this result by parts gives the solution

$$u(x,t) = \frac{T_0}{\kappa}\left[2\sqrt{\frac{kt}{\pi}} \exp\left(-\frac{x^2}{4\kappa t}\right) - x \; erfc\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]. \tag{4.3.23}$$

Alternatively, the heat conduction problem (4.3.9)–(4.3.12) can be solved by using fractional derivatives (see Chapter 5 or Debnath, 1978). We recall (4.3.15) and rewrite it

$$\frac{\partial \bar{u}}{\partial x} = -\sqrt{\frac{s}{\kappa}}\ \bar{u}. \tag{4.3.24}$$

In view of (3.9.21), this can be expressed in terms of fractional derivative of order $\frac{1}{2}$ as

$$\frac{\partial u}{\partial x} = -\frac{1}{\sqrt{\kappa}}\mathscr{L}^{-1}\left\{\sqrt{s}\,\bar{u}(x,s)\right\} = -\frac{1}{\sqrt{\kappa}}\ {}_0D_t^{\frac{1}{2}}\,u(x,t). \tag{4.3.25}$$

Thus, the heat flux is expressed in terms of the fractional derivative. In particular, when $u(0,t) =$ constant $= T_0$, then the heat flux at the surface is

$$-k\left(\frac{\partial u}{\partial x}\right)_{x=0} = \frac{k}{\sqrt{\kappa}}D_t^{\frac{1}{2}}\,T_0 = \frac{kT_0}{\sqrt{\pi\kappa\, t}}. \tag{4.3.26}$$

□

***Example 4.3.4***

(*Diffusion Equation in a Finite Medium*). Solve the diffusion equation

$$u_t = \kappa\, u_{xx}, \quad 0 < x < a, \quad t > 0, \tag{4.3.27}$$

with the initial and boundary conditions

$$u(x,0) = 0, \quad 0 < x < a, \tag{4.3.28}$$

$$u(0,t) = U, \quad t > 0, \tag{4.3.29}$$

$$u_x(a,t) = 0, \quad t > 0, \tag{4.3.30}$$

where $U$ is a constant.

We introduce the Laplace transform of $u(x,t)$ with respect to $t$ to obtain

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{\kappa}\,\bar{u} = 0, \qquad 0 < x < a, \tag{4.3.31}$$

$$\bar{u}(0,s) = \frac{U}{s}, \quad \left(\frac{d\bar{u}}{dx}\right)_{x=a} = 0. \tag{4.3.32ab}$$

The general solution of (4.3.31) is

$$\bar{u}(x,s) = A\cosh\left(x\sqrt{\frac{s}{\kappa}}\right) + B\sinh\left(x\sqrt{\frac{s}{\kappa}}\right), \tag{4.3.33}$$

where $A$ and $B$ are constants of integration. Using (4.3.32ab), we obtain the values of $A$ and $B$ so that the solution (4.3.33) becomes

$$\bar{u}(x,s) = \frac{U}{s} \cdot \frac{\cosh\left[(a-x)\sqrt{\frac{s}{\kappa}}\right]}{\cosh\left(a\sqrt{\frac{s}{\kappa}}\right)}. \tag{4.3.34}$$

The inverse Laplace transform gives the solution

$$u(x,t) = U\mathscr{L}^{-1}\left\{\frac{\cosh(a-x)\sqrt{\frac{s}{\kappa}}}{s\cosh\left(a\sqrt{\frac{s}{\kappa}}\right)}\right\}. \tag{4.3.35}$$

The inversion can be carried out by the Cauchy Residue Theorem to obtain

$$u(x,t) = U\left[1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\left\{\frac{(2n-1)(a-x)\pi}{2a}\right\} \times \exp\left\{-(2n-1)^2\left(\frac{\pi}{2a}\right)^2\kappa t\right\}\right], \tag{4.3.36}$$

which is, by expanding the cosine term,

$$= U\left[1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left\{\left(\frac{2n-1}{2a}\right)\pi x\right\} \times \exp\left\{-(2n-1)^2\left(\frac{\pi}{2a}\right)^2\kappa t\right\}\right]. \tag{4.3.37}$$

This result can be obtained by the method of separation of variables. ▯

***Example 4.3.5***

(*Diffusion in a Finite Medium*). Solve the one-dimensional diffusion equation in a finite medium $0 < z < a$, where the concentration function $C(z,t)$ satisfies the equation

$$C_t = \kappa C_{zz}, \quad 0 < z < a, \quad t > 0, \tag{4.3.38}$$

and the initial and boundary data

$$C(z,0) = 0 \quad \text{for } 0 < z < a, \tag{4.3.39}$$

$$C(z,t) = C_0 \quad \text{for } z = a, \quad t > 0, \tag{4.3.40}$$

$$\frac{\partial C}{\partial z} = 0 \quad \text{for } z = 0, \quad t > 0, \tag{4.3.41}$$

where $C_0$ is a constant.

Application of the Laplace transform of $C(z,t)$ with respect to $t$ gives

$$\frac{d^2\bar{C}}{dz^2} - \left(\frac{s}{\kappa}\right)\bar{C} = 0, \quad 0 < z < a,$$
$$\bar{C}(a,s) = \frac{C_0}{s}, \quad \left(\frac{d\bar{C}}{dz}\right)_{z=0} = 0.$$

The solution of this system is

$$\bar{C}(z,s) = \frac{C_0 \cosh\left(z\sqrt{\frac{s}{\kappa}}\right)}{s \cosh\left(a\sqrt{\frac{s}{\kappa}}\right)}, \tag{4.3.42}$$

which is, by writing $\alpha = \sqrt{\frac{s}{\kappa}}$,

$$= \frac{C_0}{s}\frac{(e^{\alpha z} + e^{-\alpha z})}{(e^{\alpha a} + e^{-\alpha a})}$$
$$= \frac{C_0}{s}[\exp\{-\alpha(a-z)\} + \exp\{-\alpha(a+z)\}]\sum_{n=0}^{\infty}(-1)^n \exp(-2n\alpha a)$$
$$= \frac{C_0}{s}\left\{\sum_{n=0}^{\infty}(-1)^n \exp[-\alpha\{(2n+1)a - z\}]\right.$$
$$\left. + \sum_{n=0}^{\infty}(-1)^n \exp[-\alpha\{(2n+1)a + z\}]\right\}. \tag{4.3.43}$$

Using the result (3.7.4), we obtain the final solution

$$C(z,t) = C_0\left\{\sum_{n=0}^{\infty}(-1)^n \left[erfc\left\{\frac{(2n+1)a - z}{2\sqrt{\kappa t}}\right\}\right.\right.$$
$$\left.\left. + erfc\left\{\frac{(2n+1)a + z}{2\sqrt{\kappa t}}\right\}\right]\right\}. \tag{4.3.44}$$

This solution represents as infinite series of complementary error functions. The successive terms of this series are in fact the concentrations at depths $a - z, a + z, 3a - z, 3a + z, ...$ in the medium. The series converges rapidly for all except large values of $\left(\frac{\kappa t}{a^2}\right)$. ▯

***Example 4.3.6***

(*The Wave Equation for the Transverse Vibration of a Semi-Infinite String*). Find the displacement of a semi-infinite string which is initially at rest in its equilibrium position. At time $t=0$, the end $x=0$ is constrained to move so that the displacement is $u(0,t)=Af(t)$ for $t\geq 0$, where $A$ is a constant. The problem is to solve the one-dimensional wave equation

$$u_{tt}=c^2u_{xx}, \quad 0\leq x<\infty, \quad t>0, \tag{4.3.45}$$

with the boundary and initial conditions

$$u(x,t)=Af(t) \qquad \text{at } x=0, \qquad t\geq 0, \tag{4.3.46}$$

$$u(x,t)\to 0 \qquad \text{as } x\to\infty, \qquad t\geq 0, \tag{4.3.47}$$

$$u(x,t)=0=\frac{\partial u}{\partial t} \qquad \text{at } t=0 \qquad \text{for } 0<x<\infty. \tag{4.3.48ab}$$

Application of the Laplace transform of $u(x,t)$ with respect to $t$ gives

$$\begin{aligned}\frac{d^2\bar{u}}{dx^2}-\frac{s^2}{c^2}\,\bar{u}=0, &\qquad \text{for } 0\leq x<\infty,\\ \bar{u}(x,s)=A\bar{f}(s) &\qquad \text{at } x=0,\\ \bar{u}(x,s)\to 0 &\qquad \text{as } x\to\infty.\end{aligned}$$

The solution of this differential system is

$$\bar{u}(x,s)=A\bar{f}(s)\exp\left(-\frac{xs}{c}\right). \tag{4.3.49}$$

Inversion gives the solution

$$u(x,t)=Af\left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right). \tag{4.3.50}$$

In other words, the solution is

$$u(x,t)=\begin{bmatrix} Af\left(t-\dfrac{x}{c}\right), & t>\dfrac{x}{c}\\ 0, & t<\dfrac{x}{c}\end{bmatrix}. \tag{4.3.51}$$

This solution represents a wave propagating at a velocity $c$ with the characteristic $x=ct$. ▯

***Example 4.3.7***

(*Potential and Current in an Electric Transmission Line*). We consider a transmission line which is a model of co-axial cable containing resistance $R$, inductance $L$, capacitance $C$, and leakage conductance $G$. The current $I(x,t)$ and potential $V(x,t)$ at a point $x$ and time $t$ in the line satisfy the coupled equations

$$L\,\frac{\partial I}{\partial t} + RI = -\frac{\partial V}{\partial x}, \tag{4.3.52}$$

$$C\,\frac{\partial V}{\partial t} + GV = -\frac{\partial I}{\partial x}. \tag{4.3.53}$$

If $I$ or $V$ is eliminated from these equations, both $I$ and $V$ satisfy the same equation in the form

$$\frac{1}{c^2}\,u_{tt} - u_{xx} + au_t + bu = 0 \tag{4.3.54}$$

where $c^2 = (LC)^{-1}, a = LG + RC$, and $b = RG$. Equation (4.3.54) is called the *telegraph equation.*

Or, equivalently, the telegraph equation can be written in the form

$$u_{tt} = c^2\,u_{xx} - (p+q)\,u_t - p\,q\,u \tag{4.3.55}$$

where $ac^2 = \frac{R}{C} + \frac{G}{C} = p + q$ and $bc^2 = pq$.

For a lossless transmission line, $R = 0$ and $G = 0$, $I$ or $V$ satisfies the classical wave equation

$$u_{tt} = c^2 u_{xx}. \tag{4.3.56}$$

The solution of this equation with the initial and boundary data is obtained from Example 4.3.6 using the boundary conditions in the potential $V(x,t)$:

$$\text{(i)}\quad V(x,t) = V_0 f(t) \qquad \text{at} \quad x = 0,\ t > 0. \tag{4.3.57}$$

This corresponds to a signal at the end $x = 0$ for $t > 0$, and $V(x,t) \to 0$ as $x \to \infty$ for $t > 0$.

A special case when $f(t) = H(t)$ is also of interest. The solution for this special case is given by

$$V(x,t) = V_0 f\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right). \tag{4.3.58}$$

This represents a wave propagating at a speed $c$ with the characteristic $x = ct$.

Similarly, the solution associated with the boundary data

$$\text{(ii)}\quad \mathrm{V(x,t)} = V_0 \cos\omega t \qquad \text{at} \quad \mathrm{x} = 0 \text{ for } \mathrm{t} > 0 \tag{4.3.59}$$

$$V(x,t) \to 0 \qquad \text{as} \quad \mathrm{x} \to \infty \text{ for } \mathrm{t} > 0 \tag{4.3.60}$$

can readily be obtained from Example 4.3.6.

For ideal submarine cable (or the *Kelvin ideal cable*), $L = 0$ and $G = 0$ equation (4.3.54) reduces to the classical diffusion equation

$$u_t = \kappa\,u_{xx}, \tag{4.3.61}$$

where $\kappa = a^{-1} = (RC)^{-1}$.

The method of solution is similar to that discussed in Example 4.3.3. Using the boundary data (i), the solution for the potential $V(x,t)$ is given by

$$V(x,t) = V_0 \, erfc\left(\frac{x}{2\sqrt{\kappa t}}\right). \tag{4.3.62}$$

The current field is given by

$$I(x,t) = -\frac{1}{R}\left(\frac{\partial V}{\partial x}\right) = \frac{V_0}{R}(\pi\kappa t)^{-1/2} \exp\left(-\frac{x^2}{4\kappa t}\right). \tag{4.3.63}$$

For very large $x$, the asymptotic representation of the complementary error function is

$$erfc(x) \sim \frac{1}{x\sqrt{\pi}} \exp(-x^2), \quad x \to \infty. \tag{4.3.64}$$

In view of this asymptotic representation, solution (4.3.62) becomes

$$V(x,t) \sim \frac{2V_0}{x}\left(\frac{\kappa t}{\pi}\right)^{1/2} \exp\left(-\frac{x^2}{4\kappa t}\right). \tag{4.3.65}$$

For any $t > 0$, no matter how small, solution (4.3.62) reveals that $V(x,t) > 0$ for all $x > 0$, even though $V(x,t) \to 0$ as $x \to \infty$ Thus, the signal applied at $t = 0$ propagates with the infinite speed although its amplitude is very small for large $x$. Physically, the infinite speed is unrealistic and is essentially caused by the neglect of the first term in equation (4.3.54). In a real cable, the presence of some inductance would set a limit to the speed of propagation.

Instead of the Kelvin cable, a non-inductive leady cable ($L = 0$ and $G \neq 0$) is of interest. The equation for this case is obtained from (4.3.54) in the form

$$V_{xx} - a\,V_t - b\,V = 0, \tag{4.3.66}$$

with zero initial conditions, and with the boundary data

$$V(0,t) = H(t) \text{ and } V(x,t) \to 0 \text{ as } x \to \infty. \tag{4.3.67ab}$$

The Laplace transformed problem is

$$\frac{d^2\overline{V}}{dx^2} = (sa + b)\overline{V}, \tag{4.3.68}$$

$$\overline{V}(0,s) = \frac{1}{s}, \quad \overline{V}(x,s) \to 0 \quad \text{as } x \to \infty. \tag{4.3.69ab}$$

Thus, the solution is given by

$$\overline{V}(x,s) = \frac{1}{s} \exp[-x(sa + b)^{1/2}]. \tag{4.3.70}$$

With the aid of a standard table of the inverse Laplace transform, the solution is given by

$$V(x,t)=\frac{1}{2}\,e^{x\sqrt{b}}\,erfc\left(\frac{x}{2}\sqrt{\frac{a}{t}}+\sqrt{\frac{bt}{a}}\right)+\frac{1}{2}\,e^{-x\sqrt{b}}\,erfc\left(\frac{x}{2}\sqrt{\frac{a}{t}}-\frac{bt}{a}\right). \tag{4.3.71}$$

When $G=0\,(b=0)$, the solution becomes identical with (4.3.62).

For the Heaviside distortionless cable, $\frac{R}{L}=\frac{G}{C}=k=$ constant, the potential $V(x,t)$ and the current $I(x,t)$ satisfies the same equation

$$u_{tt}+2ku_t+k^2u=c^2u_{xx}, \quad 0\le x<\infty, \quad t>0. \tag{4.3.72}$$

We solve this equation with the initial data (4.3.48ab) and the boundary condition (4.3.57). Application of the Laplace transform with respect to $t$ to (4.3.72) gives

$$\frac{d^2\overline{V}}{dx^2}=\left(\frac{s+k}{c}\right)^2\overline{V}. \tag{4.3.73}$$

The solution for $\overline{V}(x,s)$ with the transformed boundary condition (4.3.56) is

$$\overline{V}(x,s)=V_0\bar{f}(s)\exp\left[-\left(\frac{s+k}{c}\right)x\right]. \tag{4.3.74}$$

This can easily be inverted to obtain the final solution

$$V(x,t)=V_0\exp\left(-\frac{kx}{c}\right)f\left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right). \tag{4.3.75}$$

This solution represents the signal that propagates with velocity $c=(LC)^{-1/2}$ with exponentially decaying amplitude, but with no distortion. Thus, the signals can propagate along the Heaviside distortionless line over long distances if appropriate boosters are placed at regular intervals in order to increase the strength of the signal so as to counteract the effects of attenuation. □

***Example 4.3.8***

Find the bounded solution of the axisymmetric heat conduction equation

$$u_t=\kappa\left(u_{rr}+\frac{1}{r}u_r\right), \quad 0\le r<a, \quad t>0, \tag{4.3.76}$$

with the initial and boundary data

$$u(r,0)=0 \qquad \text{for } 0<r<a, \tag{4.3.77}$$

$$u(r,t)=f(t) \quad \text{at } r=a \text{ for } t>0, \tag{4.3.78}$$

where $\kappa$ and $T_0$ are constants.

Application of the Laplace transform to (4.3.76) gives

$$\frac{d^2\bar{u}}{dr^2}+\frac{1}{r}\frac{d\bar{u}}{dr}-\frac{s}{\kappa}\bar{u}=0.$$

Or,

$$r^2\frac{d^2\bar{u}}{dr^2}+r\frac{d\bar{u}}{dr}-r^2\left(\frac{s}{\kappa}\right)\bar{u}=0. \tag{4.3.79}$$

This is the standard Bessel equation with the solution

$$\bar{u}(r,s)=AI_0\left(r\sqrt{\frac{s}{\kappa}}\right)+BK_0\left(r\sqrt{\frac{s}{\kappa}}\right), \tag{4.3.80}$$

where $A$ and $B$ are constants of integration and $I_0(x)$ and $K_0(x)$ are the modified Bessel functions of zero order.

Since $K_0(\alpha r)$ is unbounded at $r=0$, for the bounded solution $B\equiv 0$, and hence, the solution is

$$\bar{u}(r,s)=AI_0(kr), \quad k=\sqrt{\frac{s}{\kappa}}.$$

In view of the transformed boundary condition $\bar{u}(a,s)=\bar{f}(s)$, we obtain

$$\bar{u}(r,s)=\bar{f}(s)\,\frac{I_0(kr)}{I_0(ka)}=\bar{f}(s)\bar{g}(s), \tag{4.3.81}$$

where $\bar{g}(s)=\dfrac{I_0(kr)}{I_0(ka)}$.

By Convolution Theorem 3.5.1, the solution takes the form

$$u(r,t)=\int_0^t f(t-\tau)g(\tau)d\tau, \tag{4.3.82}$$

where

$$g(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\frac{I_0(kr)}{I_0(ka)}ds. \tag{4.3.83}$$

This complex integral can be evaluated by the theory of residues where the poles of the integrand are at the points $s=s_n=-\kappa\alpha_n^2$, $n=1,2,3,\ldots$ and $\alpha_n$ are the roots of $J_0(a\alpha)=0$. The residue at pole $s=s_n$ is

$$\left(\frac{2i\kappa\alpha_n}{a}\right)\frac{I_0(ir\alpha_n)}{I_0'(ia\alpha_n)}\exp(-\kappa t\alpha_n^2)=\left(\frac{2\kappa\alpha_n}{a}\right)\frac{J_0(r\alpha_n)}{J_1(a\alpha_n)}\exp(-\kappa\,t\,\alpha_n^2),$$

so that

$$g(t) = \left(\frac{2\kappa}{a}\right)\sum_{n=1}^{\infty}\frac{\alpha_n J_0(r\alpha_n)}{J_1(a\alpha_n)}\exp(-\kappa\, t\,\alpha_n^2).$$

Thus, solution (4.3.82) becomes

$$u(r,t) = \left(\frac{2\kappa}{a}\right)\sum_{n=1}^{\infty}\frac{\alpha_n J_0(r\alpha_n)}{J_1(a\alpha_n)}\int_0^t f(t-\tau)\exp(-\kappa\,\tau\,\alpha_n^2)\,d\tau, \tag{4.3.84}$$

where the summation is taken over the positive roots of $J_0(a\alpha)=0$.

In particular, if $f(t)=T_0$, then the solution (4.3.84) reduces to

$$\begin{aligned} u(r,t) &= \left(\frac{2T_0}{a}\right)\sum_{n=1}^{\infty}\frac{J_0(r\alpha_n)}{\alpha_n J_1(a\alpha_n)}(1-e^{-\kappa\,t\,\alpha_n^2}) \\ &= T_0\left[1-\frac{2}{a}\sum_{n=1}^{\infty}\frac{J_0(r\alpha_n)}{\alpha_n J_1(a\alpha_n)}e^{-\kappa\,t\,\alpha_n^2}\right]. \end{aligned} \tag{4.3.85}$$

▯

***Example 4.3.9***

(*Inhomogeneous Partial Differential Equation*). We solve the inhomogeneous problem

$$u_{xt} = -\omega\sin\omega t, \quad t>0 \tag{4.3.86}$$

$$u(x,0)=x, \quad u(0,t)=0. \tag{4.3.87ab}$$

Application of the Laplace transform with respect to $t$ gives

$$\frac{d\bar{u}}{dx} = \frac{s}{s^2+\omega^2},$$

which admits the general solution

$$\bar{u}(x,s) = \frac{sx}{s^2+\omega^2} + A,$$

where $A$ is a constant. Since $\bar{u}(0,s)=0$, $A=0$ and hence, the solution is obtained by inversion as

$$u(x,t) = x\cos\omega t. \tag{4.3.88}$$

▯

***Example 4.3.10***

(*Inhomogeneous Wave Equation*). Find the solution of

$$\frac{1}{c^2}u_{tt} - u_{xx} = k\sin\left(\frac{\pi x}{a}\right), \quad 0<x<a,\ t>0, \tag{4.3.89}$$

$$u(x,0) = 0 = u_t(x,0), \qquad 0<x<a, \tag{4.3.90}$$

$$u(0,t) = 0 = u(a,t), \qquad t>0, \tag{4.3.91}$$

where $c, k$, and $a$ are constants.

Application of the Laplace transforms gives

$$\frac{d^2\bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u} = -\frac{k}{s}\sin\left(\frac{\pi x}{a}\right), \tag{4.3.92}$$

$$\bar{u}(0,s) = 0 = \bar{u}(a,s). \tag{4.3.93}$$

The general solution of equation (4.3.92) is

$$\bar{u}(x,s) = A\exp\left(\frac{sx}{c}\right) + B\exp\left(-\frac{sx}{c}\right) + \frac{k\sin\left(\frac{\pi x}{a}\right)}{a^2 s\left(s^2 + \frac{\pi^2 c^2}{a^2}\right)}. \tag{4.3.94}$$

In view of (4.3.93), $A = B = 0$, and hence, the solution (4.3.94) becomes

$$\bar{u}(x,s) = \frac{k}{\pi^2 c^2}\sin\left(\frac{\pi x}{a}\right)\left[\frac{1}{s} - \frac{s}{s^2 + \frac{\pi^2 c^2}{a^2}}\right], \tag{4.3.95}$$

which, by inversion, gives the solution,

$$u(x,t) = \frac{k}{(\pi c)^2}\left[1 - \cos\left(\frac{\pi ct}{a}\right)\right]\sin\left(\frac{\pi x}{a}\right). \tag{4.3.96}$$

□

***Example 4.3.11***

(*The Stokes Problem and the Rayleigh Problem in Fluid Dynamics*). Solve the Stokes problem, which is concerned with the unsteady boundary layer flows induced in a semi-infinite viscous fluid bounded by an infinite horizontal disk at $z = 0$ due to non-torsional oscillations of the disk in its own plane with a given frequency $\omega$.

We solve the boundary layer equation in fluid dynamics

$$u_t = \nu\, u_{zz}, \quad z > 0, \quad t > 0, \tag{4.3.97}$$

with the boundary and initial conditions

$$u(z,t) = U_0\, e^{i\omega t} \qquad \text{on } z = 0,\ t > 0, \tag{4.3.98}$$

$$u(z,t) \to 0 \qquad \text{as } z \to \infty,\ t > 0, \tag{4.3.99}$$

$$u(z,t) \to 0 \qquad \text{at } t \le 0 \text{ for all } z > 0, \tag{4.3.100}$$

where $u(z,t)$ is the velocity of fluid of kinematic viscosity $\nu$ and $U_0$ is a constant.

The Laplace transform solution of the problem with the transformed boundary conditions is

$$\overline{u}(z,s) = \frac{U_0}{(s-i\omega)} \exp\left(-z\sqrt{\frac{s}{\nu}}\right). \tag{4.3.101}$$

Using a standard table of inverse Laplace transforms, we obtain the solution

$$u(z,t) = \frac{U_0}{2}\, e^{i\omega t}[\exp(-\lambda z)\, erfc(\zeta - \sqrt{i\omega t}) + \exp(\lambda z)\, erfc(\zeta + \sqrt{i\omega t})], \tag{4.3.102}$$

where $\zeta = z/(2\sqrt{\nu t})$ is called the *similarity variable* of the viscous boundary layer theory and $\lambda = (i\omega/\nu)^{\frac{1}{2}}$. The result (4.3.101) describes the unsteady boundary layer flow.

In view of the asymptotic formula for the complementary error function

$$erfc(\zeta \mp \sqrt{i\omega t}) \sim (2,0) \quad \text{as} \quad t \to \infty, \tag{4.3.103}$$

the above solution for $u(z,t)$ has the asymptotic representation

$$u(z,t) \sim U_0 \exp(i\omega t - \lambda z) = U_0 \exp\left[i\omega t - \left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}(1+i)z\right]. \tag{4.3.104}$$

This is called the *Stokes steady-state solution.* This represents the propagation of shear waves which spread out from the oscillating disk with velocity $(\omega/k) = \sqrt{2\nu\omega}$ and exponentially decaying amplitude. The boundary layer associated with the solution has thickness of the order $\sqrt{\nu/\omega}$ in which the shear oscillations imposed by the disk decay exponentially with distance $z$ from the disk. This boundary layer is called the *Stokes layer.* In other words, the thickness of the Stokes layer is equal to the depth of penetration of vorticity which is essentially confined to the immediate vicinity of the disk for high frequency $\omega$.

The Stokes problem with $\omega = 0$ becomes the *Rayleigh problem.* In other words, the motion is generated in the fluid from rest by moving the disk impulsively in its own plane with constant velocity $U_0$. In this case, the Laplace transformed solution is

$$\overline{u}(z,s) = \frac{U_0}{s} \exp\left(-z\sqrt{\frac{s}{\nu}}\right). \tag{4.3.105}$$

Hence, the inversion gives the Rayleigh solution

$$u(z,t) = U_0\, erfc\left(\frac{z}{2\sqrt{\nu t}}\right). \tag{4.3.106}$$

This describes the growth of a boundary layer adjacent to the disk. The associated boundary layer is called the *Rayleigh layer* of thickness of the order

$\delta \sim \sqrt{\nu t}$, which grows with increasing time. The rate of growth is of the order $d\delta/dt \sim \sqrt{\nu/t}$, which diminishes with increasing time.

The vorticity of the unsteady flow is given by

$$\frac{\partial u}{\partial z} = \frac{U_0}{\sqrt{\pi \nu t}} \exp(-\zeta^2), \tag{4.3.107}$$

which decays exponentially to zero as $z >> \delta$.

Note that the vorticity is everywhere zero at $t = 0$. This implies that it is generated at the disk and diffuses outward within the Rayleigh layer. The total viscous diffusion time is $T_d \sim \left(\delta^2/\nu\right)$.

Another physical quantity related to the Stokes and Rayleigh problems is the *skin friction* on the disk defined by

$$\tau_0 = \mu \left(\frac{\partial u}{\partial z}\right)_{z=0}, \tag{4.3.108}$$

where $\mu = \nu\rho$ is the dynamic viscosity and $\rho$ is the density of the fluid. The skin friction can readily be calculated from the flow field given by (4.3.104) or (4.3.106). ▯

## 4.4 Solutions of Integral Equations

***DEFINITION 4.4.1*** *An equation in which the unknown function occurs under an integral is called an integral equation.*

*An equation of the form*

$$f(t) = h(t) + \lambda \int_a^b k(t,\tau) f(\tau)\, d\tau, \tag{4.4.1}$$

*in which $f$ is the unknown function, $h(t), k(t,\tau)$; and the limits of integration $a$ and $b$ are known; and $\lambda$ is a constant, is called the linear integral equation of the second kind or the linear Volterra integral equation. The function $k(t,\tau)$ is called the kernel of the equation. Such an equation is said to be homogeneous or inhomogeneous according to $h(t) = 0$ or $h(t) \neq 0$. If the kernel of the equation has the form $k(t,\tau) = g(t-\tau)$, the equation is referred to as the convolution integral equation.*

In this section, we show how the Laplace transform method can be applied successfully to solve the convolution integral equations. This method is simple and straightforward, and can be illustrated by examples.

To solve the convolution integral equation of the form

$$f(t) = h(t) + \lambda \int_0^t g(t-\tau) f(\tau)\, d\tau, \tag{4.4.2}$$

we take the Laplace transform of this equation to obtain

$$\bar{f}(s) = \bar{h}(s) + \lambda \mathscr{L}\left\{ \int_0^t g(t-\tau) f(\tau)\, d\tau \right\},$$

which is, by the Convolution Theorem,

$$\bar{f}(s) = \bar{h}(s) + \lambda \bar{f}(s)\bar{g}(s).$$

Or,

$$\bar{f}(s) = \frac{\bar{h}(s)}{1 - \lambda \bar{g}(s)}. \tag{4.4.3}$$

Inversion gives the formal solution

$$f(t) = \mathscr{L}^{-1}\left\{ \frac{\bar{h}(s)}{1 - \lambda \bar{g}(s)} \right\}. \tag{4.4.4}$$

In many simple cases, the right-hand side can be inverted by using partial fractions or the theory of residues. Hence, the solution can readily be found.

***Example 4.4.1***
Solve the integral equation

$$f(t) = a + \lambda \int_0^t f(\tau)\, d\tau. \tag{4.4.5}$$

We take the Laplace transform of (4.4.5) to find

$$\bar{f}(s) = \frac{a}{s - \lambda},$$

whence, by inversion, it follows that

$$f(t) = a \exp(\lambda t). \tag{4.4.6}$$

$\square$

***Example 4.4.2***
Solve the integro-differential equation

$$f(t) = a \sin t + 2 \int_0^t f'(\tau) \sin(t-\tau)\, d\tau, \quad f(0) = 0. \tag{4.4.7}$$

Taking the Laplace transform, we obtain

$$\bar{f}(s) = \frac{a}{s^2+1} + 2\mathscr{L}\{f'(t)\}\mathscr{L}\{\sin t\}$$

Or,

$$\bar{f}(s) = \frac{a}{s^2+1} + 2\frac{\{s\bar{f}(s) - f(0)\}}{s^2+1}.$$

Hence, by the initial condition,

$$\bar{f}(s) = \frac{a}{(s-1)^2}.$$

Inversion yields the solution

$$f(t) = a\,t\exp(t). \tag{4.4.8}$$

□

***Example 4.4.3***
Solve the integral equation

$$f(t) = a\,t^n - e^{-bt} - c\int_0^t f(\tau)\,e^{c(t-\tau)}\,d\tau. \tag{4.4.9}$$

Taking the Laplace transform, we obtain

$$\bar{f}(s) = \frac{a\,n!}{s^{n+1}} - \frac{1}{s+b} - \bar{f}(s)\frac{c}{s-c}$$

so that we have

$$\begin{aligned}
\bar{f}(s) &= \left(\frac{s-c}{s}\right)\left[\frac{a\,n!}{s^{n+1}} - \frac{1}{s+b}\right] \\
&= \frac{a\,n!}{s^{n+1}} - \frac{(ac)n!}{s^{n+2}} - \frac{1}{s}\left[\frac{s+b-c-b}{s+b}\right] \\
&= \frac{a\,n!}{s^{n+1}} - \frac{(ac)n!}{s^{n+2}} - \frac{1}{s} + \frac{c+b}{b}\left[\frac{1}{s} - \frac{1}{s+b}\right] \\
&= \frac{a\,n!}{s^{n+1}} - \frac{(ac)n!}{s^{n+2}} - \frac{1}{s} + \left(1+\frac{c}{b}\right)\frac{1}{s} - \left(1+\frac{c}{b}\right)\frac{1}{s+b} \\
&= \frac{a\,n!}{s^{n+1}} - \frac{(ac)n!}{s^{n+2}} + \frac{c}{bs} - \left(1+\frac{c}{b}\right)\frac{1}{s+b}
\end{aligned}$$

Inversion yields the solution

$$f(t) = at^n - \frac{n!ac}{(n+1)!}t^{n+1} + \frac{c}{b} - \left(1+\frac{c}{b}\right)e^{-bt}.$$

□

## 4.5 Solutions of Boundary Value Problems

The Laplace transform technique is also very useful in finding solutions of certain simple boundary value problems that arise in many areas of applied mathematics and engineering sciences. We illustrate the method by solving boundary value problems in the theory of deflection of elastic beams.

A horizontal beam experiences a vertical deflection due to the combined effect of its own weight and the applied load on the beam. We consider a beam of length $\ell$ and its equilibrium position is taken along the horizontal $x$-axis.

***Example 4.5.1***

(*Deflection of Beams*). The differential equation for the vertical deflection $y(x)$ of a uniform beam under the action of a transverse load $W(x)$ per unit length at a distance $x$ from the origin on the $x$-axis of the beam is

$$El\frac{d^4y}{dx^4} = W(x), \qquad \text{for} \quad 0 < x < \ell, \tag{4.5.1}$$

where $E$ is *Young's modulus*, $I$ is the *moment of inertia* of the cross section about an axis normal to the plane of bending and $EI$ is called the *flexural rigidity* of the beam.

Some physical quantities associated with the problem are $y'(x), M(x) = EIy''(x)$ and $S(x) = M'(x) = EIy'''(x)$, which respectively represent the slope, bending moment, and shear at a point.

It is of interest to find the solution of (4.5.1) subject to a given loading function and simple boundary conditions involving the deflection, slope, bending moment and shear. We consider the following cases:

(i) Concentrated load on a clamped beam of length $\ell$, that is,
$W(x) \equiv W\delta(x - a)$,
$y(0) = y'(0) = 0$ and $y(\ell) = y'(\ell) = 0$,
where $W$ is a constant and $0 < a < \ell$.

(ii) Distributed load on a uniform beam of length $\ell$ clamped at $x = 0$ and unsupported at $x = \ell$, that is,
$W(x) = WH(x - a)$,
$y(0) = y'(0) = 0$, and $M(\ell) = S(\ell) = 0$.

(iii) A uniform semi-infinite beam freely hinged at $x = 0$ resting horizontally on an elastic foundation and carrying a load $W$ per unit length.

In order to solve the problem, we use the Laplace transform $\bar{y}(s)$ of $y(x)$ defined by

$$\bar{y}(s) = \int_0^\infty e^{-sx} y(x)\, dx. \tag{4.5.2}$$

In view of this transformation, equation (4.5.1) becomes

$$EI[s^4\bar{y}(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)] = \overline{W}(s). \tag{4.5.3}$$

The solution of the transformed deflection function $\bar{y}(s)$ for case (i) is

$$\bar{y}(s) = \frac{y''(0)}{s^3} + \frac{y'''(0)}{s^4} + \frac{W}{EI}\frac{e^{-as}}{s^4}. \tag{4.5.4}$$

Inversion gives

$$y(x) = y''(0)\frac{x^2}{2} + y'''(0)\frac{x^3}{6} + \frac{W}{6EI}(x-a)^3 H(x-a). \tag{4.5.5}$$

$$y'(x) = y''(0)x + \frac{1}{2}x^2 y'''(0) + \frac{W}{2EI}(x-a)^2 H(x-a). \tag{4.5.6}$$

The conditions $y(\ell) = y'(\ell) = 0$ require that

$$y''(0)\frac{\ell^2}{2} + y'''(0)\frac{\ell^3}{6} + \frac{W}{6EI}(\ell - a)^3 = 0,$$

$$y''(0)\ell + y'''(0)\frac{\ell^2}{2} + \frac{W}{2EI}(\ell - a)^2 = 0.$$

These algebraic equations determine the value of $y''(0)$ and $y'''(0)$. Solving these equations, it turns out that

$$y''(0) = \frac{Wa(\ell - a)^2}{EI\,\ell^2} \quad \text{and} \quad y'''(0) = -\frac{W(\ell - a)^2(\ell + 2a)}{EI\,\ell^3}.$$

Thus, the final solution for case (i) is

$$y(x) = \frac{W}{2EI}\left[\frac{a(\ell - a)^2 x^2}{\ell^2} - \frac{(\ell - a)^2(\ell + 2a)x^3}{3\ell^3} + \frac{(x-a)^3 H(x-a)}{3}\right]. \tag{4.5.7}$$

It is now possible to calculate the bending moment and shear at any point of the beam, and, in particular, at the ends.

The solution for case (ii) follows directly from (4.5.3) in the form

$$\overline{y}(s) = \frac{y''(0)}{s^3} + \frac{y'''(0)}{s^4} + \frac{W}{EI}\frac{e^{-as}}{s^5}. \tag{4.5.8}$$

The inverse transformation yields

$$y(x) = \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{W}{24EI}(x-a)^4 H(x-a), \tag{4.5.9}$$

where $y''(0)$ and $y'''(0)$ are to be determined from the remaining boundary conditions $M(\ell) = S(\ell) = 0$, that is, $y''(\ell) = y'''(\ell) = 0$.

From (4.5.9) with $y''(\ell) = y'''(\ell) = 0$, it follows that

$$y''(0) + y'''(0)\ell + \frac{W}{2EI}(\ell - a)^2 = 0$$
$$y'''(0) + \frac{W}{EI}(\ell - a) = 0$$

which give

$$y''(0) = \frac{W(\ell - a)(\ell + a)}{2EI} \text{ and } y'''(0) = -\frac{W}{EI}(\ell - a).$$

Hence, the solution for $y(x)$ for case (ii) is

$$y(x) = \frac{W}{2EI}\left[\frac{(\ell^2 - a^2)x^2}{2} - (\ell - a)\frac{x^3}{3} + \frac{W}{12}(x-a)^4 H(x-a)\right]. \qquad (4.5.10)$$

The shear, $S$, and the bending moment, $M$, at the origin, can readily be calculated from the solution.

The differential equation for case (iii) takes the form

$$EI\frac{d^4y}{dx^4} + ky = W, \quad x > 0, \qquad (4.5.11)$$

where the second term on the left-hand side represents the effect of elastic foundation and $k$ is a positive constant.

Writing $\left(\frac{k}{EI}\right) = 4\omega^4$, equation (4.5.11) becomes

$$\left(\frac{d^4}{dx^4} + 4\omega^4\right) y(x) = \frac{W}{EI}, \quad x > 0. \qquad (4.5.12)$$

This has to be solved subject to the boundary conditions

$$y(0) = y''(0) = 0, \qquad (4.5.13)$$
$$y(x) \text{ is finite as } x \to \infty. \qquad (4.5.14)$$

Using the Laplace transform with respect to $x$ to (4.5.12), we obtain

$$(s^4 + 4\omega^4)\,\bar{y}(s) = \left(\frac{W}{EI}\right)\frac{1}{s} + sy'(0) + y'''(0). \qquad (4.5.15)$$

In view of the Tauberian Theorem 3.8.2 (ii), that is,

$$\lim_{s\to 0} s\,\bar{y}(s) = \lim_{x\to\infty} y(x),$$

it follows that $\bar{y}(s)$ must be of the form

$$\bar{y}(s) = \frac{W}{EI}\frac{1}{s(s^4+4\omega^4)}, \tag{4.5.16}$$

which gives

$$\lim_{x\to\infty} y(x) = \frac{W}{k}. \tag{4.5.17}$$

We now write (4.5.16) as

$$\bar{y}(s) = \frac{W}{EI}\frac{1}{4\omega^4}\left[\frac{1}{s} - \frac{s^3}{s^4+4\omega^4}\right]. \tag{4.5.18}$$

Using the standard table of inverse Laplace transforms, we obtain

$$\begin{aligned} y(x) &= \frac{W}{k}(1-\cos\omega x\, \cosh\,\omega x) \\ &= \frac{W}{k}\left[1-\frac{1}{2}\,e^{-\omega x}\cos\omega x - \frac{1}{2}\,e^{\omega x}\cos\omega x\right]. \end{aligned} \tag{4.5.19}$$

In view of (4.5.17), the final solution is

$$y(x) = \frac{W}{k}\left(1-\frac{1}{2}\,e^{-\omega x}\cos\omega x\right). \tag{4.5.20}$$

□

## 4.6 Evaluation of Definite Integrals

The Laplace transform can be employed to evaluate easily certain definite integrals containing a parameter. Although the method of evaluation may not be very rigorous, it is quite simple and straightforward. The method is essentially based upon the permissibility of interchange of the order of integration, that is,

$$\mathscr{L}\int_a^b f(t,x)\,dx = \int_a^b \mathscr{L} f(t,x)\,dx, \tag{4.6.1}$$

and may be well described by considering some important integrals.

***Example 4.6.1***
Evaluate the integral

$$f(t) = \int_0^\infty \frac{\sin tx}{x(a^2+x^2)}\,dx. \tag{4.6.2}$$

We take the Laplace transform of (4.6.2) with respect to $t$ and interchange the order of integration, which is permissible due to uniform convergence, to obtain

$$\begin{aligned}\bar{f}(s) &= \int_0^\infty \frac{dx}{x(a^2+x^2)} \int_0^\infty e^{-st} \sin tx\, dt \\ &= \int_0^\infty \frac{dx}{(a^2+x^2)(x^2+s^2)} \\ &= \frac{1}{s^2-a^2} \int_0^\infty \left( \frac{1}{a^2+x^2} - \frac{1}{x^2+s^2} \right) dx \\ &= \frac{1}{s^2-a^2} \left( \frac{1}{a} - \frac{1}{s} \right) \frac{\pi}{2} \\ &= \frac{\pi}{2} \frac{1}{s(s+a)} = \frac{\pi}{2} \left( \frac{1}{s} - \frac{1}{s+a} \right).\end{aligned}$$

Inversion gives the value of the given integral

$$f(t) = \frac{\pi}{2a}(1 - e^{-at}). \tag{4.6.3}$$

▯

***Example 4.6.2***

Evaluate the integral

$$f(t) = \int_0^\infty \frac{\sin^2 tx}{x^2}\, dx. \tag{4.6.4}$$

A procedure similar to the above integral with $2\sin^2 tx = 1 - \cos(2\,tx)$ gives

$$\begin{aligned}\bar{f}(s) &= \frac{1}{2} \int_0^\infty \frac{1}{x^2} \left( \frac{1}{s} - \frac{s}{4x^2+s^2} \right) dx = \frac{2}{s} \int_0^\infty \frac{dx}{4x^2+s^2} \\ &= \frac{1}{s} \int_0^\infty \frac{dy}{y^2+s^2} = \frac{1}{s^2} \left[ \tan^{-1} \frac{y}{s} \right]_0^\infty = \pm \frac{\pi}{2s^2}\end{aligned}$$

according as $s >$ or $< 0$. The inverse transform yields

$$f(t) = \frac{\pi t}{2}\, sgn\, t. \tag{4.6.5}$$

▯

***Example 4.6.3***
Show that

$$\int_0^\infty \frac{x\sin xt}{x^2+a^2}\,dx = \frac{\pi}{2}\,e^{-at}, \quad (a, t > 0). \tag{4.6.6}$$

Suppose

$$f(t) = \int_0^\infty \frac{x\sin xt}{x^2+a^2}\,dx.$$

Taking the Laplace transform with respect to $t$ gives

$$\begin{aligned}\bar{f}(s) &= \int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+s^2)} \\ &= \int_0^\infty \frac{dx}{x^2+s^2} - \frac{a^2}{s^2-a^2}\int_0^\infty \left(\frac{1}{x^2+a^2} - \frac{1}{x^2+s^2}\right)dx \\ &= \frac{\pi}{2s}\left(1 - \frac{a}{s+a}\right) = \frac{\pi}{2}\frac{1}{(s+a)}.\end{aligned}$$

Taking the inverse transform, we obtain

$$f(t) = \frac{\pi}{2}\,e^{-at}.$$

□

## 4.7 Solutions of Difference and Differential-Difference Equations

Like differential equations, the difference and differential-difference equations describe mechanical, electrical, and electronic systems of interest. These equations also arise frequently in problems of economics and business, and particularly in problems concerning interest, annuities, amortization, loans, and mortgages. Thus, for the study of the above systems or problems, it is often necessary to solve difference or differential-difference equations with prescribed initial data. This section is essentially devoted to the solution of simple difference and differential-difference equations by the Laplace transform technique.

Suppose $\{u_r\}_{r=1}^{\infty}$ is a given sequence. We introduce the difference operators $\Delta$, $\Delta^2$, $\Delta^3$,...,$\Delta^n$ defined by

$$\Delta u_r = u_{r+1} - u_r, \tag{4.7.1}$$
$$\Delta^2 u_r = \Delta(\Delta u_r) = \Delta(u_{r+1} - u_r) = u_{r+2} - 2u_{r+1} + u_r, \tag{4.7.2}$$
$$\Delta^3 u_r = \Delta^2(u_{r+1} - u_r) = u_{r+3} - 3\,u_{r+2} + 3\,u_{r+1} - u_r. \tag{4.7.3}$$

More generally,

$$\Delta^n u_r = \Delta^{n-1}(u_{r+1} - u_r) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_{r+n-k}. \tag{4.7.4}$$

These expressions are usually called the *first, second, third,* and *nth finite* differences respectively. Any equation expressing a relation between finite differences is called a *difference equation.* The highest order finite difference involved in the equation is referred to as its *order.* A difference equation containing the derivatives of the unknown function is called the *differential-difference equation.* Thus, the differential-difference equation has two distinct orders—one is related to the highest order finite difference and the other is associated with the highest order derivatives. Equations

$$\Delta u_r - u_r = 0, \tag{4.7.5}$$
$$\Delta^2 u_r - 2\Delta u_r = 0, \tag{4.7.6}$$

are the examples of difference equations of the first and second order, respectively. The most general linear $n$th order difference equation has the form

$$a_0\Delta^n u_r + a_1\Delta^{n-1} u_r + \cdots + a_{n-1}\Delta u_r + a_n u_r = f(n), \tag{4.7.7}$$

where $a_0, a_1, \ldots, a_n$ and $f(n)$ are either constants or functions of non-negative integer $n$. Like ordinary differential equations, (4.7.7) is called a *homogeneous* or *inhomogeneous* according to $f(n) = 0$ or $\neq 0$.

The following equations

$$u'(t) - u(t-1) = 0, \tag{4.7.8}$$
$$u'(t) - au(t-1) = f(t), \tag{4.7.9}$$

are the examples of the differential-difference equations, where $f(t)$ is a given function of $t$. The study of the above equation is facilitated by introducing the function

$$S_n(t) = H(t-n) - H(t-n-1), \quad n \le t < n+1, \tag{4.7.10}$$

where $n$ is a non-negative integer and $H(t)$ is the Heaviside unit step function.

The Laplace transform of $S_n(t)$ is given by

$$\overline{S}_n(s) = \mathscr{L}\{S_n(t)\} = \int_0^\infty e^{-st}\{H(t-n) - H(t-n-1)\}\,dt$$
$$= \int_n^{n+1} e^{-st}dt = \frac{1}{s}(1-e^{-s})e^{-ns} = \overline{S}_0(s)\exp(-ns), \qquad (4.7.11)$$

where $\overline{S}_0(s)$ is equal to $\frac{1}{s}(1-e^{-s})$.

We next define the function $u(t)$ by a series

$$u(t) = \sum_{n=0}^{\infty} u_n S_n(t), \qquad (4.7.12)$$

where $\{u_n\}_{n=0}^{\infty}$ is a given sequence. It follows that $u(t) = u_n$ in $n \le t < n+1$ and represents a staircase function. Further

$$u(t+1) = \sum_{n=0}^{\infty} u_n S_n(t+1) = \sum_{n=0}^{\infty} u_n[H(t+1-n) - H(t-n)]$$
$$= \sum_{n=1}^{\infty} u_n S_{n-1}(t) = \sum_{n=0}^{\infty} u_{n+1} S_n(t). \qquad (4.7.13)$$

Similarly,

$$u(t+2) = \sum_{n=0}^{\infty} u_{n+2} S_n(t). \qquad (4.7.14)$$

More generally,

$$u(t+k) = \sum_{n=0}^{\infty} u_{n+k} S_n(t). \qquad (4.7.15)$$

The Laplace transform of $u(t)$ is given by

$$\bar{u}(s) = \mathscr{L}\{u(t)\} = \int_0^\infty e^{-st}u(t)\,dt = \sum_{n=0}^{\infty} u_n \int_0^\infty e^{-st} S_n(t)\,dt$$
$$= \frac{1}{s}(1-e^{-s}) \sum_{n=0}^{\infty} u_n \exp(-ns).$$

Thus,

$$\bar{u}(s) = \frac{1}{s}(1-e^{-s})\zeta(s) = \bar{S}_0(s)\zeta(s), \qquad (4.7.16)$$

where $\zeta(s)$ represents the *Dirichlet function* defined by

$$\zeta(s) = \sum_{n=0}^{\infty} u_n \exp(-ns). \qquad (4.7.17)$$

We thus deduce

$$u(t) = \mathscr{L}^{-1}\{\bar{S}_0(s)\zeta(s)\}. \tag{4.7.18}$$

In particular, if $u_n = a^n$ is a geometric sequence, then

$$\zeta(s) = \sum_{n=0}^{\infty}(ae^{-s})^n = \frac{1}{1-ae^{-s}} = \frac{e^s}{e^s - a}. \tag{4.7.19}$$

Thus, we obtain from (4.7.16) that

$$\mathscr{L}\{a^n\} = \bar{S}_0(s)\zeta(s) = \bar{S}_0(s)\,\frac{e^s}{e^s-a}, \tag{4.7.20}$$

so that

$$\mathscr{L}^{-1}\left\{\bar{S}_0(s)\,\frac{e^s}{e^s-a}\right\} = a^n. \tag{4.7.21}$$

From the identity,

$$\sum_{n=0}^{\infty}(n+1)(ae^{-s})^n = (1-ae^{-s})^{-2}, \tag{4.7.22}$$

it further follows that

$$\mathscr{L}\{(n+1)a^n\} = \bar{S}_0(s)(1-ae^{-s})^{-2} = \frac{e^{2s}\,\bar{S}_0(s)}{(e^s-a)^2}. \tag{4.7.23}$$

Thus,

$$\mathscr{L}^{-1}\left\{\frac{e^{2s}\,\bar{S}_0(s)}{(e^s-a)^2}\right\} = (n+1)a^n. \tag{4.7.24}$$

We deduce from (4.7.22) that

$$\sum_{n=0}^{\infty} na^n e^{-ns} = \frac{ae^s}{(1-ae^{-s})^2}. \tag{4.7.25}$$

Hence,

$$\mathscr{L}\{na^n\} = \bar{S}_0(s)\,\frac{ae^s}{(e^s-a)^2}. \tag{4.7.26}$$

Therefore,

$$\mathscr{L}^{-1}\left\{\frac{a\bar{S}_0(s)e^s}{(e^s-a)^2}\right\} = na^n. \tag{4.7.27}$$

***THEOREM 4.7.1***

If $\bar{u}(s) = \mathscr{L}\{u(t)\}$, then

$$\mathscr{L}\{u(t+1)\} = e^s[\bar{u}(s) - u_0\bar{S}_0(s)], \qquad u_0 = u(0). \tag{4.7.28}$$

**PROOF** We have

$$\begin{aligned}\mathscr{L}\{u(t+1)\} &= \int_0^\infty e^{-st}u(t+1)\,dt = e^s\int_1^\infty e^{-s\tau}u(\tau)\,d\tau \\ &= e^s\left[\bar{u}(s) - \int_0^1 e^{-s\tau}u(\tau)\,d\tau\right] \\ &= e^s\left[\bar{u}(s) - u(0)\int_0^1 e^{-s\tau}\,d\tau\right] = e^s[\bar{u}(s) - u_0\bar{S}_0(s)].\end{aligned}$$

This proves the theorem.

In view of this theorem, we derive

$$\begin{aligned}\mathscr{L}\{u(t+2)\} &= e^s[\mathscr{L}\{u(t+1)\} - u(1)\bar{S}_0(s)] \\ &= e^{2s}[\bar{u}(s) - u(0)\bar{S}_0(s)] - e^s u_1\bar{S}_0(s) \\ &= e^{2s}[\bar{u}(s) - (u_0 + u_1e^{-s})\bar{S}_0(s)], \qquad u(1) = u_1. \qquad (4.7.29)\end{aligned}$$

Similarly,

$$\mathscr{L}\{u(t+3)\} = e^{3s}[\bar{u}(s) - (u_0 + u_1e^{-s} + u_2e^{-2s})\bar{S}_0(s)]. \qquad (4.7.30)$$

More generally, if $k$ is an integer,

$$\mathscr{L}\{u(t+k)\} = e^{ks}\left(\bar{u}(s) - \bar{S}_0(s)\sum_{r=0}^{k-1}u_re^{-rs}\right). \qquad (4.7.31)$$

▮

***Example 4.7.1***

Solve the difference equation

$$\Delta u_n - u_n = 0, \qquad (4.7.32)$$

with the initial condition $u_0 = 1$.

We take the Laplace transform of the equation to obtain

$$\mathscr{L}\{u_{n+1}\} - 2\mathscr{L}\{u_n\} = 0,$$

which is, by (4.7.28),

$$e^s[\bar{u}(s) - u_0\bar{S}_0(s)] - 2\,\bar{u}(s) = 0.$$

Thus,

$$\bar{u}(s) = \frac{e^s\,\bar{S}_0(s)}{e^s - 2}.$$

Inversion with (4.7.21) gives the solution

$$u_n = 2^n. \tag{4.7.33}$$

□

***Example 4.7.2***
Show that the solution of the difference equation

$$\Delta^2 u_n - 2\Delta u_n = 0 \tag{4.7.34}$$

is

$$u_n = A + B\,3^n, \tag{4.7.35}$$

where $A = \frac{1}{2}(3u_0 - u_1)$ and $B = \frac{1}{2}(u_1 - u_0)$.

The given equation is

$$u_{n+2} - 4u_{n+1} + 3u_n = 0.$$

Taking the Laplace transform, we obtain

$$e^{2s}[\bar{u}(s) - (u_0 + u_1 e^{-s})\bar{S}_0(s)] - 4\,e^s[\bar{u}(s) - u_0\bar{S}_0(s)] + 3\bar{u}(s) = 0$$

or,

$$(e^{2s} - 4e^s + 3)\bar{u}(s) = [u_0(e^{2s} - 4e^s) + u_1 e^s]\bar{S}_0(s).$$

Hence,

$$\begin{aligned}\bar{u}(s) &= \bar{S}_0(s)\left[\frac{u_0(e^{2s} - 4e^s) + u_1 e^s}{(e^s - 1)(e^s - 3)}\right]\\ &= \bar{S}_0(s)\left[\frac{(3u_0 - u_1)e^s}{2(e^s - 1)} + \frac{(u_1 - u_0)e^s}{2(e^s - 3)}\right].\end{aligned}$$

The inverse Laplace transform combined with (4.7.21) gives

$$u_n = A + B\,3^n.$$

□

***Example 4.7.3***
Solve the difference equation

$$u_{n+2} - 2\lambda u_{n+1} + \lambda^2 u_n = 0, \tag{4.7.36}$$

with $u_0 = 0$ and $u_1 = 1$.

The Laplace transformed equation is

$$e^{2s}[\bar{u}(s) - e^{-s}\,\bar{S}_0(s)] - 2\lambda\bar{u}(s)e^s + \lambda^2\bar{u}(s) = 0$$

or,

$$\bar{u}(s) = \frac{e^s \,\bar{S}_0(s)}{(e^s - \lambda)^2}.$$

The inverse transform gives the solution

$$u_n = \frac{1}{\lambda}\, n\, \lambda^n = n\, \lambda^{n-1}. \tag{4.7.37}$$

▯

***Example 4.7.4***
Solve the differential-difference equation

$$u'(t) = u(t-1), \quad u(0) = 1. \tag{4.7.38}$$

Application of the Laplace transform gives

$$s\bar{u}(s) - u(0) = e^{-s}[\bar{u}(s) - u(0)\bar{S}_0(s)],$$

or,

$$\bar{u}(s)(s - e^{-s}) = 1 + \frac{e^{-s}}{s}(e^{-s} - 1).$$

Or,

$$\begin{aligned}
\bar{u}(s) &= \left\{\frac{1}{s - e^{-s}} - \frac{e^{-s}}{s(s - e^{-s})}\right\} + \frac{e^{-2s}}{s(s - e^{-s})} \\
&= \frac{1}{s} + \frac{e^{-2s}}{s^2}\left(1 - \frac{e^{-s}}{s}\right)^{-1} \\
&= \frac{1}{s} + \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^3} + \frac{e^{-4s}}{s^4} + \cdots + \frac{e^{-ns}}{s^n} + \cdots .
\end{aligned}$$

In view of the result

$$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s^n}\right\} = \frac{(t-a)^{n-1}}{\Gamma(n)} H(t-a), \tag{4.7.39}$$

we obtain the solution

$$u(t) = 1 + \frac{(t-2)}{1!} + \frac{(t-3)^2}{2!} + \cdots + \frac{(t-n)^{n-1}}{(n-1)!}, \quad t > n. \tag{4.7.40}$$

▯

***Example 4.7.5***
Solve the differential-difference equation

$$u'(t) - \alpha\, u(t-1) = \beta, \quad u(0) = 0. \tag{4.7.41}$$

Application of the Laplace transform yields

$$s\,\bar{u}(s) - u(0) - \alpha\, e^{-s}[\bar{u}(s) - u(0)\bar{S}_0(s)] = \frac{\beta}{s}.$$

Or,

$$\begin{aligned}\bar{u}(s) &= \frac{\beta}{s(s-\alpha e^{-s})} = \frac{\beta}{s^2}\left(1 - \frac{\alpha}{s}e^{-s}\right)^{-1} \\ &= \beta\left[\frac{1}{s^2} + \frac{\alpha\, e^{-s}}{s^3} + \frac{\alpha^2\, e^{-2s}}{s^4} + \cdots + \frac{\alpha^n\, e^{-ns}}{s^{n+2}} + \cdots\right].\end{aligned}$$

Inverting with the help of (4.7.39), we obtain the solution

$$u(t) = \beta\left[t + \frac{\alpha(t-1)^2}{\Gamma(3)} + \alpha^2\frac{(t-2)^3}{\Gamma(4)} + \cdots + \frac{\alpha^n(t-n)^{n+1}}{\Gamma(n+2)}\right],\ t > n. \quad (4.7.42)$$

$\square$

## 4.8 Applications of the Joint Laplace and Fourier Transform

***Example 4.8.1***

(*The Inhomogeneous Cauchy Problem for the Wave Equation*). Use the joint Fourier and Laplace transform method to solve the Cauchy problem for the wave equation as stated in Example 2.12.4. with an inhomogeneous term, $q(x,t)$.

We define the joint Fourier and Laplace transform of $u(x,t)$ by

$$\bar{U}(k,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,dx \int_0^{\infty} e^{-st}u(x,t)\,dt. \quad (4.8.1)$$

The transformed inhomogeneous Cauchy problem has the solution in the form

$$\bar{U}(k,s) = \frac{sF(k) + G(k) + \bar{Q}(k,s)}{(s^2 + c^2k^2)}, \quad (4.8.2)$$

where $\bar{Q}(k,s)$ is the joint transform of the inhomogeneous term, $q(x,t)$ present on the right side of the wave equation.

The joint inverse transform gives the solution as

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \mathscr{L}^{-1}\left[\frac{sF(k)+G(k)+\bar{Q}(k,s)}{s^2+c^2k^2}\right] dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[F(k)\cos ckt + \frac{G(k)}{ck}\sin ckt\right] e^{ikx}\, dk$$

$$+ \frac{1}{ck}\int_0^t \sin ck(t-\tau)\, Q(k,\tau)\, d\tau$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)(e^{ickt}+e^{-ickt})e^{ikx}\, dk$$

$$+ \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{G(k)}{ick}(e^{ickt}-e^{-ickt})e^{ikx}\, dk$$

$$+ \frac{1}{\sqrt{2\pi}}\frac{1}{2c}\int_0^t d\tau \int_{-\infty}^{\infty} \frac{Q(k,\tau)}{ik}\left[e^{ick(t-\tau)}+e^{-ick(t-\tau)}\right] e^{ikx} dk$$

$$= \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{\sqrt{2\pi}}\frac{1}{2c}\int_{-\infty}^{\infty} G(k)\, dk \int_{x-ct}^{x+ct} e^{ik\xi}\, d\xi$$

$$+ \frac{1}{2c}\int_0^t d\tau \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, Q(k,\tau)\, dk \int_{x-c(t-\tau)}^{x+c(t-\tau)} e^{ik\xi}\, d\xi$$

$$= \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

$$+ \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} q(\xi,\tau)\, d\xi. \qquad (4.8.3)$$

This is identical with the d'Alembert solution (2.12.41) when $q(x,t) \equiv 0$. ☐

***Example 4.8.2***

(*Dispersive Long Water Waves in a Rotating Ocean*). We use the joint Laplace and Fourier transform to solve the linearized horizontal equations of motion and the continuity equation in a rotating inviscid ocean. These equations in a rotating coordinate system (see Proudman, 1953; Debnath and Kulchar, 1972) are given by

$$\frac{\partial \mathbf{u}}{\partial t} + f\,\hat{\mathbf{k}} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho h}\boldsymbol{\tau}, \qquad (4.8.4)$$

$$\nabla \cdot \mathbf{u} = -\frac{1}{h}\frac{\partial \zeta}{\partial t}, \qquad (4.8.5)$$

where $\mathbf{u} = (u, v)$ is the horizontal velocity field, $\hat{\mathbf{k}}$ is the unit vector normal to the horizontal plane, $f = 2\Omega \sin\phi$ is the constant Coriolis parameter, $\rho$ is the constant density of water, $\zeta(x,t)$ is the vertical free surface elevation, $\tau = (\tau^x, \tau^y)$ represents the components of wind stress in the $x$ and $y$ directions, and the pressure is given by the hydrostatic equation

$$p = p_0 + g\rho(\zeta - z), \tag{4.8.6}$$

where $z$ is the depth of water below the mean free surface and $g$ is the acceleration due to gravity.

Equation (4.8.4)–(4.8.5) combined with (4.8.6) reduce to the form

$$\frac{\partial u}{\partial t} - f v = -g\,\frac{\partial \zeta}{\partial x} + \frac{\tau^x}{\rho h}, \tag{4.8.7}$$

$$\frac{\partial v}{\partial t} + f u = -g\,\frac{\partial \zeta}{\partial y} + \frac{\tau^y}{\rho h}, \tag{4.8.8}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{h}\frac{\partial \zeta}{\partial t}. \tag{4.8.9}$$

It follows from (4.8.7)–(4.8.8) that

$$Du = -g\left(\frac{\partial^2}{\partial x \partial t} + f\frac{\partial}{\partial y}\right)\zeta + \frac{1}{\rho h}\left(\frac{\partial \tau^x}{\partial t} + f\tau^y\right), \tag{4.8.10}$$

$$Dv = -g\left(\frac{\partial^2}{\partial y \partial t} - f\frac{\partial}{\partial x}\right)\zeta + \frac{1}{\rho h}\left(\frac{\partial \tau^y}{\partial t} - f\tau^x\right), \tag{4.8.11}$$

where the differential operator $D$ is

$$D \equiv \left(\frac{\partial^2}{\partial t^2} + f^2\right). \tag{4.8.12}$$

Elimination of $u$ and $v$ from (4.8.9)–(4.8.11) gives

$$\left(\nabla^2 - \frac{1}{c^2}D\right)\zeta_t = E(x,y,t), \tag{4.8.13}$$

where $c^2 = gh$ and $\nabla^2$ is the horizontal Laplacian, and $E(x,y,t)$ is a known forcing function given by

$$E(x,y,t) = \frac{1}{\rho c^2}\left[\frac{\partial^2 \tau^x}{\partial x \partial t} + \frac{\partial^2 \tau^y}{\partial y \partial t} + f\left(\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\right)\right]. \tag{4.8.14}$$

Further, we assume that the conditions are uniform in the $y$ direction and the wind stress acts only in the $x$ direction so that $\tau^x$ and $E$ are given functions of $x$ and $t$ only. Consequently, equation (4.8.13) becomes

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\right]\zeta_t = \frac{1}{\rho c^2}\left(\frac{\partial^2 \tau^x}{\partial x \partial t}\right).$$

Integrating this equation with respect to $t$ gives

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\right]\zeta = \frac{1}{\rho c^2}\left(\frac{\partial \tau^x}{\partial x}\right). \tag{4.8.15}$$

Similarly, the velocity $u(x,t)$ satisfies the equation

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\right]u = -\frac{1}{\rho h c^2}\left(\frac{\partial \tau^x}{\partial t}\right). \tag{4.8.16}$$

If the right-hand side of equations (4.8.15) and (4.8.16) is zero, these equations are known as the *Klein-Gordon equations*, which have received extensive attention in quantum mechanics and in applied mathematics.

Equation (4.8.15) is to be solved subject to the following boundary and initial conditions

$$|\zeta| \quad \text{is} \quad \text{bounded} \quad \text{as } |x| \to \infty, \tag{4.8.17}$$

$$\zeta(x,t) = 0 \quad \text{at} \quad t=0 \text{ for all real } x. \tag{4.8.18}$$

Before we solve the initial value problem, we seek a plane wave solution of the homogeneous equation (4.8.15) in the form

$$\zeta(x,t) = A\exp\{i(\omega t - kx)\}, \tag{4.8.19}$$

where $A$ is a constant amplitude, $\omega$ is the frequency, and $k$ is the wavenumber. Such a solution exists provided the dispersion relation

$$\omega^2 = c^2k^2 + f^2 \tag{4.8.20}$$

is satisfied. Thus, the phase and the group velocities of waves are given by

$$C_p = \frac{\omega}{k} = \left(c^2 + \frac{f^2}{k^2}\right)^{\frac{1}{2}}, \quad C_g = \frac{\partial \omega}{\partial k} = \frac{c^2k}{(c^2k^2+f^2)^{\frac{1}{2}}}. \tag{4.8.21ab}$$

Thus, the waves are dispersive in a rotating ocean ($f \neq 0$). However, in a non-rotating ocean ($f = 0$) all waves would propagate with constant velocity $c$, and they are non-dispersive shallow water waves. Further, $C_pC_g = c^2$ whence it follows that the phase velocity has a minimum of $c$ and the group velocity a maximum. The short waves will be observed first at a given point, even though they have the smallest phase velocity.

Application of the joint Laplace and Fourier transform to (4.8.15) together with (4.8.17)–(4.8.18) give the transformed solution

$$\tilde{\bar{\zeta}}(k,s) = -\frac{Ac^2}{(s^2+a^2)}\tilde{\bar{f}}(k,s), \qquad a^2 = (c^2k^2+f^2), \tag{4.8.22}$$

where

$$f(x,t) = \frac{1}{\rho c^2}\left(\frac{\partial \tau^x}{\partial x}\right)H(t). \tag{4.8.23}$$

The inverse transforms combined with the Convolution Theorem of the Laplace transform lead to the formal solution

$$\zeta(x,t) = -\frac{Ac}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(k^2 + \frac{f^2}{c^2}\right)^{-\frac{1}{2}} e^{ikx}\, dk \int_0^t \tilde{f}(k, t-\tau) \sin a\tau \, d\tau. \qquad (4.8.24)$$

In general, this integral cannot be evaluated unless $f(x,t)$ is prescribed. Even if some particular form of $f$ is given, an exact evaluation of (4.8.24) is almost a formidable task. Hence, it is necessary to resort to asymptotic methods (see Debnath and Kulchar, 1972).

To investigate the solution, we choose a particular form of the wind stress distribution

$$\frac{\tau^x}{\rho c^2} = A\, e^{i\omega t} H(t) H(-x), \qquad (4.8.25)$$

where $A$ is a constant and $\omega$ is the frequency of the applied disturbance. Thus,

$$\frac{1}{\rho c^2}\left(\frac{\partial \tau^x}{\partial x}\right) = -A e^{i\omega t} H(t)\delta(-x). \qquad (4.8.26)$$

In this case, solution (4.8.24) reduces to the form

$$\begin{aligned}\zeta(x,t) &= \frac{Ac}{\sqrt{2\pi}} \int_0^t e^{i\omega(t-\tau)} H(t-\tau) \mathscr{F}^{-1}\left[\frac{\sin a\tau}{\sqrt{k^2 + \frac{f^2}{c^2}}}\right] d\tau \\ &= \frac{Ac}{2} \int_0^t e^{i\omega(t-\tau)} H(t-\tau) J_0\left\{\frac{f}{c}(c^2\tau^2 - x^2)^{\frac{1}{2}}\right\} \\ &\qquad \times H(c\tau - |x|)\, d\tau, \end{aligned} \qquad (4.8.27)$$

where $J_0(z)$ is the zero-order Bessel function of the first kind.

When $\omega \equiv 0$, this solution is identical with that of Crease (1956) who obtained the solution using the Green's function method. In this case, the solution becomes

$$\zeta = \frac{Ac}{2} \int_0^t H(t-\tau) J_0\left[f\left\{\tau^2 - \frac{x^2}{c^2}\right\}^{\frac{1}{2}}\right] H\left(\tau - \frac{|x|}{c}\right) d\tau. \qquad (4.8.28)$$

In terms of non-dimensional parameters $f\tau = \alpha$, $ft = a$, and $\frac{fx}{c} = b$, solution (4.8.28) assumes the form

$$\left(\frac{2f}{Ac}\right)\zeta = \int_0^a H(a-\alpha) J_0\left[(\alpha^2 - b^2)^{\frac{1}{2}}\right] H(\alpha - |b|)\, d\alpha. \qquad (4.8.29)$$

Or, equivalently,

$$\left(\frac{2f}{Ac}\right)\zeta = \int_{|b|}^{d} J_0\left[(\alpha^2 - b^2)^{\frac{1}{2}}\right] d\alpha, \tag{4.8.30}$$

where $d = \max(|b|, a)$. This is the basic solution of the problem.

In order to find the solution of (4.8.16), we first choose

$$\frac{1}{\rho c^2}\left(\frac{\partial \tau^x}{\partial t}\right) = A\delta(t)H(-x), \tag{4.8.31}$$

so that the joint Laplace and Fourier transform of this result is $A\mathscr{F}\{H(-x)\}$. Thus, the transformed solution of (4.8.16) is

$$\bar{\tilde{u}}(k,s) = \frac{Ac^2}{h}\mathscr{F}\{H(-x)\}\frac{1}{(s^2+\omega^2)}, \quad \omega^2 = (ck)^2 + f^2. \tag{4.8.32}$$

The inverse transforms combined with the Convolution Theorem lead to the solution

$$u(x,t) = \frac{Ac}{2h}\int_{-\infty}^{\infty} H(-\xi)\, J_0\left[f\left\{t^2 - \left(\frac{x-\xi}{c}\right)^2\right\}^{\frac{1}{2}}\right] \times H\left(t - \frac{(x-\xi)}{c}\right) d\xi, \tag{4.8.33}$$

which is, by the change of variable $(x-\xi)f = c\alpha$, with $a = ft$ and $b = (fx/c)$,

$$= \frac{Ac^2}{2hf}\int_{b}^{\infty} J_0\left[(a^2 - \alpha^2)^{\frac{1}{2}}\right] H(a - |\alpha|)\, d\alpha. \tag{4.8.34}$$

For the case $b > 0$, solution (4.8.34) becomes

$$u(x,t) = \frac{Ac^2}{2hf}H(a-b)\int_{b}^{a} J_0\left\{(a^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha. \tag{4.8.35}$$

When $b < 0$, the velocity field is

$$\begin{aligned} u(x,t) &= \frac{Ac^2}{2hf}\left[\int_{-a}^{a} J_0\left\{(a^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha - H(a-|b|)\int_{-a}^{b} J_0\left\{(a^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha\right] \\ &= \frac{gA}{2f}\left[2\sin a - H(a-|b|)\int_{|b|}^{a} J_0\left\{(a^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha\right], \end{aligned} \tag{4.8.36}$$

which is, for $a < |b|$,

$$u(x,t) = \left(\frac{gA}{2f}\right)\sin a. \tag{4.8.37}$$

Finally, it can be shown that the velocity transverse to the direction of propagation is

$$v = \left(-\frac{gA}{2f}\right)\int_0^a d\beta \int_b^\infty J_0\left\{(\beta^2-\alpha^2)^{\frac{1}{2}}\right\} H(\beta - |\alpha|)\, d\alpha. \tag{4.8.38}$$

If $b > 0$, that is, $x$ is outside the generating region, then

$$\left(\frac{2f}{gA}\right) v = -H(a-b)\int_b^a d\beta \int_b^\beta J_0\left\{(\beta^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha,$$

which becomes, after some simplification,

$$= -\left[(1-\cos a) - \int_0^b d\alpha \int_\alpha^a J_0\left\{(\beta^2-\alpha^2)^{\frac{1}{2}}\right\}\right] H(a-b). \tag{4.8.39}$$

For $b < 0$, it is necessary to consider two cases: (i) $a < |b|$ and (ii) $a > |b|$. In the former case, (4.8.38) takes the form

$$\left(\frac{2f}{gA}\right) v = -\int_0^a d\beta \int_{-\beta}^\beta J_0\left\{(\beta^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha = -2(1-\cos b). \tag{4.8.40}$$

In the latter case, the final form of the solution is

$$\left(\frac{2f}{gA}\right) v = -(1-\cos b) + \int_0^{|b|} d\alpha \int_\alpha^a J_0\left\{(\beta^2-\alpha^2)^{\frac{1}{2}}\right\} d\beta. \tag{4.8.41}$$

Finally, the steady-state solutions are obtained in the limit as $t \to \infty (b \to \infty)$

$$\begin{aligned}
\zeta &= \frac{Ac}{2f}\exp(-|b|),\\
u &= \frac{Ag}{2f}\sin ft,\\
v &= \frac{Ag}{2f}\begin{bmatrix}\cos ft - \exp(-b), & b > 0\\ \cos ft + \exp(-|b|) - 2, & b < 0\end{bmatrix}.
\end{aligned} \tag{4.8.42}$$

Thus, the steady-state solutions are attained in a rotating ocean. This shows a striking contrast with the corresponding solutions in the non-rotating ocean where an ever-increasing free surface elevation is found. The terms $\sin ft$ and $\cos ft$ involved in the steady-state velocity field represent inertial oscillations with frequency $f$. ▯

***Example 4.8.3***

(*One-Dimensional Diffusion Equation on a Half Line*). Solve the equation

$$u_t = \kappa u_{xx}, \quad 0 < x < \infty, \quad t > 0, \tag{4.8.43}$$

with the boundary data

$$\left.\begin{aligned} u(x,t) &= f(t) && \text{for } x = 0 \\ u(x,t) &\to 0 && \text{as } x \to \infty \end{aligned}\right\} \quad t > 0 \tag{4.8.44ab}$$

and the initial condition

$$u(x,t) = 0 \quad \text{at } t = 0 \quad \text{for} \quad 0 < x < \infty. \tag{4.8.45}$$

We use the joint Fourier sine and Laplace transform defined by

$$\overline{U}_s(k,s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-st}\, dt \int_0^\infty u(x,t) \sin kx\, dx, \tag{4.8.46}$$

so that the solution of the transformed problem is

$$\overline{U}_s(k,s) = \sqrt{\frac{2}{\pi}}\, (\kappa k) \frac{\bar{f}(s)}{(s + k^2\kappa)}. \tag{4.8.47}$$

The inverse transform yields the solution

$$u(x,t) = \left(\frac{2\kappa}{\pi}\right) \int_0^\infty k \sin kx\, dk \int_0^t f(t-\tau) \exp(-\kappa\tau k^2)\, d\tau.$$

In particular, if $f(t) = T_0 = \text{constant}$, then the solution becomes

$$u(x,t) = \frac{2T_0}{\pi} \int_0^\infty \frac{\sin kx}{k} (1 - e^{-\kappa k^2 t})\, dk. \tag{4.8.48}$$

Making use of the integral (2.15.11) gives the solution

$$\begin{aligned} u(x,t) &= \frac{2T_0}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\, erf\left(\frac{x}{2\sqrt{\kappa t}}\right)\right] \\ &= T_0\, erfc\left(\frac{x}{2\sqrt{\kappa t}}\right). \end{aligned} \tag{4.8.49}$$

This is identical with (2.15.12). ▯

**Example 4.8.4**

(*The Bernoulli-Euler Equation on an Elastic Foundation*). Solve the equation

$$EI\,\frac{\partial^4 u}{\partial x^4} + \kappa u + m\,\frac{\partial^2 u}{\partial t^2} = W\delta(t)\delta(x), \quad -\infty < x < \infty, \quad t > 0, \tag{4.8.50}$$

with the initial data

$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = 0. \tag{4.8.51}$$

We use the joint Laplace and Fourier transform (4.8.1) to find the solution of the transformed problem in the form

$$\overline{U}(k,s) = \frac{W}{m\sqrt{2\pi}}\frac{1}{(s^2 + a^2k^4 + \omega^2)}, \tag{4.8.52}$$

where

$$a^2 = \frac{EI}{m} \quad \text{and} \quad \omega^2 = \frac{\kappa}{m}.$$

The inverse Laplace transform gives

$$U(k,t) = \frac{W}{m\sqrt{2\pi}}\left(\frac{\sin\alpha t}{\alpha}\right), \quad \alpha = (a^2k^4 + \omega^2)^{\frac{1}{2}}. \tag{4.8.53ab}$$

Then the inverse Fourier transform yields the formal solution

$$u(x,t) = \frac{W}{2\pi m}\int_{-\infty}^{\infty} e^{ikx}\left(\frac{\sin\alpha t}{\alpha}\right) dk. \tag{4.8.54}$$

▯

**Example 4.8.5**

(*The Cauchy-Poisson Wave Problem in Fluid Dynamics*). We consider the two-dimensional Cauchy-Poisson problem for an inviscid liquid of infinite depth with a horizontal free surface. We assume that the liquid has constant density $\rho$ and negligible surface tension. Waves are generated on the surface of water initially at rest for time $t < 0$ by the prescribed free surface displacement at $t = 0$.

In terms of the velocity potential $\phi(x,z,t)$ and the free surface elevation $\eta(x,t)$, the linearized surface wave motion in Cartesian coordinates $(x,y,z)$ is governed by the following equation and free surface and boundary conditions:

$$\nabla^2\phi = \phi_{xx} + \phi_{zz} = 0, \quad -\infty < z \le 0, \quad -\infty < x < \infty,\ t > 0, \tag{4.8.55}$$

$$\left.\begin{aligned}\phi_z - \eta_t = 0\\ \phi_t + g\eta = 0\end{aligned}\right\} \quad \text{on } z = 0, t > 0, \tag{4.8.56ab}$$

$$\phi_z \to 0 \quad \text{as } z \to -\infty. \tag{4.8.57}$$

The initial conditions are

$$\phi(x, 0, 0) = 0 \quad \text{and} \quad \eta(x, 0) = \eta_0(x), \tag{4.8.58}$$

where $\eta_0(x)$ is a given function with compact support.

We introduce the Laplace transform with respect to $t$ and the Fourier transform with respect to $x$ defined by

$$[\bar{\tilde{\phi}}(k, z, s), \bar{\tilde{\eta}}(k, s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, dx \int_0^{\infty} e^{-st} [\phi, \eta]\, dt. \tag{4.8.59}$$

The use of joint transform to the above system gives

$$\bar{\tilde{\phi}}_{zz} - k^2 \bar{\tilde{\phi}} = 0, \quad -\infty < z \le 0, \tag{4.8.60}$$

$$\left.\begin{aligned}\bar{\tilde{\phi}}_z = s\bar{\tilde{\eta}} - \tilde{\eta}_0(k)\\ s\bar{\tilde{\phi}} + g\bar{\tilde{\eta}} = 0\end{aligned}\right\} \quad \text{on } z = 0, \tag{4.8.61ab}$$

$$\bar{\tilde{\phi}}_z \to 0 \quad \text{as } z \to -\infty. \tag{4.8.62}$$

The bounded solution of (4.8.60) is

$$\bar{\tilde{\phi}}(k, s) = \bar{A} \exp(|k|z) \tag{4.8.63}$$

where $\overline{A} = \overline{A}(s)$ is an arbitrary function of $s$, and $\tilde{\eta}_0(k) = \mathscr{F}\{\eta_0(x)\}$.

Substituting (4.8.63) into (4.8.61ab) and eliminating $\bar{\tilde{\eta}}$ from the resulting equations gives $\bar{A}$. Hence, the solutions for $\bar{\tilde{\phi}}$ and $\bar{\tilde{\eta}}$ are

$$[\bar{\tilde{\phi}}, \bar{\tilde{\eta}}] = \left[ -\frac{g\, \tilde{\eta}_0 \exp(|k|z)}{s^2 + \omega^2}, \frac{s\, \tilde{\eta}_0}{s^2 + \omega^2} \right], \tag{4.8.64ab}$$

where the dispersion relation for deep water waves is

$$\omega^2 = g|k|. \tag{4.8.65}$$

The inverse Laplace and Fourier transforms give the solutions

$$\phi(x,z,t) = -\frac{g}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin\omega t}{\omega} \exp(ikx + |k|z)\, \tilde{\eta}_0(k)\, dk \qquad (4.8.66)$$

$$\eta(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\eta}_0(k) \cos\omega t\, e^{ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \tilde{\eta}_0(k)[e^{i(kx-\omega t)} + e^{i(kx+\omega t)}]\, dk, \qquad (4.8.67)$$

in which $\tilde{\eta}_0(-k) = \tilde{\eta}_0(k)$ is assumed.

Physically, the first and second integrals of (4.8.67) represent waves traveling in the positive and negative directions of $x$ respectively with phase velocity $\left(\frac{\omega}{k}\right)$. These integrals describe superposition of all such waves over the wavenumber spectrum $0 < k < \infty$.

For the classical Cauchy-Poisson wave problem, $\eta(x) = a\,\delta(x)$ where $\delta(x)$ is the Dirac delta function so that $\tilde{\eta}_0(k) = \left(a/\sqrt{2\pi}\right)$. Thus, solution (4.8.67) becomes

$$\eta(x,t) = \frac{a}{2\pi} \int_{0}^{\infty} \left[e^{i(kx-\omega t)} + e^{i(kx+\omega t)}\right] dk. \qquad (4.8.68)$$

The wave integrals (4.8.66) and (4.8.67) represent the exact solution for the velocity potential $\phi$ and the free surface elevation $\eta$ for all $x$ and $t > 0$. However, they do not lend any physical interpretations. In general, the exact evaluation of these integrals is almost a formidable task. So it is necessary to resort to asymptotic methods. It would be sufficient for the determination of the principal features of the wave motions to investigate (4.8.67) or (4.8.68) asymptotically for large time $t$ and large distance $x$ with $(x/t)$ held fixed. The asymptotic solution for this kind of problem is available in many standard books (for example, see Debnath, 1994, p 85). We state the stationary phase approximation of a typical wave integral, for $t \to \infty$,

$$\eta(x,t) = \int_{a}^{b} f(k) \exp[itW(k)]\, dk \qquad (4.8.69)$$

$$\sim f(k_1) \left[\frac{2\pi}{t|W''(k_1)|}\right]^{\frac{1}{2}} \exp\left[i\left\{tW(k_1) + \frac{\pi}{4} sgn\, W''(k_1)\right\}\right], \qquad (4.8.70)$$

where $W(k) = \dfrac{kx}{t} - \omega(k), x > 0$ and $k = k_1$ is a stationary point that satisfies the equation

$$W'(k_1) = \frac{x}{t} - \omega'(k_1) = 0, \quad a < k_1 < b. \qquad (4.8.71)$$

Application of (4.8.70) to (4.8.67) shows that only the first integral in (4.8.67) has a stationary point for $x > 0$. Hence, the stationary phase approximation gives the asymptotic solution, as $t \to \infty, x > 0$,

$$\eta(x,t) \sim \left[\frac{1}{t|\omega''(k_1)|}\right]^{\frac{1}{2}} \tilde{\eta}_0(k_1) \exp[i\{(k_1 x - t\omega(k_1)\} + \frac{i\pi}{4} sgn\{-\omega''(k_1)\}], \quad (4.8.72)$$

where $k_1 = (gt^2/4x^2)$ is the root of the equation $\omega'(k) = \dfrac{x}{t}$.

On the other hand, when $x < 0$, only the second integral of (4.8.67) has a stationary point $k_1 = (gt^2/4x^2)$, and hence, the same result (4.8.70) can be used to obtain the asymptotic solution for $t \to \infty$ and $x < 0$ as

$$\eta(x,t) \sim \left[\frac{1}{t|\omega''(k_1)|}\right]^{\frac{1}{2}} \tilde{\eta}_0(k_1) \exp[i\{t\omega(k_1) - k_1|x|\} + \frac{i\pi}{4} sgn\, \omega''(k_1)]. \quad (4.8.73)$$

In particular, for the classical Cauchy-Poisson solution (4.8.68), the asymptotic representation for $\eta(x,t)$ follows from (4.8.73) in the form

$$\eta(x,t) \sim \frac{at}{2\sqrt{2\pi}} \frac{\sqrt{g}}{x^{3/2}} \cos\left(\frac{gt^2}{4x}\right), \quad gt^2 >> 4x \quad (4.8.74)$$

and a similar result for $x < 0$ and $t \to \infty$. ▯

## 4.9 Summation of Infinite Series

With the aid of Laplace transforms, Wheelon (1954) first developed a direct method to the problem of summing infinite series in closed form. His method is essentially based on the operation that is contained in the summation of both sides of a Laplace transform with respect to the transform variable $s$, which is treated as the dummy index of summation $n$. This is followed by an interchange of summation and integration that leads to the desired sum as the integral of a geometric or exponential series, which can be summed in closed form. We next discuss this procedure in some detail.

If $\bar{f}(s) = \mathscr{L}\{f(x)\}$, then

$$\sum_{n=1}^{\infty} a_n \bar{f}(n) = \sum_{n=1}^{\infty} a_n \int_0^{\infty} f(x) e^{-nx}\, dx. \quad (4.9.1)$$

In many cases, it is possible to interchange the order of summation and integration so that (4.9.1) gives

$$\sum_{n=1}^{\infty} a_n \bar{f}(n) = \int_0^{\infty} f(t)b(t)\,dt, \tag{4.9.2}$$

where

$$b(t) = \sum_{n=1}^{\infty} a_n \exp(-nt). \tag{4.9.3}$$

We now assume $f(t) = \frac{1}{\Gamma(p)} t^{p-1} \exp(-xt)$ so that $\bar{f}(n) = (n+x)^{-p}$. Consequently, (4.9.2) becomes

$$\sum_{n=1}^{\infty} a_n \bar{f}(n) = \sum_{n=1}^{\infty} \frac{a_n}{(n+x)^p} = \frac{1}{\Gamma(p)} \int_0^{\infty} b(t)\, t^{p-1} \exp(-xt)\,dt. \tag{4.9.4}$$

This shows that a general series has been expressed in terms of an integral. We next illustrate the method by simple examples.

***Example 4.9.1***
Show that the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{4.9.5}$$

Putting $x=0, p=2$, and $a_n = 1$ for all $n$, we find, from (4.9.3) and (4.9.4),

$$b(t) = \sum_{n=1}^{\infty} \exp(-nt) = \frac{1}{e^t - 1}, \tag{4.9.6}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \int_0^{\infty} \frac{t\,dt}{e^t - 1} = \zeta(2) = \frac{\pi^2}{6}, \tag{4.9.7}$$

in which the following standard result is used

$$\int_0^{\infty} \frac{t^{p-1}}{e^{at} - 1}\,dt = \frac{\Gamma(p)}{a^p} \zeta(p), \tag{4.9.8}$$

where $\zeta(p)$ is the *Riemann zeta function* defined below by (4.9.10).

Similarly, we can show

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{\Gamma(3)} \int_0^{\infty} \frac{t^2\,dt}{e^t - 1} = \zeta(3). \tag{4.9.9}$$

More generally, we obtain, from (4.9.8),

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \int_0^{\infty} \frac{t^{p-1}\,dt}{e^t - 1} = \zeta(p). \tag{4.9.10}$$

□

***Example 4.9.2***
Show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \exp(-an) = -\log(1 - e^{-a}). \tag{4.9.11}$$

We put $x = 0, p = 1$, and $a_n = \exp(-an)$ so that

$$b(t) = \sum_{n=1}^{\infty} \exp[-n(t+a)] = \frac{1}{e^{a+t} - 1}. \tag{4.9.12}$$

Then result (4.9.4) gives

$$\sum_{n=1}^{\infty} \frac{1}{n} \exp(-an) = \int_0^{\infty} \frac{dt}{e^{a+t} - 1}, \qquad \exp(-t) = x,$$

$$= \int_0^{1} \frac{dx}{e^a - x} = -\log(1 - e^{-a}).$$

□

***Example 4.9.3***
Show that

$$\sum_{n=1}^{\infty} \frac{1}{(n^2 + x^2)} = \frac{1}{2x^2}(\pi x \coth \pi x - 1). \tag{4.9.13}$$

We set

$$f(t) = \frac{1}{x} \sin xt, \; \bar{f}(n) = \frac{1}{n^2 + x^2}, \text{ and } a_n = 1 \quad \text{for all } n.$$

Clearly

$$b(t) = \sum_{n=1}^{\infty} \exp(-nt) = \frac{1}{e^t - 1}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{(n^2 + x^2)} = \frac{1}{x} \int_0^{\infty} \frac{\sin xt}{e^t - 1}\, dt = \frac{1}{2x^2}(\pi x \;\; \coth \;\; \pi x - 1).$$

□

## 4.10 Transfer Function and Impulse Response Function of a Linear System

Many science and engineering systems are described by initial value problems that are governed by linear ordinary differential equations. In general, a linear system is governed by an $n$th order linear ordinary differential equation with constant coefficients in the form

$$L(D)[x(t)] \equiv a_n\, x^{(n)}(t) + a_{n-1}\, x^{(n-1)}(t) + \ldots + a_0\, x(t) = f(t), \quad (4.10.1)$$

where $a_n$, $a_{n-1}$, ..., $a_0$ are real constants with $a_n \neq 0$ and the initial conditions are

$$x(0) = x_0, \quad x'(0) = x_1, \quad \ldots, \quad x^{(n-1)}(0) = x_{n-1}. \qquad (4.10.2)$$

The solution, $x(t)$ of the system (4.10.1)–(4.10.2) is called the *output* or the *response function*, and the given $f(t)$ is called the *input function* (or *driving function*) of time $t$.

The *transfer function* $\overline{h}(s)$ of a linear system is defined as the ratio of the Laplace transform of the output function $x(t)$ to the Laplace transform of the input function $f(t)$, under the assumption that all initial conditions are zero.

More generally, however, the Laplace transform of the system (4.10.1)–(4.10.2) gives

$$\begin{aligned} a_n & \left[s^n\, \overline{x}(s) - s^{n-1}x(0) - \ldots - x^{(n-1)}(0)\right] \\ & + a_{n-1}\left[s^{n-1}\, \overline{x}(s) - s^{n-2}\, x(0) - \ldots - x^{(n-2)}\right] \\ & + \ldots + a_1\left[s\, \overline{x}(s) - x(0)\right] + a_0\, \overline{x}(s) = \overline{f}(s). \end{aligned} \qquad (4.10.3)$$

Or, equivalently,

$$\left(a_n\, s^n + a_{n-1}\, s^{n-1} + \ldots + a_0\right) \overline{x}(s) = \overline{f}(s) + \overline{g}(s),$$

or,

$$\overline{p_n}(s)\, \overline{x}(s) = \overline{f}(s) + \overline{g}(s), \qquad (4.10.4)$$

where

$$\overline{p_n}(s) = \left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0\right) \qquad (4.10.5)$$

is a polynomial of degree $n$, $\overline{g}(s)$ is a polynomial of degree less than or equal to $(n-1)$ consisting of the various products of the coefficients $a_r$ $(r = 1, 2, \ldots, n)$ and the given initial conditions $x_0$, $x_1$, ..., $x_{n-1}$.

The *transfer function* (or *system function*) is denoted by $\overline{h}(s)$ and defined by

$$\overline{h}(s) = \frac{1}{\overline{p_n}(s)} = \frac{1}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}. \tag{4.10.6}$$

Consequently, equation (4.10.4) becomes

$$\overline{x}(s) = \frac{\overline{f}(s)}{\overline{p_n}(s)} + \frac{\overline{g}(s)}{\overline{p_n}(s)} = \overline{h}(s)\left[\overline{f}(s) + \overline{g}(s)\right]. \tag{4.10.7}$$

The inverse Laplace transform of (4.10.7) provides the response function $x(t)$ of the system which is the superposition of two responses as follows:

$$x(t) = \mathscr{L}^{-1}\left\{\overline{h}(s)\,\overline{g}(s)\right\} + \mathscr{L}^{-1}\left\{\overline{h}(s)\,\overline{f}(s)\right\} \tag{4.10.8}$$

$$= \int_0^t h(t-\tau)\,g(\tau)\,d\tau + \int_0^t h(t-\tau)\,f(\tau)\,d\tau \tag{4.10.9}$$

$$= x_0(t) + x_1(t), \tag{4.10.10}$$

where

$$x_0(t) = \mathscr{L}^{-1}\left\{\overline{h}(s)\,\overline{g}(s)\right\}, \quad x_1(t) = \mathscr{L}^{-1}\left\{\overline{h}(s)\,\overline{f}(s)\right\},$$

and

$$h(t) = \mathscr{L}^{-1}\left\{\overline{h}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{1}{\overline{p_n}(s)}\right\}, \tag{4.10.11}$$

are often called the *impulse response function* of the linear system.

If the input is $f(t) \equiv 0$, the solution of the problem is $x_0(t)$, which is called the *zero-input response* of the system. On the other hand, $x_1(t)$ is the output due to the input $f(t)$ and is called the *zero-state response* of the system. If all initial conditions are zero, that is, $x_0 = x_1 = \ldots = x_{n-1} = 0$, then $\overline{g}(s) = 0$ and so, the unique solution of the nonhomogeneous equation (4.10.1) is $x_1(t)$.

For example, $h(t) = \mathscr{L}^{-1}\left\{\overline{h}(s)\right\}$ describes the solution for a mass-spring system when it is struck by a hammer. For an electric circuit, the function $\overline{z}(s) = \left[s\,\overline{h}(s)\right]^{-1}$ is called the *impendence* of the circuit.

The polynomial $\overline{p_n}(s) = \left(a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0\right)$ in $s$ of degree $n$ is called the *characteristic polynomial* of the system, and $\overline{p_n}(s) = 0$ is called the *characteristic equation* of the system. Since the coefficients of $\overline{p_n}(s)$ are real, it follows that roots of the characteristic equation are all real or, if complex, they must occur in complex conjugate pairs. If $\overline{h}(s)$ is expressed in partial fractions, the system is said to be *stable* provided all roots of the characteristic equation have negative real parts. From a physical point of view, when every root of $\overline{p_n}(s) = 0$ has a negative real part, any bounded input to a system that is stable will lead to an output that is also bounded for all time $t$.

We close this section by adding the following examples:

***Example 4.10.1***

Find the transfer function for each of the following linear systems. Determine the order of each system and find which is stable.

$$\text{(a)} \quad L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(\tau)\,d\tau = E(t), \tag{4.10.12}$$

$$\text{(b)} \quad x''(t) + 2\,x'(t) + 5\,x(t) = 3f'(t) + 2f(t), \tag{4.10.13}$$

$$\text{(c)} \quad x'''(t) + x''(t) + 3\,x'(t) - 5x(t) = 6f''(t) - 13f'(t) + 6f(t),$$

where $L$, $R$, and $C$ are constants. (4.10.14)

(a) This current equation is solved in Example 4.2.13. The Laplace transformed equation with the zero initial condition is given by

$$\left(Ls + R + \frac{1}{Cs}\right)\overline{I}(s) = \overline{E}(s)$$

so that the transfer equation is

$$\overline{h}(s) = \frac{1}{\left(Ls + R + \frac{1}{Cs}\right)} = \frac{1}{L}\,\frac{s}{\left(s^2 + \frac{R}{L}\,s + \frac{1}{CL}\right)}.$$

The system is of order 2 and its characteristic equation is

$$s^2 + \frac{R}{L}\,s + \frac{1}{CL} = 0.$$

Or,

$$(s+k)^2 + n^2 = 0$$

where

$$k = \frac{R}{2L}, \qquad n^2 = \frac{1}{CL} - \frac{R^2}{4L^2}.$$

The roots of the characteristic equation are complex and they are $s = -k \pm i\,n$ with the negative real part. So, the system is stable.

(b) We take the Laplace transform of the equation (4.10.13) with zero initial conditions so that

$$\left(s^2 + 2s + 5\right)\overline{x}(s) = (3s+2)\,\overline{f}(s).$$

Thus, the transfer function is

$$\overline{h}(s) = \frac{\overline{x}(s)}{\overline{f}(s)} = \frac{(3s+2)}{s^2 + 2s + 5}.$$

The system is of order 2 and its characteristic equation is

$$s^2 + 2s + 5 = 0$$

with complex roots $s = -1 \pm 2i$. Since the real part of these roots is negative, the system is stable.
(c) Similarly,

$$\overline{h}(s) = \frac{\overline{x}(s)}{\overline{f}(s)} = \frac{\left(6s^2 - 13s + 6\right)}{(s^3 + s^2 + 3s - 5)}.$$

The system is of order 3 and its characteristic equation is

$$s^3 + s^2 + 3s - 5 = 0$$

with roots $s_1 = 1$, $s_2$, $s_3 = -1 \pm 2i$. Since the real parts of all roots are not negative, the system is *unstable*. ▯

***Example 4.10.2***
Find the transfer function, the impulse response function, and the solution of a linear system described by

$$x''(t) + 2a\,x'(t) + \left(a^2 + 4\right) x(t) = f(t) \tag{4.10.15}$$

$$x(0) = 1, \qquad x'(0) = -a. \tag{4.10.16ab}$$

According to formula (4.10.4), the transfer function of this system is

$$\overline{h}(s) = \frac{1}{(s^2 + 2as + a^2 + 4)} = \frac{1}{(s+a)^2 + 2^2}.$$

The inverse Laplace transform of the transform function $\overline{h}(s)$ is the impulse response function

$$\overline{h}(t) = \mathscr{L}^{-1}\left\{\overline{h}(s)\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s^2+a)^2 + 2^2}\right\} = \frac{1}{2}\,e^{-at}\sin 2t. \tag{4.10.17}$$

Solving the homogeneous initial value problem gives

$$x_0(t) = e^{-at}\cos(2t). \tag{4.10.18}$$

The solution of the problem (4.10.15)–(4.10.16ab) is

$$\begin{aligned} x(t) &= x_0(t) + h(t) * f(t) \\ &= e^{-at}\cos(2t) + \int_0^t e^{-at} f(t-\tau)\sin 2\tau\, d\tau. \end{aligned} \tag{4.10.19}$$

▯

***Example 4.10.3***
Consider a linear system governed by the differential equation

$$a_2\, x''(t) + a_1\, x'(t) + a_0\, x(t) = H(t), \tag{4.10.20}$$

where $H(t)$ is the Heaviside unit step function.
Derive Duhamel's formulas

$$\text{(a)} \quad x(t) = \int_0^t A'(t-\tau)\, f(\tau)\, d\tau, \tag{4.10.21}$$

$$\text{(b)} \quad x(t) = \int_0^t A(\tau)\, f'(t-\tau)\, d\tau + A(t)\, f(0). \tag{4.10.22}$$

The transfer function for this system (4.10.20) is

$$\overline{h}(s) = \frac{\overline{x}(s)}{\overline{f}(s)} = s\, \overline{x}(s). \tag{4.10.23}$$

Or,

$$\overline{x}(s) = \frac{\overline{h}(s)}{s}. \tag{4.10.24}$$

The output function in this special case is called the *indicial admittance* and is denoted by $A(t)$ so that

$$\overline{A}(s) = \frac{\overline{h}(s)}{s}. \tag{4.10.25}$$

We next derive Duhamel's formulas. We have, from (4.10.7) with $\overline{g}(s) = 0$,

$$\overline{x}(s) = s\left[\frac{\overline{h}(s)}{s}\right]\overline{f}(s) = s\,\overline{A}(s)\,\overline{f}(s). \tag{4.10.26}$$

Using the convolution theorem gives

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\left\{s\,\overline{A}(s)\cdot\overline{f}(s)\right\} \\ &= \int_0^t A'(t-\tau)\, f(\tau)\, d\tau = \frac{d}{dt}\int_0^t A(\tau)\, f(t-\tau)\, d\tau \end{aligned}$$

which is, by Leibniz's rule,

$$= \int_0^t A(\tau)\, f'(t-\tau)\, d\tau + A(t)\, f(0),$$

where the initial conditions $A(0) = A'(0) = 0$ are used. ▯

## 4.11 Exercises

1. Using the Laplace transform, solve the following initial value problems

   (a) $\dfrac{dx}{dt} + ax = e^{-bt}$, $t > 0$, $a \neq b$ with $x(0) = 0$.

   (b) $\dfrac{dx}{dt} - x = t^2$, $t > 0$, $x(0) = 0$.

   (c) $\dfrac{dx}{dt} + 2x = \cos t$, $t > 0$, $x(0) = 1$.

   (d) $\dfrac{dx}{dt} - 2x = 4$, $t > 0$, $x(0) = 0$.

2. Solve the initial value problem for the radioactive decay of an element

$$\frac{dx}{dt} = -kx, \;\; (k > 0), \;\; t > 0, \;\; x(0) = x_0.$$

   Prove that the half-life time $T$ of the element, which is defined as the time taken for half a given amount of the element to decay, is

$$T = \frac{1}{k} \log 2.$$

3. Find the solutions of the following systems of equations with the initial data:

   (a) $\dfrac{dx}{dt} = x - 2y, \quad \dfrac{dy}{dt} = y - 2x, \quad x(0) = 1, \quad y(0) = 0.$

   (b) $\dfrac{dx_1}{dt} = x_1 + 2x_2 + t, \quad \dfrac{dx_2}{dt} = x_2 + 2x_1 + t; \quad x_1(0) = 2, \quad x_2(0) = 4.$

   (c) $\dfrac{dx}{dt} = 6x - 7y + 4z, \quad \dfrac{dy}{dt} = 3x - 4y + 2z, \quad \dfrac{dz}{dt} = -5x + 5y - 3z,$ with $x(0) = 5, \quad y(0) = z(0) = 0.$

   (d) $\dfrac{dx}{dt} = 2x - 3y, \quad \dfrac{dy}{dt} = y - 2x; \quad x(0) = 2, \quad y(0) = 1.$

   (e) $\dfrac{dx}{dt} + x = y, \quad \dfrac{dy}{dt} - y = x, \quad x(0) = y(0) = 1.$

   (f) $\dfrac{dx}{dt} + \dfrac{dy}{dt} + x = 0, \quad \dfrac{dx}{dt} + 2\,\dfrac{dy}{dt} - x = e^{-at}, \quad x(0) = y(0) = 1.$

4. Solve the matrix differential system

$$\frac{dx}{dt} = Ax \text{ with } x(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix},$$

   where $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $A = \begin{pmatrix} -3 & -2 \\ 3 & 2 \end{pmatrix}$.

5. Find the solution of the autonomous system described by

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = x + 2y \quad \text{with} \quad x(0) = x_0, \quad y(0) = y_0.$$

6. Solve the differential systems

(a)

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2k\frac{dy}{dt} + lx = 0 \\ \frac{d^2y}{dt^2} + 2k\frac{dx}{dt} + ly = 0 \end{aligned}\right\} \quad t > 0$$

with the initial conditions

$$x(0) = a, \quad \dot{x}(0) = 0; \quad y(0) = 0, \quad \dot{y}(0) = \upsilon,$$

where $k, l, a$, and $\upsilon$ are constants.

(b)

$$\left.\begin{aligned} \frac{d^2x}{dt^2} = y - 2\,x \\ \frac{d^2y}{dt^2} = x - 2\,y \end{aligned}\right\} \quad t > 0$$

with the initial conditions

$$x(0) = y(0) = 1, \quad \text{and} \quad \dot{x}(0) = \dot{y}(0) = 0.$$

7. The glucose concentration in the blood during continuous intravenous injection of glucose is $C(t)$, which is in excess of the initial value at the start of the infusion. The function $C(t)$ satisfies the initial value problem

$$\frac{dC}{dt} + k\,C = \frac{\alpha}{V}, \quad t > 0, \quad C(0) = 0,$$

where $k$ is the constant velocity of elimination, $\alpha$ is the rate of infusion (in mg/min), and $V$ is the volume in which glucose is distributed. Solve this problem.

8. The blood is pumped into the aorta by the contraction of the heart. The pressure $p(t)$ in the aorta satisfies the initial value problem

$$\frac{dp}{dt} + \frac{c}{k}\,p = cA\,\sin\omega t, \quad t > 0; \quad p(0) = p_0$$

where $c, k, A$, and $p_0$ are constants. Solve this initial value problem.

9. The zero-order chemical reaction satisfies the initial value problem

$$\frac{dc}{dt} = -k_0, \quad t > 0, \quad \text{with} \quad c = c_0 \quad \text{at} \quad t = 0$$

where $k_0$ is a positive constant and $c(t)$ is the concentration of a reacting substance at time $t$. Show that

$$c(t) = c_0 - k_0\, t.$$

10. Solve the equation governing the first order chemical reaction

$$\frac{dc}{dt} = -k_1 c \quad \text{with} \quad c(t) = c_0 \quad \text{at} \quad t = 0 \quad (k_1 > 0).$$

11. Obtain the solutions of the systems of differential equations governing the consecutive chemical reactions of the first order

$$\frac{dc_1}{dt} = -k_1 c_1, \quad \frac{dc_2}{dt} = k_1 c_1 - k_2 c_2, \quad \frac{dc_3}{dt} = k_2 c_2, \quad t > 0,$$

with the initial conditions

$$c_1(0) = c_1, \quad c_2(0) = c_3(0) = 0,$$

where $c_1(t)$ is the concentration of a substance $A$ at time $t$, which breaks down to form a new substance $A_2$ with concentration $c_2(t)$, and $c_3(t)$ is the concentration of a new element originated from $A_2$.

12. Solve the following initial value problems

(a) $\ddot{x} + \omega^2 x = \cos nt, \quad (\omega \neq n) \qquad x(0) = 1, \quad \dot{x}(0) = 0.$

(b) $\ddot{x} + x = \sin 2t, \quad x(0) = \dot{x}(0) = 0.$

(c) $\dfrac{d^3x}{dt^3} + \dfrac{d^2x}{dt^2} = 3e^{-4t}$, $x(0) = 0$, $\dot{x}(0) = -1$, $\ddot{x}(0) = 1$.

(d) $\dfrac{d^4x}{dt^4} = 16x$, $x(t) = \ddot{x}(t) = 0$, $\dot{x}(t) = \dddot{x}(t) = 1$ at $t = 0$.

(e) $(D^4 + 2D^3 - D^2 - 2D + 10)x(t) = 0$, $t > 0$,
$x(0) = -1$, $\dot{x}(0) = 3$, $\ddot{x}(0) = -1$, $\dddot{x}(0) = 4$.

(f) $\dfrac{d^2x}{dt^2} + b\,\dfrac{dx}{dt} = \delta(t-a)$, $x(0) = \alpha$, $\dot{x}(0) = \beta$.

(g) $C\,\dfrac{d^2v}{dt^2} + \dfrac{1}{R}\dfrac{dv}{dt} + \dfrac{v}{L} = \dfrac{di}{dt}$, $\quad v(0) = \dot{v}(0) = 0$; $\quad i(t) = H(t-1) - H(t)$,
where $R$, $L$, and $C$ are constants.

(h) $\dfrac{d^2x}{dt^2} + 2\,t\,\dfrac{dx}{dt} - 4\,x = 2, \quad x(0) = 0 = \dot{x}(0).$

(i) $\dfrac{d^2x}{dt^2} - 2\,a\,\dfrac{dx}{dt} + a^2\,x = t - (t-a)\,H(t-a) - a\,H(t-a),$
$x(0) = 0 = \dot{x}(0).$

13. Solve the following systems of equations:

(a) $\ddot{x} - 2\dot{y} - x = 0,\ \ddot{y} + 2\dot{x} - y = 0,$
$x(t) = y(t) = 0, \quad \dot{x}(t) = \dot{y}(t) = 1$ at $t = 0$.

(b) $\ddot{x}_1 + 3\dot{x}_1 - 2x_1 + \dot{x}_2 - 3x_2 = 2e^{-t}, \qquad 2\dot{x}_1 - x_1 + \dot{x}_2 - 2x_2 = 0,$
with $x_1(0) = \dot{x}_1(0) = 0$ and $x_2(0) = 4$.

14. With the aid of the Laplace transform, investigate the motion of a particle governed by the equations of motion $\ddot{x} - \omega\dot{y} = 0, \ddot{y} + \omega\dot{x} = \omega^2 a$ and the initial conditions $x(0) = y(0) = \dot{x}(0) = \dot{y}(0) = 0$.

15. Show that the solution of the equation

$$\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = e^{-ax},\ x > 0$$

with the initial data $y(x) = \dfrac{1}{a^2}$ and $\dfrac{dy}{dx} = 0$ at $x = 0$ is

$$y(x) = \frac{1}{a^2(a-b)}(ae^{-bx} - be^{-ax} - xa^2e^{-ax}) + \frac{e^{-bx} - e^{-ax}}{(a-b)^2}.$$

16. The motion of an electron of charge $-e$ in a static electric field $\mathbf{E} = (E, 0, 0)$ and a static magnetic field $\mathbf{H} = (0, 0, H)$ is governed by the vector equation

$$m\ddot{\mathbf{r}} = -e\,\mathbf{E} + \frac{e}{c}\,(\dot{\mathbf{r}} \times \mathbf{H}), \quad t > 0,$$

with zero initial velocity and displacement ($\mathbf{r} = \dot{\mathbf{r}} = \mathbf{0}$ at $t = 0$) where $\mathbf{r} = (x, y, z)$ and $c$ is the velocity of light. Show that the displacement fields are

$$x(t) = \frac{eE}{m\omega^2}\,(\cos\omega t - 1), \quad y(t) = \frac{eE}{m\omega^2}\,(\sin\omega t - \omega t), \quad z(t) = 0,$$

where $\omega = \dfrac{eH}{mc}$. Hence, calculate the velocity field.

17. An electron of mass $m$ and charge $-e$ is acted on by a periodic electric field $E\sin\omega_0 t$ along the $x$-axis and a constant magnetic field $H$ along the $z$-axis. Initially, the electron is emitted at the origin with zero velocity. With the same $\omega$ as given in exercise 16, show that

$$x(t) = \frac{eE}{m\omega(\omega^2 - \omega_0^2)}\,(\omega_0 \sin\,\omega t - \omega \sin\,\omega_0 t)\,,$$

$$y(t) = \frac{eE}{m\omega\,(\omega^2 - \omega_0^2)\,\omega_0}\left\{(\omega^2 - \omega_0^2) + \left(\omega_0^2 \cos\,\omega t - \omega^2 \cos\,\omega_0 t\right)\right\}.$$

18. The stress-strain relation and equation of motion for a viscoelastic rod in the absence of external force are

$$\frac{\partial e}{\partial t} = \frac{1}{E}\frac{\partial \sigma}{\partial t} + \frac{\sigma}{\eta}, \qquad \frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2},$$

where $e$ is the strain, $\eta$ is the coefficient of viscosity, and the displacement $u(x,t)$ is related to the strain by $e = \dfrac{\partial u}{\partial x}$. Prove that the stress $\sigma(x,t)$ satisfies the equation

$$\frac{\partial^2 \sigma}{\partial x^2} - \frac{\rho}{\eta}\frac{\partial \sigma}{\partial t} = \frac{1}{c^2}\frac{\partial^2 \sigma}{\partial t^2}.$$

Show that the stress distribution in a semi-infinite viscoelastic rod subject to the boundary and initial conditions

$$\dot{u}(0,t) = U H(t), \quad \sigma(x,t) \to 0 \quad \text{as} \quad x \to \infty,$$
$$\sigma(x,0) = 0, \quad \dot{u}(x,0) = 0, \quad \text{for} \quad 0 < x < \infty,$$

is given by

$$\sigma(x,t) = -U \rho\, c \exp\left(-\frac{Et}{2\eta}\right) I_0\left[\frac{E}{2\eta}\left(t^2 - \frac{x^2}{c^2}\right)^{1/2}\right] H\left(t - \frac{x}{c}\right).$$

19. An elastic string is stretched between $x = 0$ and $x = \ell$ and is initially at rest in the equilibrium position. Find the Laplace transform solution for the displacement subject to the boundary conditions $y(0,t) = f(t)$ and $y(l,t) = 0$, $t > 0$.

20. The end $x = 0$ of a semi-infinite submarine cable is maintained at a potential $V_0 H(t)$. If the cable has no initial current and potential, determine the potential $V(x,t)$ at a point $x$ and at time $t$.

21. A semi-infinite lossless transmission line has no initial current or potential. A time-dependent electromagnetic force, $V_0(t)H(t)$ is applied at the end $x = 0$. Find the potential $V(x,t)$. Hence, determine the potential for cases (i) $V_0(t) = V_0 = \text{constant}$, and (ii) $V_0(t) = V_0 \cos \omega t$.

22. Solve the Blasius problem of an unsteady boundary layer flow in a semi-infinite body of viscous fluid enclosed by an infinite horizontal disk at $z = 0$. The governing equation and the boundary and initial conditions are

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \nu \frac{\partial^2 u}{\partial z^2}, & z > 0, \quad t > 0,\\
u(z,t) &= Ut & \text{on } z = 0, \quad t > 0,\\
u(z,t) &\to 0 & \text{as } z \to \infty, \quad t > 0,\\
u(z,t) &= 0 & \text{at } t \le 0, \quad z > 0.
\end{aligned}$$

Explain the significance of the solution.

23. Obtain the solution of the Stokes-Ekman problem of an unsteady boundary layer flow in a semi-infinite body of viscous fluid bounded by an infinite horizontal disk at $z=0$, when both the fluid and the disk rotate with a uniform angular velocity $\Omega$ about the $z$-axis. The governing boundary layer equation, the boundary and the initial conditions are

$$\frac{\partial q}{\partial t} + 2\Omega i q = \nu \frac{\partial^2 q}{\partial z^2}, \qquad z>0,$$
$$q(z,t) = ae^{i\omega t} + be^{-i\omega t} \quad \text{on} \quad z=0,\ t>0,$$
$$q(z,t) \to 0 \quad \text{as} \quad z\to\infty,\ t>0,$$
$$q(z,t) = 0 \quad \text{at} \quad t\le 0 \quad \text{for all} \quad z>0,$$

where $q=u+iv$, $\omega$ is the frequency of oscillations of the disk and $a, b$ are complex constants. Hence, deduce the steady-state solution and determine the structure of the associated boundary layers.

24. Show that, when $\omega=0$ in exercise 23, the steady flow field is given by

$$q(z,t) \sim (a+b)\exp\left\{\left(-\frac{2i\Omega}{\nu}\right)^{1/2} z\right\}.$$

Hence, determine the thickness of the Ekman layer.

25. Solve the following integral and integro-differential equations:

(a) $f(t) = \sin 2t + \int_0^t f(t-\tau)\sin\tau\, d\tau.$

(b) $f(t) = \frac{t}{2}\sin t + \int_0^t f(\tau)\sin(t-\tau)\, d\tau.$

(c) $\int_0^t f(\tau) J_0[a(t-\tau)]\, d\tau = \sin at.$

(d) $f(t) = \sin t + \int_0^t f(\tau)\sin\{2(t-\tau)\}\, d\tau.$

(e) $f(t) = t^2 + \int_0^t f'(t-\tau)\exp(-a\tau)\, d\tau, \quad f(0)=0.$

(f) $x(t) = 1 + a^2\int_0^t (t-\tau)\, x(\tau)\, d\tau.$

(g) $x(t) = t + \frac{1}{a} \int_0^t (t-\tau)^3\, x(\tau)\, d\tau.$

26. Prove that the solution of the integro-differential equation

$$f(t) = \frac{2}{\sqrt{\pi}} \left[ \sqrt{t} + \sqrt{a} \int_0^t (t-\tau)^{1/2} f'(\tau)\, d\tau \right], \quad f(0) = 0$$

is

$$f(t) = \frac{e^{at}}{\sqrt{a}} [1 + erf\sqrt{at}] - \frac{1}{\sqrt{a}}.$$

27. Solve the integro-differential equations

(a) $\frac{d^2x}{dt^2} = \exp(-2t) - \int_0^t \exp\{-2(t-\tau)\} \left(\frac{dx}{d\tau}\right) d\tau, \quad x(0) = 0 \text{ and } \dot{x}(0) = 0.$

(b) $\frac{dx}{dt} = \int_0^t x(\tau) \cos(t-\tau) d\tau, \qquad x(0) = 1.$

28. Using the Laplace transform, evaluate the following integrals:

(a) $\int_0^\infty \frac{\sin tx}{x(x^2+a^2)} dx, \ (a, t > 0),$

(b) $\int_0^\infty \frac{\sin tx}{x} dx,$

(c) $\int_{-\infty}^\infty \frac{\cos tx}{x^2+a^2} dx, \ (a, t > 0),$

(d) $\int_{-\infty}^\infty \frac{x \sin xt}{x^2+a^2} dx, \ (a, t > 0),$

(e) $\int_0^\infty \exp(-tx^2) dx, \quad t > 0,$

(f) $\int_0^\infty \cos(tx^2)\, dx.$

29. Show that

(a) $\int_0^\infty e^{-ax} \left(\frac{\cos px - \cos qx}{x}\right) dx = \frac{1}{2} \log\left(\frac{a^2+q^2}{a^2+p^2}\right), \ (a > 0).$

(b) $\int_0^\infty e^{-ax} \left(\frac{\sin qx - \sin px}{x}\right) dx = \tan^{-1}\left(\frac{q}{a}\right) - \tan^{-1}\left(\frac{p}{a}\right), \quad a > 0.$

30. Establish the following results:

(a) $\int_{-\infty}^\infty \frac{\cos tx\, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2} \left(\frac{e^{-bt}}{b} - \frac{e^{-at}}{a}\right), \quad a, b, t > 0.$

(b) $\displaystyle\int_0^\infty \frac{\sin(\pi\, tx)}{x(1+x^2)}\,dx = \frac{\pi}{2}(1-e^{-\pi t}), \quad t>0.$

(c) $\displaystyle\int_0^\infty \cos(tu^2)\,du = \int_0^\infty \sin(tu^2)\,du = \frac{1}{2}\left(\frac{\pi}{2t}\right)^{1/2}, \quad t>0.$

31. In Example 4.5.1(i), write the solution when the point load is applied at the mid point of the beam.

32. A uniform horizontal beam of length $2\ell$ is clamped at the end $x=0$ and freely supported at $x=2\ell$. It carries a distributed load of constant value $W$ in $\frac{\ell}{2}<x<\frac{3\ell}{2}$ and zero elsewhere. Obtain the deflection of the beam which satisfies the boundary value problem

$$EI\,\frac{d^4y}{dx^4} = W\left[H\left(x-\frac{\ell}{2}\right) - H\left(x-\frac{3\ell}{2}\right)\right], \quad 0<x<2\ell,$$
$$y(0)=0=y'(0), \quad y''(2\ell)=0=y'''(2\ell).$$

33. Solve exercise 32 if the beam carries a constant distributed load $W$ per unit length in $0<x<\ell$ and zero in $\ell<x<2\ell$. Find the bending moment and shear at $x=\frac{\ell}{2}$.

34. A horizontal cantilever beam of length $2\ell$ is deflected under the combined effect of its own constant weight $W$ and a point load of magnitude $P$ located at the midpoint. Obtain the deflection of the beam which satisfies the boundary value problem

$$EI\,\frac{d^4y}{dx^4} = W[H(x)-H(x-2\ell)] + P\,\delta(x-\ell), \quad 0<x<2\ell,$$
$$y(0)=0=y'(0), \quad y''(2\ell)=0=y'''(2\ell).$$

Find the bending moment and shear at $x=\frac{\ell}{2}$.

35. Using the Laplace transform, solve the following difference equations:

(a) $\Delta u_n - 2u_n = 0, \quad u_0=1,$

(b) $\Delta^2 u_n - 2u_{n+1} + 3u_n = 0, \quad u_0=0 \quad \text{and} \quad u_1=1,$

(c) $u_{n+2} - 4u_{n+1} + 4u_n = 0, \quad u_0=1 \quad \text{and} \quad u_1=4,$

(d) $u_{n+2} - 5u_{n+1} + 6u_n = 0, \quad u_0=1 \quad \text{and} \quad u_1=4,$

(e) $\Delta^2 u_n + 3u_n = 0, \quad u_0=0, \quad u_1=1,$

(f) $u_{n+2} - 4u_{n+1} + 3u_n = 0,$

(g) $u_{n+2} - 9u_n = 0, \quad u_0=1 \text{ and } u_1=3,$

(h) $\Delta u_n - (a-1)u_n = 0, \quad u_0 = \text{constant}.$

36. Show that the solution of the difference equation

$$u_{n+2} + 4u_{n+1} + u_n = 0, \quad \text{with} \quad u_0 = 0 \quad \text{and} \quad u_1 = 1,$$

is

$$u_n = \frac{1}{2\sqrt{3}}\left[\left(\sqrt{3}-2\right)^n + (-1)^{n+1}\left(2+\sqrt{3}\right)^n\right].$$

37. Show that the solution of the differential-difference equation

$$\dot{u}(t) - u(t-1) = 2, \quad u(0) = 0$$

is

$$u(t) = 2\left[t - \frac{(t-1)^2}{2!} + \frac{(t-2)^3}{3!} + \cdots + \frac{(t-n)^{n+1}}{(n+1)!}\right], \quad t > n.$$

38. Obtain the solution of the differential-difference equation

$$\dot{u} = u(t-1), \quad u(0) = 1, \quad 0 < t < \infty \text{ with } u(t) = 1 \text{ when } -1 \le t < 0.$$

39. Use the Laplace transform to solve the initial-boundary value problem

$$\begin{aligned} &u_{tt} - u_{xx} = k^2\, u_{xxtt}, \quad 0 < x < \infty, \quad t > 0, \\ &u(x,0) = 0, \quad \left(\frac{\partial u}{\partial x}\right)_{t=0} = 0, \quad \text{for} \quad x > 0, \\ &u(x,t) \to 0 \quad \text{as} \quad x \to \infty, \quad t > 0, \\ &u(0,t) = 1 \quad \text{for} \quad t > 0. \end{aligned}$$

Hence, show that

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = -\frac{1}{k}\, J_0\left(\frac{t}{k}\right).$$

40. Solve the telegraph equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} + 2a u_t = 0, \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = g(x). \end{aligned}$$

41. Use the joint Laplace and Fourier transform to solve Example 2.12.3 in Chapter 2.

42. Use the Laplace transform to solve the initial-boundary value problem

$$\begin{aligned} u_t = c^2 u_{xx}, \quad 0 < x < a, \quad t > 0, \\ u(x,0) = x + \sin\left(\frac{3\pi x}{a}\right) \quad \text{for} \quad 0 < x < a, \\ u(0,t) = 0 = u(a,t) \quad \text{for} \quad t > 0. \end{aligned}$$

43. Solve the diffusion equation

$$u_t = k u_{xx}, \quad -a < x < a, \quad t > 0,$$
$$u(x,0) = 1 \quad \text{for} \quad -a < x < a,$$
$$u(-a,t) = 0 = u(a,t) \quad \text{for} \quad t > 0.$$

44. Use the joint Laplace and Fourier transform to solve the initial value problem for water waves which satisfies (see Debnath, 1994, p. 92)

$$\nabla^2\phi = \phi_{xx} + \phi_{zz} = 0, \quad -\infty < z < 0, \quad -\infty < x < \infty, \quad t > 0$$
$$\left.\begin{aligned} \phi_z &= \eta_t \\ \phi_t + g\eta &= -\frac{P}{\rho} p(x) e^{i\omega t} \end{aligned}\right\} \text{on} \quad z = 0, \quad t > 0,$$
$$\phi(x,z,0) = 0 = \eta(x,0) \quad \text{for all } x \text{ and } z,$$

where $P$ and $\rho$ are constants.

45. Show that

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{\sqrt{n^2+x^2}} = \int_0^{\infty} b(t) J_0(xt)\, dt$, where $b(t)$ is given by (4.9.3).

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2-a^2} = \frac{1}{2a^2}(1 - \pi a \cot \pi a)$.

46. Show that

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{(n^2-a^2)} = \frac{1}{2a^2}\left[1 - \frac{\pi a \cos ax}{\sin a\pi}\right]$.

(b) $\displaystyle\sum_{n=1}^{\infty} \log\left(1 + \frac{a^2}{n^2}\right) = \log\left(\frac{\sinh \pi a}{\pi a}\right)$.

47. (a) If $f(t) = 1$ in Example 4.3.3, show that

$$u(x,t) = \frac{x}{\sqrt{4\pi\kappa}} \int_o^t \tau^{-\frac{3}{2}} \exp\left(-\frac{x^2}{4\,\kappa\,\tau}\right) d\tau = u_0(x,t) \qquad \text{(say)}$$

(b) Hence or otherwise derive the *Duhamel's formula* from (4.3.16):

$$u(x,t) = \int_o^t f(t-\tau) \left(\frac{\partial u_0}{\partial \tau}\right) d\tau,$$
$$\text{where} \quad \frac{\partial u_0}{\partial t} = \frac{x}{\sqrt{4\pi\kappa}} \tau^{-\frac{3}{2}} \exp\left(-\frac{x^2}{4\,\kappa\,t}\right).$$

48. Consider a progressive plane wave solution that propagates to the right with the phase velocity $\left(\frac{\omega}{k}\right)$ of the telegraph equation (4.3.55)

(a) Derive the dispersion relation

$$\omega^2 + i\,(p+q)\,\omega - \left(c^2k^2 + p\,q\right) = 0.$$

(b) If $4p\,q \neq (p+q)^2$, show that the plane wave solution is given by

$$u(x,t) = A\exp\left[-\frac{1}{2}\,(p+q)\,t\right]\exp\left[i(kx \pm \sigma t)\right],$$

$$\text{where}\quad \sigma = \frac{1}{2}\sqrt{4c^2k^2 + 4p\,q - (p+q)^2}.$$

(c) If $4p\,q = (p+q)^2$, show that the plane wave solution is given by

$$u(x,t) = A\exp\left[-\frac{1}{2}\,(p+q)\,t\right]\exp\left[ik(x \pm ct)\right].$$

Explain the physical significance of the solutions given in cases (b) and (c).

49. (a) Use the substitution $v(x,t) = \exp\left[\frac{1}{2}\,(p+q)\,t\right]u(x,t)$ into (4.3.55) to show that $v(x,t)$ satisfies the wave equation

$$v_{tt} - c^2\,v_{xx} = \frac{1}{4}\,(p-q)^2\,v.$$

(b) Show that the undistorted wave solution exists if $p = q$ and that a progressive wave of the form $\exp(-at)f(x \pm ct)$ propagates in either direction where $f$ is an arbitrary twice differentiable function of its argument.

50. (a) Use the joint Laplace and Fourier transform to solve the inhomogeneous diffusion problem

$$u_t - \kappa\,u_{xx} = q(x,t) \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x,0) = f(x), \quad \text{for all } x \in \mathbb{R}.$$

(b) Solve the initial-boundary value problem for the diffusion equation

$$u_t - \kappa\,u_{xx} = 0, \quad 0 < x < l, \quad t > 0$$
$$u(x,0) = 0, \quad \text{and} \quad u(0,t) = 1 = u(l,t).$$

51. Use the Laplace transform to solve for the small displacement $y(x,\,t)$ of a semi-infinite string fixed at $x = 0$ under the action of gravity $g$ that

satisfies the wave equation and the initial-boundary conditions

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} &= -g, \quad 0 < x < \infty, \quad t > 0, \\ y(x, 0) &= 0 = y_t(x, 0), \quad x \geq 0, \\ \frac{\partial y}{\partial x} &\to 0 \quad \text{as} \quad \text{x} \to \infty.\end{aligned}$$

52. Use the Laplace transform to solve the boundary layer equation (4.3.97) subject to the boundary and initial conditions

$$\begin{aligned}u(z, t) &= U_0 f(t), \quad \text{on z} = 0, \quad \text{t} > 0, \\ u(z, t) &\to 0 \quad \text{as} \quad z \to \infty, \quad t > 0, \\ u(z, t) &\to 0 \quad \text{at} \quad t \leq 0 \quad \text{for all } z > 0.\end{aligned}$$

Consider the special case where $f(t) = \sin \omega t$.

53. Find the transfer function, the impulse response function and a formula for the solution of the following systems:

(a) $x''(t) + 2\,x'(t) + 5\,x(t) = f(t), \quad x(0) = 2,\ x'(0) = -2.$

(b) $x''(t) - 2\,x'(t) + 5\,x(t) = f(t), \quad x(0) = 0,\ x'(0) = 2.$

(c) $x''(t) + 9\,x'(t) = f(t), \quad x(0) = 2,\ x'(0) = -3.$

(d) $x''(t) - 2\,x'(t) + 5\,x(t) = f(t), \quad x(0) = x_0,\ x'(0) = x_1.$

54. Determine the transfer function for each of the following systems. Obtain the order of each system and find which is stable.

(a) $x''(t) + 2\,x'(t) + 2\,x(t) = 3\,f'(t) + 2\,f(t).$

(b) $4\,x''(t) + 16\,x'(t) + 25\,x(t) = 2\,f'(t) + 3\,f(t).$

(c) $36\,x''(t) + 12\,x'(t) + 37\,x(t) = 2\,f''(t) + f'(t) - 6\,f(t).$

(d) $x''(t) - 6\,x'(t) + 10\,x(t) = 2\,f'(t) + 5\,f(t).$

55. Examine the stability of a system for real constants $a$ and $b$ with zero initial data

$$x'''(t) - a\,x''(t) + b^2\,x'(t) - ab^2 x(t) = f(t),$$

where $x(t)$ is the output corresponding to input $f(t)$.

Discuss three cases: (a) $a > 0$, (b) $a \leq 0, \quad b \neq 0$, (c) $a \neq 0, \quad b = 0$.

# 5

# *Fractional Calculus and Its Applications*

"In his discovery of calculus, Leibniz first introduced the idea of a symbolic method and used the symbol $\frac{d^n y}{dx^n} = D^n y$ for the $n$th derivative, where $n$ is a non-negative integer. L'Hospital asked Leibniz about the possibility that $n$ be a fraction. 'What if $n = \frac{1}{2}$.' Leibniz (1695) replied, 'It will lead to a paradox.' But he added prophetically, 'From this apparent paradox, one day useful consequences will be drawn'."

Gottfried Wilhelm Leibniz

"The mathematician's best work is art, a high perfect art, as daring as the most secret dreams of imagination, clear and limpid. Mathematical genius and artistic genius touch one another."

Gȯsta Mittag-Leffler

## 5.1 Introduction

This chapter deals with fractional derivatives and fractional integrals and their basic properties. Several methods including the Laplace transform are discussed to introduce the Riemann-Liouville fractional integrals. Attention is given to the Weyl fractional integral and its properties. Finally, the fractional derivative is applied to solve the celebrated Abel integral equation. This is followed by brief comments on the Heaviside operational calculus and modern applications of fractional calculus to science and engineering. This chapter is based on two articles of Debnath (2003, 2004) and hence, the reader is referred to these articles for all references cited in this chapter.

## 5.2 Historical Comments

Historically, Isaac Newton (1642-1727) and Gottfried Wihelm Leibniz (1646-1716) independently discovered calculus in the seventieth century. In recognition of this remarkable discovery, John Von Neumann's (1903-1957) thought seems to worth quoting. "...the calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more equivocally than anything else the inception of modern mathematics, and the system of mathematical analysis, which is its logical development, still constitute the greatest technical advance in exact thinking." In his discovery of calculus, Leibniz first introduced the idea of a symbolic method and used the symbol $\frac{d^n y}{dx^n} = D^n y$ for the $n$th derivative, where $n$ is a non-negative integer. L'Hospital asked Leibniz about the possibility that $n$ be a fraction. "What if $n = \frac{1}{2}$." Leibniz (1695) replied, "It will lead to a paradox." But he added prophetically, "From this apparent paradox, one day useful consequences will be drawn." Can the meaning of derivatives of integral order $D^n y$ be extended to have meaning where $n$ is any number — rational, irrational, or complex? In his 700-page long book on calculus published in 1819, Lacroix developed the formula for the $n$th derivative of $y = x^m$, $m$ is a positive integer,

$$D^n y = \frac{m!}{(m-n)!} x^{m-n}, \tag{5.2.1}$$

where $n\,(\le m)$ is an integer. Replacing the factorial symbol by the gamma function, he further obtained the formula for the fractional derivative

$$D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \tag{5.2.2}$$

where $\alpha$ and $\beta$ are fractional numbers. In particular, he calculated

$$D^{\frac{1}{2}} x = \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)} x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}. \tag{5.2.3}$$

On the other hand, in 1832, Joseph Liouville (1809-1882) formally extended the formula for the derivative of integral order $n$

$$D^n e^{ax} = a^n e^{ax} \tag{5.2.4}$$

to the derivative of arbitrary order $\alpha$

$$D^\alpha e^{ax} = a^\alpha e^{ax}. \tag{5.2.5}$$

Using the series expansion of a function $f(x)$, Liouville derived the formula

$$D^\alpha f(x) = \sum_{n=0}^{\infty} c_n\, a_n^\alpha\, e^{a_n x}, \tag{5.2.6}$$

where

$$f(x) = \sum_{n=0}^{\infty} c_n \exp(a_n x), \quad Re\, a_n > 0. \tag{5.2.7}$$

Formula (5.2.6) is referred to as *Liouville's first formula* for fractional derivative. It can be used as a formula for derivative of arbitrary order $\alpha$, which may be rational, irrational or complex. However, it can only be used for functions of the form (5.2.7). In order to extend his first definition (5.2.6), Liouville formulated another definition of a fractional derivative based on the gamma function (see Debnath and Speight (1971))

$$\Gamma(\beta)\, x^{-\beta} = \int_0^{\infty} t^{\beta-1} e^{-xt} dt, \quad \beta > 0, \tag{5.2.8}$$

$$D^{\alpha} x^{-\beta} = (-1)^{\alpha} \, \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\alpha-\beta}, \quad \beta > 0. \tag{5.2.9}$$

This is called the *Liouville's second definition* of fractional derivative. He successfully applied both his definitions to problems in potential theory. However, Liouville's first definition is restricted to a certain class of function in the form (5.2.7), and his second definition is useful only for rational functions. Neither of his definitions was found to be suitable for a wide class of functions. According to (5.2.9), the derivative of a constant function ($\beta = 0$) is zero because $\Gamma(0) = \infty$. On the other hand, the Lacroix definition (5.2.2) gives a nonzero value for the fractional derivative of a constant function ($\beta = 0$) in the form

$$D^{\alpha} 1 = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \neq 0. \tag{5.2.10}$$

Peacock (1833) favored Lacroix formula (5.2.2) for fractional derivatives, but other mathematicians preferred Liouville's definitions. This led to a discrepancy between the two definitions of a fractional derivative. In spite of a lot of subsequent progress on the subject of fractional calculus, this controversy has hardly been resolved.

In 1822, Fourier obtained the following integral representations for $f(x)$ and its derivatives.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, d\xi \int_{-\infty}^{\infty} \cos t\,(x-\xi)\, dt, \tag{5.2.11}$$

and

$$D^n f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, d\xi \int_{-\infty}^{\infty} t^n \cos\left\{ t(x-\xi) + \frac{n\pi}{2} \right\} dt. \tag{5.2.12}$$

Replacing integer $n$ by arbitrary real $\alpha$ yields formally

$$D^{\alpha} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, d\xi \int_{-\infty}^{\infty} t^{\alpha} \cos\left\{ t(x-\xi) + \frac{\pi\alpha}{2} \right\} dt. \tag{5.2.13}$$

Greer (1858-1859) derived formulas for the fractional derivatives of trigonometric functions based on (5.2.4) in the form

$$\begin{aligned} D^{\alpha}e^{iax} &= i^{\alpha}a^{\alpha}e^{iax} = i^{\alpha}a^{\alpha}\left(\cos ax + i\sin ax\right) \\ &= a^{\alpha}\left(\cos\frac{\pi\alpha}{2} + i\sin\frac{\pi\alpha}{2}\right)\left(\cos ax + i\sin ax\right) \end{aligned} \tag{5.2.14}$$

so that the fractional derivatives of trigonometric functions are given by

$$\begin{aligned} D^{\alpha}\left(\cos ax\right) &= a^{\alpha}\left(\cos\frac{\pi\alpha}{2}\cos ax - \sin\frac{\pi\alpha}{2}\sin ax\right) \\ &= a^{\alpha}\cos\left(ax + \frac{\pi\alpha}{2}\right), \end{aligned} \tag{5.2.15}$$

$$\begin{aligned} D^{\alpha}\left(\sin ax\right) &= a^{\alpha}\left(\cos ax\sin\frac{\pi\alpha}{2} + \sin ax\cos\frac{\pi\alpha}{2}\right) \\ &= a^{\alpha}\sin\left(ax + \frac{\pi\alpha}{2}\right). \end{aligned} \tag{5.2.16}$$

When $\alpha = \frac{1}{2}$ and $a = 1$, Greer's formulas are as follows:

$$D^{\frac{1}{2}}\cos x = \cos\left(x + \frac{\pi}{4}\right), \quad D^{\frac{1}{2}}\sin x = \sin\left(x + \frac{\pi}{4}\right). \tag{5.2.17}$$

Similarly, fractional derivatives for hyperbolic functions can be obtained.

## 5.3 Fractional Derivatives and Integrals

The idea of fractional derivative or fractional integral can be described in different ways. First, we consider a linear nonhomogenerous $n$th order ordinary differential equation

$$D^{n}y = f\left(x\right), \quad b \le x \le c. \tag{5.3.1}$$

Then $\left\{1, x, x^2, ...x^{n-1}\right\}$ is a fundamental set of the corresponding homogeneous equation, $D^n y = 0$. If $f\left(x\right)$ is any continuous on $b \le x \le c$, then for any $a \in (b, c)$,

$$y\left(x\right) = \int_a^x \frac{\left(x-t\right)^{n-1}}{(n-1)!} f\left(t\right)dt \tag{5.3.2}$$

is the unique solution of equation (5.3.1) with the initial data

$$y^{(k)}\left(a\right) = 0, \;\; 0 \le k \le n-1.$$

Or, equivalently,

$$y = {}_aD_x^{-n}f\left(x\right) = \frac{1}{\Gamma\left(n\right)}\int_a^x \left(x-t\right)^{n-1} f\left(t\right)dt. \tag{5.3.3}$$

Replacing $n$ by $\alpha$, where $Re\,\alpha > 0$ in the above formula, we obtain the *Riemann-Liouville definition of fractional integral* that was reported by Liouville in 1832 and by Riemann in 1876 as

$$ {}_aD_x^{-\alpha} f(x) = {}_aJ_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \qquad (5.3.4)$$

where ${}_aD_x^{-\alpha} = {}_aJ_x^{\alpha}$ is the Riemann-Liouville integral operator. When $a = 0$, (5.3.4) is the Riemann definition of fractional integral, and if $a = -\infty$, (5.3.4) represents the Liouville definition. Integrals of this type were found to arise in the theory of linear ordinary differential equations where they are known as Euler transforms of the first kind.

If $a = 0$ and $x > 0$, then the Laplace transform solution of the initial value problem

$$D^n y(x) = f(x), \quad x > 0, \quad y^{(k)}(0) = 0, \quad 0 \le k \le n-1,$$

is

$$\overline{y}(s) = s^{-n}\,\overline{f}(s),$$

where $\overline{y}(s)$ is the Laplace transform of $y(x)$ defined by (3.2.5).

The inverse Laplace transform gives the solution of the initial value problem

$$y(x) = {}_0D_x^{-n} f(x) = \mathscr{L}^{-1}\left\{s^{-n}\,\overline{f}(s)\right\} = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t)\, dt.$$

This is the Riemann-Liouville integral formula for an integer $n$. Replacing $n$ by real $\alpha$ gives the Riemann-Liouville fractional integral (5.3.4) with $a = 0$.

We consider a definite integral in the form

$$f_n(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt \qquad (5.3.5)$$

with $f_0(x) = f(x)$ so that

$$ {}_aD_x f_n(x) = \frac{1}{(n-2)!} \int_a^x (x-t)^{n-2} f(t)\, dt = f_{n-1}(x), \qquad (5.3.6)$$

and hence,

$$\begin{aligned} f_n(x) &= \int_a^x f_{n-1}(t)\, dt = {}_aJ_x f_{n-1}(x) = {}_aJ_x^2 f_{n-2}(x) \\ &= \ldots = {}_aJ_x^n f(x). \end{aligned} \qquad (5.3.7)$$

Thus, for a positive integer $n$, it follows that

$$ {}_aJ_x^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt. \qquad (5.3.8)$$

Replacing $n$ by $\alpha$ where $Re\,\alpha > 0$ in (5.3.8) leads to the definition of the Riemann-Liouville fractional integral

$$_aJ_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt. \tag{5.3.9}$$

Thus, this fractional-order integral formula is a natural extension of an iterated integral. The fractional integral formula (5.3.9) can also be obtained from the Euler integral formula

$$\int_0^x (x-t)^r\, t^s dt = \frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)} x^{r+s+1}, \qquad r, s > -1. \tag{5.3.10}$$

Replacing $r$ by $n-1$ and $s$ by $n$ gives

$$\int_0^x (x-t)^{n-1}\, t^n dt = \frac{\Gamma(n)}{(n+1)\ldots(2n)} x^{2n} = \Gamma(n)\; {_0D_x^{-n}} x^n. \tag{5.3.11}$$

Consequently, (5.3.8) follows from (5.3.11) when $f(t) = t^n$ and $a = 0$. In general, (5.3.11) gives (5.3.8) replacing $t^n$ by $f(t)$. Hence, when $n$ is replaced by $\alpha$, we derive (5.3.9).

It may be interesting to point out that Euler's integral expression for the $_2F_1(a, b, c; x)$ hypergeometric series can now be expressed as a fractional integral of order $(c-b)$ as

$$\begin{aligned} {_2F_1}(a, b, c; x) &= \frac{\Gamma(c)\, x^{1-c}}{\Gamma(b)\,\Gamma(c-b)} \int_0^x t^{b-1} (x-t)^{c-b-1} (1-t)^{-a}\, dt \\ &= \frac{\Gamma(c)}{\Gamma(b)}\, x^{1-c} \quad {_0J_x^{c-b}} f(x), \end{aligned} \tag{5.3.12}$$

where $f(t) = t^{b-1}(1-t)^{-a}$.

In complex analysis, the Cauchy integral formula for the $n$th derivative of an analytic function $f(z)$ is given by

$$D^n f(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)\, dt}{(t-z)^{n+1}}, \tag{5.3.13}$$

where $C$ is a closed contour on which $f(z)$ is analytic, and $t = z$ is any point inside $C$, and $t = z$ is a pole.

If $n$ is replaced by an arbitrary number $\alpha$ and $n!$ by $\Gamma(\alpha+1)$, then a derivative of arbitrary order $\alpha$ can be defined by

$$D^{\alpha} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_C \frac{f(t)\, dt}{(t-z)^{\alpha+1}}, \tag{5.3.14}$$

where $t = z$ is no longer a pole but a branch point. In (5.3.14) $C$ is no longer an appropriate contour, and it is necessary to make a branch cut along the real axis from the point $z = x > 0$ to negative infinity. Thus, we can define a derivative of arbitrary order $\alpha$ by a loop integral

$$ {}_aD_x^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_a^x (t-z)^{-\alpha-1} f(t)\, dt, \qquad (5.3.15) $$

where $(t-z)^{-\alpha-1} = \exp[-(\alpha+1)\, ln\,(t-z)]$ and $ln\,(t-z)$ is real when $t-z>0$. Using the classical method of contour integration along the branch cut contour $D$, it can be shown that

$$ \begin{aligned} {}_0D_z^\alpha f(z) &= \frac{\Gamma(\alpha+1)}{2\pi i} \int_D (t-z)^{-\alpha-1} f(t)\, dt \\ &= \frac{\Gamma(\alpha+1)}{2\pi i} \left[1-\exp\{-2\pi i(\alpha+1)\}\right] \int_0^z (t-z)^{-\alpha-1} f(t)\, dt \\ &= \frac{1}{\Gamma(-\alpha)} \int_0^z (t-z)^{-\alpha-1} f(t)\, dt \qquad (5.3.16) \end{aligned} $$

which agrees with the Riemann-Liouville definition (5.3.4) with $z = x$, and $a = 0$ when $\alpha$ is replaced $-\alpha$.

On the other hand, the *Weyl fractional integral* of order $\alpha$ was introduced by Weyl (1917) by

$$ {}_xW_\infty^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\, dt, \qquad Re\,\alpha > 0. \quad (5.3.17) $$

The major difference between this definition and the Riemann-Liouville definition are the limits of integration with the kernel here being $(t-x)^{\alpha-1}$.

If $D^n y = f(x)$ is the $n$th order nonhomogeneous differential equations and its adjoint equation is $(-1)^n D^n y = f(x)$ whose solution with the initial conditions $D^k y(c) = 0$, $0 \le k \le n-1$ is given by

$$ y(x) = {}_xW_c^{-n} f(x) = \frac{1}{\Gamma(n)} \int_x^c (t-x)^{n-1} f(t)\, dt. \qquad (5.3.18) $$

Replacing $n$ by $\alpha$ so that $Re\,\alpha > 0$, and $c = \infty$, we can define the Weyl adjoint fractional integral by (5.3.18). For a class of good functions $G$ consisting of functions $f$ which are everywhere differentiable any number of times and all of its derivatives are $0\left(x^{-N}\right)$ as $x \to \infty$ for all $N$ (see Lighthill, 1958), ${}_xW_\infty^{-\alpha} f(x)$ defined by (5.3.17) exists. Putting $t - x = \xi$ in (5.3.17) gives

$$ {}_xW_\infty^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \xi^{\alpha-1} f(\xi+x)\, d\xi. \qquad (5.3.19) $$

Application of the operator $D^n$ to both sides of (5.3.19), dropping the subscripts $x$ and $\infty$ in the Weyl operator, gives

$$ D^n W^{-\alpha} f(x) = W^{-\alpha} D^n f(x). \qquad (5.3.20) $$

Similarly, we can prove

$$E^n W^{-\alpha} f(x) = W^{-\alpha} E^n f(x), \tag{5.3.21}$$

where $E^n = (-1)^n D^n$.

If $f \in G$, an $n$-fold integration of (5.3.17) by parts gives

$$W^{-\alpha} f(x) = W^{-(\alpha+n)} \left[E^n f(x)\right] \tag{5.3.22}$$

$$= E^n \left[W^{-(\alpha+n)} f(x)\right] \qquad \text{by } (5.3.21). \tag{5.3.23}$$

In order to define the Weyl fractional derivatives, we assume that $\nu > 0$ and $n$ is the smallest integer greater than $\nu$ so that $\beta = n - \nu > 0$. If, for any function $f$, $W^{-\beta} f$ exists and has continuous derivatives. We then define the *Weyl fractional derivative* of $f$ of order $\nu$ by

$$W^{\nu} f(x) = W^{-(\beta-n)} f(x) = E^n \left[W^{-(\beta-n+n)} f(x)\right], \text{by } (5.3.23)$$

$$= E^n \left[W^{-\beta} f(x)\right] = E^n \left[W^{-(n-\nu)} f(x)\right]. \tag{5.3.24}$$

$$= E^n \left[\frac{1}{\Gamma(n-\nu)} \int_x^{\infty} (t-x)^{n-\nu-1} f(t)\, dt\right]. \tag{5.3.25}$$

It may be relevant to mention that Liouville's classical problem of potential theory can be described by an integral equation involving the Weyl fractional integral as

$$(\pi x)^{\frac{1}{2}} W^{-\frac{1}{2}} f(x) = W^{-1} f(x), \tag{5.3.26}$$

where the force field $\phi(r) = r f\left(r^2\right)$, $r = \sqrt{x}$. Using a series expansion $f(x)$, Liouville (1832) obtained the desired law of force as

$$\phi(r) = \frac{a}{r^2}, \tag{5.3.27}$$

where $a$ is any constant.

Grünwald (1867) introduced the idea of fractional derivative as the limit of a sum given by

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{n} (-1)^r \frac{\Gamma(\alpha+1) f(x-rh)}{\Gamma(r+1)\Gamma(\alpha-r+1)} \tag{5.3.28}$$

provided the limit exists. Using the identity

$$(-1)^r \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1)} = \frac{\Gamma(r-\alpha)}{\Gamma(-\alpha)}, \tag{5.3.29}$$

the result (5.3.28) becomes

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{h^{-\alpha}}{\Gamma(-\alpha)} \sum_{r=0}^{n} \frac{\Gamma(r-\alpha)}{\Gamma(r+1)} f(x-rh). \tag{5.3.30}$$

When $\alpha$ is equal to an integer $m$, definition (5.3.28) reduces to the derivative of integral order $m$ as

$$D^m f(x) = \lim_{h\to 0} \frac{1}{h^m} \sum_{r=0}^{n} (-1)^r \binom{m}{r} f(x - rh), \tag{5.3.31}$$

where $\binom{m}{r}$ is the usual binomial coefficient.

Result (5.3.31) follows from the classical definitions of $f'(x)$, $f''(x)$, ... as

$$Df(x) = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h} = \lim_{h\to 0} \frac{\Delta f(x)}{h}, \tag{5.3.32}$$

$$D^2 f(x) = \lim_{h\to 0} \frac{f'(x) - f'(x-h)}{h} = \lim_{h\to 0} \frac{\Delta^2 f(x)}{h^2}, \tag{5.3.33}$$

where

$$\Delta f(x - rh) = f(x - rh) - f(x - (r+1)h). \tag{5.3.34}$$

On the other hand, Marchaud (1927) formulated the fractional derivative of arbitrary order $\alpha$ in the form

$$D^\alpha f(x) = \frac{f(x)}{\Gamma(1-\alpha)\, x^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^x \frac{f(x) - f(t)}{(x-t)^{\alpha+1}} dt, \tag{5.3.35}$$

where $0 < \alpha < 1$. It has been shown by Samko et al. (1987) that (5.3.35) and (5.3.28) are equivalent.

Replacing $m$ by $-m$ in (5.3.31), it can be shown inductively that

$$\begin{aligned} {}_0D_x^{-m} f(x) &= \lim_{h\to 0} h^m \sum_{r=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix} f(t - rh) \\ &= \frac{1}{\Gamma(m)} \int_0^x (x-t)^{m-1} f(t)\, dt, \end{aligned} \tag{5.3.36}$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{m(m+1)\ldots(m+r+1)}{r!}. \tag{5.3.37}$$

It is important to point out that Hargreave (1848) extended the Leibniz product rule for the $n$th derivative to the fractional order $n = \alpha$ in the form

$$D^\alpha [f(x)\, g(x)] = \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+1)}{r!\,\Gamma(\alpha - r + 1)} D^{\alpha - r} f(x)\, D^r g(x), \tag{5.3.38}$$

provided the series converges, where $D^r$ is the differential operator of integral order $r$ and $D^{\alpha-r}$ is a fractional operator.

In a series of papers, Osler (1970, 1971, 1972) and others thoroughly studied the Leibniz product rule of derivatives of arbitrary order $\alpha$ and proved a general result

$$D^{\alpha}\left[f(x)\,g(x)\right] = \sum_{r=-\infty}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma-r+1)\,\Gamma(\gamma+r+1)} \times D^{\alpha-\gamma-r} f(x)\, D^{r+\gamma} g(x), \tag{5.3.39}$$

where $\gamma$ is arbitrary. When $\gamma=0$, Osler's result (5.3.39) reduces to (5.3.38). Osler also proved a generalization of the Leibniz product rule (5.3.38) in the integral form

$$D^{\alpha}\left[f(x)\,g(x)\right] = \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma-r+1)\,\Gamma(\gamma+r+1)} \times D^{\alpha-\gamma-r} f(r)\; D^{\gamma+r} g(r)\; dr. \tag{5.3.40}$$

This is a very useful formula for evaluating many definite integrals including a generalized version of Parseval's formula in Fourier analysis.

Using the Cauchy integral formula for fractional derivatives, Nishimoto (1991) gave a new proof of the Leibniz product formula $D_z^{\alpha}\left[f(z)\,g(z)\right]$ for analytic functions $f(z)$ and $g(z)$. Watanabe (1931) also derived the Leibniz product rule by using formula (5.3.38). It may be important to point out Nishimoto's (1991) formula for the fractional derivatives and integrals of logarithm function

$$D^{\alpha}(\log az) = -e^{-i\pi\alpha}\Gamma(\alpha)\,z^{-\alpha}, \quad D^{-\alpha}\left(z^{-\alpha}\right) = -\frac{e^{-i\pi\alpha}}{\Gamma(\alpha)}\log z, \tag{5.3.41}$$

where $a \neq 0$, $|\arg a| < \frac{\pi}{2}$, $z$ and $\alpha$ are complex numbers.

In addition to definition (5.3.15), it is revelant to mention Nishimoto's definition (1991) and properties of fractional calculus of functions of a single complex variable and several complex variables. On the other hand, in their paper, Hardy and Littlewood (1928, 1932) proved the formula

$$D^{\alpha} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_C \frac{f(t)}{(t-z)^{\alpha+1}}\,dt, \tag{5.3.42}$$

where $|z|<1$ and $C$ is the Hardy-Littlewood loop from $t=0$ round $t=z$ in a positive sense.

The Riemann-Liouville integral operator ${}_aD_x^{\alpha}$ defined by (5.3.9) satisfies the following properties:

$${}_0D_x^0 f(x) = If(x) = f(x) \qquad \text{(Identity)}, \tag{5.3.43}$$

$${}_aD_x^{\alpha}\left[cf(x) + dg(x)\right] = c\,{}_aD_x^{\alpha} f(x) + d\,{}_aD_x^{\alpha} g(x) \qquad \text{(Linearity)}, \tag{5.3.44}$$

where $c$ and $d$ are arbitrary constants.

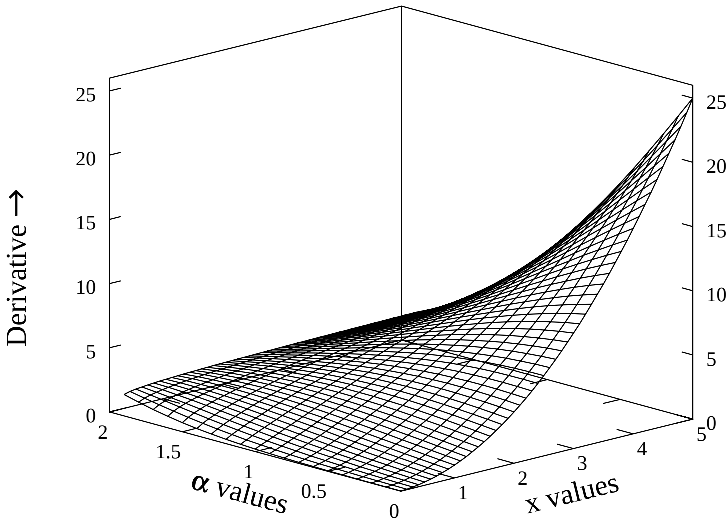


**Figure 5.1** Fractional derivative of $D^{\alpha}x^{2}$ for $0 \leq \alpha \leq 2$.

Computational results of fractional derivative of $x^2$ for $0 \leq \alpha \leq 2$ are shown in the Figure 5.1.

Computational results of fractional derivative of $\sin x$ and $\cos x$ are shown in Figure 5.2 and Figure 5.3, respectively (Bhatta 2006).

## 5.4 Applications of Fractional Calculus

It may be important to point out that the first application of fractional calculus was made by Abel (1802-1829) in the solution of an integral equation that arises in the formulation of the *tautochronous problem*. This problem deals with the determination of the shape of a frictionless plane curve through the origin in a vertical plane along which a particle of mass $m$ can fall in a time that is independent of the starting position. If the sliding time is constant $T$, then the Abel integral equation (1823) is

$$\sqrt{2g}\,T = \int_0^{\eta} (\eta - y)^{-\frac{1}{2}} f'(y)\,dy, \tag{5.4.1}$$

where $g$ is the acceleration due to gravity, $(\xi, \eta)$ is the initial position and $s = f(y)$ is the equation of the sliding curve. It turns out that (5.4.1) is equivalent

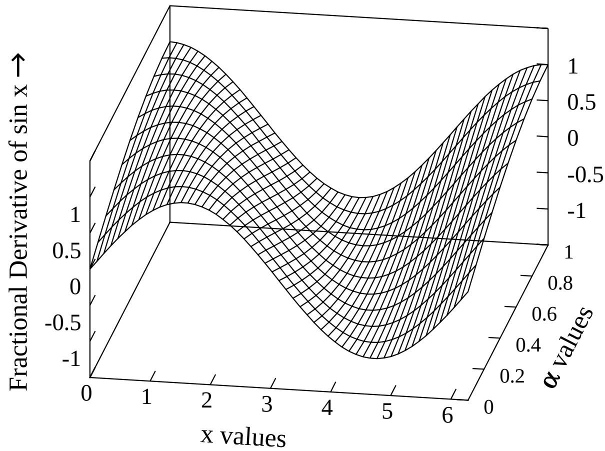


**Figure 5.2** Fractional derivative of $D^{\alpha}\sin x$ for $0 \leq \alpha \leq 1$.

to the fractional integral equation

$$T\sqrt{2g} = \Gamma\left(\frac{1}{2}\right)\,{}_0D_{\eta}^{-\frac{1}{2}} f'(\eta)\,. \tag{5.4.2}$$

Or, equivalently,

$$f'(\eta) = T\sqrt{\frac{2g}{\pi}}\,{}_0D_{\eta}^{\frac{1}{2}} 1 = \sqrt{\frac{2a}{\eta}}, \tag{5.4.3}$$

where $a = \left(\frac{gT^2}{\pi^2}\right)$. Finally, the solution is

$$f(\eta) = \sqrt{8a\eta} = 4a\sin\psi, \tag{5.4.4}$$

where $\frac{d\eta}{ds} = \sin\psi$. This curve is the cycloid with the vertex at the origin and the tangent at the vertex as the x-axis. The solution of the Abel problem is based on the fact that the derivative of a constant is not always equal to zero.

During the last decades of the nineteenth century, Heaviside successfully developed his operational calculus without rigorous mathematical arguments. In 1892 he introduced the idea of fractional derivatives in his study of electric transmission lines. Based on the symbolic operator form solution of heat equation due to Gregory (1846), Heaviside introduced the letter $p$ for the differential operator $\frac{d}{dt}$ and gave the solution of the diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = a^2 p \tag{5.4.5}$$

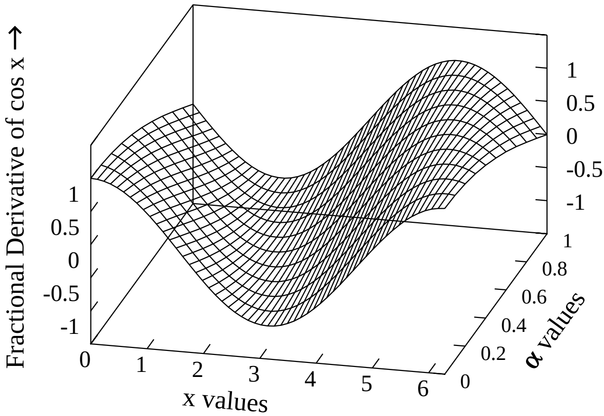


**Figure 5.3** Fractional derivative of $D^{\alpha}\cos x$ for $0 \leq \alpha \leq 1$.

for the temperature distribution $u\,(x,t)$ in the symbolic form

$$u\,(x,t) = A\exp\,(ax\sqrt{p}) + B\exp\,(-ax\sqrt{p})\,, \tag{5.4.6}$$

in which $p \equiv \frac{d}{dx}$ was treated as constant, where $a$, $A$, and $B$ are also constants. Indeed, Heaviside gave an interpretation of $\sqrt{p} = D^{\frac{1}{2}}$ so that ${}_0D_t^{\frac{1}{2}}1 = \frac{1}{\sqrt{\pi t}}$, which is in complete agreement with (5.2.10) The development of the Heaviside operational calculus was somewhat similar to that of calculus. Both Newton and Leibniz who discovered calculus did not provide a rigorous formulation of it. The rigorous theory had been developed in the nineteenth century, even though it is the transition of the non-rigorous development of the calculus that is still admired. It is well known that twentieth-century mathematicians have provided a rigorous foundation of the Heaviside operational calculus. In his book, Davis (1936) described the theory of linear operators with fractional calculus and its applications. He also states "The period of the formal development of operational methods may be regarded as having ended by 1900. The theory of integral equations was just beginning to stir the imagination of mathematicians and to reveal the possibilities of operational methods."

During the second half of the twentieth century, considerable amount of research in fractional calculus was published in engineering literature. Indeed, recent advances of fractional calculus are dominated by modern examples of applications in differential and integral equations, physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology and electrochemistry. There is no doubt that fractional calculus has become an exciting new mathematical method of solution of diverse problems in mathematics, science, and engineering. In a recent article by Debnath (2003), he presented numerous new and recent applications of fractional calculus in mathematics, science, and engineering. For more details, the reader is referred to the paper by Debnath (2003).

## 5.5 Exercises

1. Show that

(a) $\displaystyle\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx = \frac{\sqrt{\pi}}{3}$ (b) $\displaystyle\int_0^\infty x^4\, e^{-x^3}\, dx = \frac{1}{3}\,\Gamma\left(\frac{5}{3}\right)$

2. If $f(x) = \sqrt{x}$, show that

(a) $\displaystyle\frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{2}$ (b) $\displaystyle\frac{d^{-\frac{1}{2}} f}{dx^{-\frac{1}{2}}} = \frac{\sqrt{\pi}x}{2}$

3. If $f(x) = x$, show that

(a) $\displaystyle\frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} = 2\sqrt{\frac{x}{\pi}}$ (b) $\displaystyle\frac{d^{-\frac{1}{2}} f}{dx^{-\frac{1}{2}}} = \frac{4x^{3/2}}{3\sqrt{\pi}}$

4. If $f(x) = x^{3/2}$, show that

(a) $\displaystyle\frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} = \frac{3}{4}\sqrt{\pi}x$ (b) $\displaystyle\frac{d^{-\frac{1}{2}} f}{dx^{-\frac{1}{2}}} = \frac{3}{8}\sqrt{\pi}x^2$

5. If $f(x) = x^2$, show that

(a) $\displaystyle\frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} = \frac{8x^{3/2}}{3\sqrt{\pi}}$ (b) $\displaystyle\frac{d^{-\frac{1}{2}} f}{dx^{-\frac{1}{2}}} = \frac{16x^{5/2}}{15\sqrt{\pi}}$

6. If $f(x) = \sin(\sqrt{x})$, show that

(a) $\displaystyle\frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{2}\, J_0(\sqrt{x})$ (b) $\displaystyle\frac{d^{-\frac{1}{2}} f}{dx^{-\frac{1}{2}}} = \sqrt{\pi x}\, J_1(\sqrt{x})$

7. If $f(x) = \sinh(\sqrt{x})$, show that

(a) $\displaystyle\frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{2}\, I_0(\sqrt{x})$ (b) $\displaystyle\frac{d^{-\frac{1}{2}} f}{dx^{-\frac{1}{2}}} = \sqrt{\pi x}\, I_1(\sqrt{x})$

8. If $f(x) = J_0(\sqrt{x})$, show that

(a) $\displaystyle\frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} = \frac{\cos(\sqrt{x})}{\sqrt{\pi x}}$ (b) $\displaystyle\frac{d^{-\frac{1}{2}} f}{dx^{-\frac{1}{2}}} = \frac{2\sin(\sqrt{x})}{\sqrt{\pi}}$

# 6

# *Applications of Integral Transforms to Fractional Differential and Integral Equations*

> "In every mathematical investigation, the question will arise whether we can apply our mathematical results to the real world."
>
> V. I. Arnold

> "All of Abel's works carry the imprint of an ingenuity and force of thought which is unusual and sometimes amazing, even if the youth of the author is not taken into consideration. One may say that he was able to penetrate all obstacles down to the very foundations of the problems, with a force which appeared irresistible; he attacked the problems with extraordinary energy; he regarded them from above and was able to soar so high over their present state that all difficulties seemed to vanish under the victorious onslaught of his genius.... But it was not only his great talent which created the respect for Abel and made his loss infinitely regrettable. He distinguished himself equally by the purity and nobility of his character and by a rare modesty which made his person cherished to the same unusual degree as was his genius."
>
> August Leopold Crelle

## 6.1 Introduction

In the proceeding chapter, the basic ideas of fractional calculus and its applications have been presented. This chapter is essentially devoted to applications of Laplace, Fourier and Hankel transforms to fractional integral equations, fractional ordinary and partial differential equations. Many examples of applications are presented in some detail. Included are also Green's functions of fractional differential equations. Most of the applications involving the partial differential equations are based on authors' recent papers, which are listed in the Bibliography.

## 6.2 Laplace Transforms of Fractional Integrals and Fractional Derivatives

The *Riemann-Liouville fractional integral* is usually defined by

$$D^{-\alpha} f(t) = {}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)dx, \quad \operatorname{Re}\alpha > 0. \tag{6.2.1}$$

Clearly, $D^{-\alpha}$ is a linear integral operator.

A simple change of variable $(t-x)^\alpha = u$ in (6.2.1) allows us to prove the following result

$$D[D^{-\alpha} f(t)] = D^{-\alpha}[Df(t)] + f(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}. \tag{6.2.2}$$

Clearly, the integral in (6.2.1) is a convolution, and hence, the Laplace transform of (6.2.1) gives

$$\mathscr{L}\{D^{-\alpha} f(t)\} = \mathscr{L}\{f(t)*g(t)\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}, \tag{6.2.3}$$

$$= s^{-\alpha}\bar{f}(s), \quad \alpha > 0. \tag{6.2.4}$$

where $g(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $\bar{g}(s) = s^{-\alpha}$.

The result (6.2.4) is also valid for $\alpha = 0$, and

$$\lim_{\alpha\to 0} \mathscr{L}\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\} = \lim_{\alpha\to 0} s^{-\alpha} = 1. \tag{6.2.5}$$

Using (6.2.4), it can readily be verified that the fractional integral operator satisfies the laws of exponents

$$D^{-\alpha}[D^{-\beta} f(t)] = D^{-(\beta+\alpha)} f(t) = D^{-\beta}[D^{-\alpha} f(t)]. \tag{6.2.6}$$

Formula (6.2.4) can be used for evaluating the fractional integral of a given function using the inverse Laplace transform. The following examples illustrate this point.

$$\mathscr{L}\{D^{-\alpha} t^\beta\} = \frac{\Gamma(\beta+1)}{s^{\alpha+\beta+1}}, \quad \beta > -1. \tag{6.2.7}$$

Or, equivalently,

$$D^{-\alpha} t^\beta = \mathscr{L}^{-1}\left\{\frac{\Gamma(\beta+1)}{s^{\alpha+\beta+1}}\right\} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \tag{6.2.8}$$

In particular, if $\alpha = \frac{1}{2}$ and $\beta(=n)$ is an integer, then (6.2.8) gives

$$D^{-1/2} t^n = \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}+1\right)} \cdot t^{n+\frac{1}{2}}, \qquad n > -1. \tag{6.2.9}$$

It also follows from (6.2.4) that

$$\mathscr{L}\left\{D^{-\alpha} e^{at}\right\} = \frac{1}{s^\alpha (s-a)}, \; a > 0. \tag{6.2.10}$$

Or,

$$D^{-\alpha} e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha (s-a)}\right\} \tag{6.2.11}$$

$$= \mathscr{L}^{-1}\left\{\frac{1}{s^{\alpha+1}}\left(1 + \frac{a}{s-a}\right)\right\}$$

$$= \frac{t^\alpha}{\Gamma(\alpha+1)} + aE(t, \alpha+1, a), \tag{6.2.12}$$

where $E(t, \alpha, a)$ is defined by

$$E(t, \alpha, a) = \frac{1}{\Gamma(\alpha)} \int_0^t \xi^{\alpha-1} \exp\{a(t-\xi)\} d\xi. \tag{6.2.13}$$

In particular, if $\alpha = \frac{1}{2}$ then

$$D^{-1/2} e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-a)}\right\} = \frac{1}{\sqrt{\pi t}} * e^{at},$$

which is, by Example 3.7.8,

$$= \frac{e^{at}}{\sqrt{a}} erf(\sqrt{at}). \tag{6.2.14}$$

The following results follow readily from (6.2.4):

$$\mathscr{L}\left\{D^{-\alpha} \sin at\right\} = \frac{a}{s^\alpha (s^2 + a^2)}, \;\; \alpha > 0. \tag{6.2.15}$$

$$\mathscr{L}\left\{D^{-\alpha} \cos at\right\} = \frac{s}{s^\alpha (s^2 + a^2)}, \;\; \alpha > 0. \tag{6.2.16}$$

$$\mathscr{L}\left\{D^{-\alpha} e^{at} t^{\beta-1}\right\} = \frac{\Gamma(\beta)}{s^\alpha (s-a)^\beta}, \;\; \alpha > 0, \, \beta > 0. \tag{6.2.17}$$

The inverse Laplace transforms of these results combined with the Convolution Theorem lead to fractional integrals of functions involved.

One of the consequences of (6.2.1) is that

$$\lim_{\alpha\to 0} D^{-\alpha} f(t) = f(t). \tag{6.2.18}$$

This follows from the inverse Laplace transform of (6.2.4) combined with the limit as $\alpha \to 0$.

We now evaluate the Laplace transform of the fractional integral of the derivative and then the Laplace transform of the derivative of the integral. In view of (6.2.4), it follows that

$$\begin{aligned}\mathscr{L}\{D^{-\alpha}[D\,f(t)]\} &= s^{-\alpha}\mathscr{L}\{D\,f(t)\}\\ &= s^{-\alpha}[s\bar{f}(s) - f(0)], \quad \alpha > 0.\end{aligned} \tag{6.2.19}$$

Although this result is proved for $\alpha > 0$, it is valid even if $\alpha = 0$.

On the other hand, the Laplace transform of (6.2.2) gives

$$\begin{aligned}\mathscr{L}\{D[D^{-\alpha} f(t)]\} &= \mathscr{L}\{D^{-\alpha} D[f(t)]\} + f(0)\mathscr{L}\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}\\ &= s^{-\alpha}[s\bar{f}(s) - f(0)] + s^{-\alpha} f(0)\\ &= s^{1-\alpha}\bar{f}(s), \quad \alpha \geq 0.\end{aligned} \tag{6.2.20}$$

Obviously, if $\alpha = 0$, this result does not agree with that obtained from (6.2.19) as $\alpha \to 0$. This disagreement is due to the fact that "$\mathscr{L}$" and "lim" do not commute, as is seen from (6.2.5).

Another consequence of (6.2.1) is that the *fractional derivative* $D^{\alpha} f(t)$ can be defined as the solution $\phi(t)$ of the integral equation

$$D^{-\alpha}\phi(t) = f(t). \tag{6.2.21}$$

The Laplace transform of this result gives the solution for $\bar{\phi}(s)$ as

$$\bar{\phi}(s) = s^{\alpha}\bar{f}(s). \tag{6.2.22}$$

Inversion gives the fractional derivative of $f(t)$ as

$$\phi(t) = D^{\alpha} f(t) = \mathscr{L}^{-1}\{s^{\alpha}\bar{f}(s)\}, \tag{6.2.23}$$

leading to the result

$$D^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)}\int_0^t (t-x)^{-\alpha-1} f(x)\,dx, \quad \alpha > 0. \tag{6.2.24}$$

This is the *Cauchy integral formula*, which is often used to define the fractional derivative. However, formula (6.2.23) can be used for finding the fractional derivatives. If $f(t) = t^{\beta}$, it is seen from (6.2.23) that

$$D^{\alpha} t^{\beta} = \mathscr{L}^{-1}\left\{\frac{\Gamma(\beta+1)}{s^{\beta-\alpha+1}}\right\} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \tag{6.2.25}$$

In particular, if $\alpha = \frac{1}{2}$ and $\beta(=n)$ is an integer,

$$\begin{aligned} D^{1/2} t^n &= \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} t^{n-\frac{1}{2}}, \quad n > -1. \\ D^{1/2} e^{at} &= \mathscr{L}^{-1}\left\{\frac{\sqrt{s}}{s-a}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}} + \frac{a}{\sqrt{s}(s-a)}\right\}, \end{aligned} \tag{6.2.26}$$

which is, by (3.2.21) and (3.7.3),

$$= \frac{1}{\sqrt{\pi t}} + \sqrt{a}\exp(at)\, erf(\sqrt{at}). \tag{6.2.27}$$

***Example 6.2.1***
Show that

$$D^{-\alpha} J_0(a\sqrt{t}) = \left(\frac{2}{a}\right)^{\alpha} t^{\alpha/2} J_\alpha(a\sqrt{t}). \tag{6.2.28}$$

We apply the Laplace transform to the left-hand side of (6.2.28) and use (6.2.22) to obtain

$$\begin{aligned} \mathscr{L}\left\{D^{-\alpha} J_0(a\sqrt{t})\right\} &= s^{-\alpha}\mathscr{L}\left\{J_0(a\sqrt{t})\right\} \\ &= s^{-(1+\alpha)} \exp\left(-\frac{a^2}{4s}\right). \end{aligned}$$

The inverse Laplace transform gives

$$\begin{aligned} D^{-\alpha} J_0(a\sqrt{t}) &= \mathscr{L}^{-1}\left\{s^{-(1+\alpha)} \exp\left(-\frac{a^2}{4s}\right)\right\} \\ &= \left(\frac{2}{a}\right)^{\alpha} t^{\alpha/2} J_\alpha(a\sqrt{t}). \end{aligned}$$

$\square$

## 6.3 Fractional Ordinary Differential Equations

We first define a *fractional differential equation* with constant coefficients of order $(n, q)$ as

$$\left[D^{n\alpha} + a_{n-1} D^{(n-1)\alpha} + \cdots + a_0 D^0\right] x(t) = 0, \; t \ge 0, \tag{6.3.1}$$

where $\alpha = \dfrac{1}{q}$. If $q = 1$, then $\alpha = 1$ and this equation is simply an ordinary differential equation of order $n$. Symbolically, we write (6.3.1) as

$$f(D^\alpha)\, x(t) = 0, \tag{6.3.2}$$

where $f(D^\alpha)$ is a fractional differential operator.

We next use the Laplace transform method to solve a simple fractional differential equation with constant coefficients of order (2, 2) in the form

$$f\left(D^{\frac{1}{2}}\right)x(t)=\left(D^1+a_1D^{\frac{1}{2}}+a_0D^0\right)x(t)=0. \tag{6.3.3}$$

Application of the Laplace transform to this equation gives

$$[s\,\bar{x}(s)-x(0)]+a_1\left[\sqrt{s}\,\bar{x}(s)-D^{-\frac{1}{2}}\,x(0)\right]+a_0\,\bar{x}(s)=0.$$

Or,

$$\bar{x}(s)=\frac{x(0)+a_1\,D^{-\frac{1}{2}}x(0)}{(s+a_1\sqrt{s}+a_0)}=\frac{A}{f(\sqrt{s})}, \tag{6.3.4}$$

where $f(x)=x^2+a_1\,x+a_0$ is an associated *indicial equation* and $A$ is assumed to be a non-zero finite constant defined by

$$A=x(0)+a_1\,D^{-\frac{1}{2}}\{x(0)\}. \tag{6.3.5}$$

We next write the following partial fractions for the right hand side of (6.3.4) so that

$$\begin{aligned}\bar{x}(s) &= \frac{A}{a-b}\left(\frac{1}{\sqrt{s}-a}-\frac{1}{\sqrt{s}-b}\right)\\ &= \frac{A}{a-b}\left(\frac{\sqrt{s}}{s-a^2}+\frac{a}{s-a^2}-\frac{\sqrt{s}}{s-b^2}-\frac{b}{s-b^2}\right),\end{aligned} \tag{6.3.6}$$

where $a$ and $b$ are two distinct roots of $f(x)=0$.

Using the inverse Laplace transform formula (6.2.11) with $\alpha=-\dfrac{1}{2}$ and $\alpha=0$, we invert (6.3.6) to obtain the formal solution

$$\begin{aligned}x(t) = \frac{A}{a-b}\Bigg[&E\left(t,-\frac{1}{2},a^2\right)+aE(t,0,a^2)\\ &-E\left(t,-\frac{1}{2},b^2\right)-bE(t,0,b^2)\Bigg].\end{aligned} \tag{6.3.7}$$

For equal roots $(a=b)$ of $f(x)=0$, we find

$$\bar{x}(s)=\frac{A}{(\sqrt{s}-a)^2}=A\left[\frac{\sqrt{s}}{(s-a^2)}+\frac{a}{(s-a^2)}\right]^2. \tag{6.3.8}$$

In view of the result

$$\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha(s-a)^2}\right\}=t\,E(t,\alpha,a)-\alpha\,E(t,\alpha+1,a), \tag{6.3.9}$$

the inverse Laplace transform of (6.3.8) gives the solution as

$$x(t) = A\left[(1+2a^2t)E(t,0,a^2) + a\,E\left(t,\frac{1}{2},a^2\right) + 2at\,E\left(t,-\frac{1}{2},\,a^2\right)\right]. \tag{6.3.10}$$

Kempfle and Gaul (1996) developed the criteria for the existence, continuity and causality of global solutions of the fractional order linear ordinary differential equation of the form

$$L\left(D\right)x\left(t\right) = f\left(t\right), \quad t \in R \tag{6.3.11}$$

with given initial or boundary data, where $L\left(D\right)$ is a fractional differential operator

$$L\left(D\right) \equiv D^{\alpha_n} + a_{n-1}D^{\alpha_{n-1}} + ... + a_1 D^{\alpha_1} + a_0, \tag{6.3.12}$$

where $D \equiv \frac{d}{dt}$, $0 \le \alpha_1 \le ... \le \alpha_n$, $\alpha_k$ are non-integers, $a_{n-1}, a_{n-2}, ..., a_1, a_0$ are real constants and $f\left(t\right)$ is a given forcing function.

Application of the Fourier transform of $x\left(t\right)$ with respect to $t$ gives the physical solution in the form

$$\begin{aligned} x\left(t\right) &= \left\{L\left(D\right)\right\}^{-1} \cdot f\left(t\right) = G\left(t\right) * f\left(t\right) \\ &= \int_{-\infty}^{\infty} f\left(t-\tau\right) G\left(\tau\right) d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f\left(t-\tau\right) d\tau \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\tilde{p}\left(\omega\right)}\,d\omega, \end{aligned} \tag{6.3.13}$$

where $*$ denotes the Fourier convolution, $G\left(t\right)$ is the impulse response function given by

$$G\left(t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\tilde{p}\left(\omega\right)}\,d\omega, \quad \omega \in R, \tag{6.3.14}$$

and

$$\tilde{p}\left(\omega\right) = \left(i\omega\right)^{\alpha_n} + a_{n-1}\left(i\omega\right)^{\alpha_{n-1}} + ... + a_1\left(i\omega\right)^{\alpha_1} + a_0. \tag{6.3.15}$$

The solution (6.3.14) exists provided $\frac{1}{\tilde{p}(\omega)} \in L^2\left(\mathbb{R}\right)$ or $\tilde{p}\left(\omega\right)$ has no real zeros and $\deg L > \frac{1}{2}$.

According to the well-known stability criteria for a linear system $(i\omega \to s)$, equation (6.3.13) has the solution provided

$$\overline{q}\left(s\right) = \tilde{p}\left(-is\right) = s^{\alpha_n} + a_{n-1}s^{\alpha_{n-1}} + ... + a_1 s^{\alpha_1} + a_0 \tag{6.3.16}$$

has no zeros in the right half s-plane. For simplicity, if $\overline{q}\left(s\right)$ is restricted to the principal branch with its zeros at $s_k = -\sigma_k \underline{+}\, i\,\Omega_k$, integral (6.3.14) can be evaluated by using the Cauchy residue theory so that (6.3.14) yields

$$G(t) = \sum_{k=1}^{\infty} A_k e^{-\sigma_k t} \cos(\Omega_k t + \varepsilon_k) - \frac{1}{\pi} \int_0^{\infty} \frac{e^{-rt} dr}{q(r)}, \qquad (6.3.17)$$

where the first series solution represents phase-shifted oscillations and the second integral term is the relaxation function which becomes dominant as $t \to \infty$.

## 6.4 Fractional Integral Equations

(a) *Abel's integral equation of the first kind* is given by

$$\int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau = g(t), \qquad 0 < \alpha < 1, \qquad (6.4.1)$$

where $g(t)$ is given function. This equation can be expressed in terms of fractional integral

$$\Gamma(\alpha)\; {}_0J_t^{\alpha} f(t) = g(t). \qquad (6.4.2)$$

Application of the Laplace transform to (6.4.1) or (6.4.2) gives the solution

$$\overline{f}(s) = \frac{1}{\Gamma(\alpha)} s^{\alpha}\, \overline{g}(s) = \frac{1}{\Gamma(\alpha)} s \left[\frac{1}{s^{1-\alpha}} \cdot \overline{g}(s)\right], \qquad (6.4.3)$$

which leads to the solution of (6.4.1) in the form

$$f(t) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} g(\tau)\, d(\tau). \qquad (6.4.4)$$

(b) *Abel's integral equation of the second kind* is given by

$$f(t) + \frac{a}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau = g(t) \qquad \alpha > 0, \qquad (6.4.5)$$

where $a$ is real or complex parameter and $g(t)$ is a given function.

Application of the Laplace transform to (6.4.5) leads to the transform solution

$$\overline{f}(s) = \left(\frac{s^{\alpha}}{s^{\alpha}+a}\right) \overline{g}(s) = \left[s \cdot \frac{s^{\alpha-1}}{s^{\alpha}+a} \cdot \overline{g}(s)\right] \qquad (6.4.6)$$

whence the inverse Laplace gives the solution

$$f(t) = \frac{d}{dt} \int_0^t E_{\alpha,1}(-a\tau^{\alpha})\, g(t-\tau)\, d\tau, \qquad (6.4.7)$$

where the Mittag-Leffler function, $E_{\alpha,\beta}(z)$ is given by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \qquad \alpha,\beta>0. \tag{6.4.8}$$

(c) *Poisson's integral equation* is given by

$$\int_0^{\frac{\pi}{2}} \phi(r\cos\theta)\sin^{2\alpha+1}\theta\, d\theta = h(r). \tag{6.4.9}$$

Substituting $x = r\cos\theta$ into (6.4.9) gives

$$\int_0^r \left(1-\frac{x^2}{r^2}\right)^{\alpha} \phi(x)\, dx = r h(r),$$

which is, by replacing $\frac{1}{r}$ by $\sqrt{z}$, and $\frac{1}{\sqrt{z}}\, h\left(\frac{1}{\sqrt{z}}\right)$ by $\Psi(z)$,

$$\int_0^{\frac{1}{\sqrt{z}}} \left(\frac{1}{z}-x^2\right)^{\alpha} \phi(x)\, dx = z^{-\alpha}\Psi(z).$$

Invoking substitution of $x^2 = \tau$ and $\frac{1}{z} = t$ yields the Abel integral equation

$$\int_0^{\sqrt{t}} (t-\tau)^{\alpha} f(\tau)\, d\tau = g(t), \tag{6.4.10}$$

where

$$f(\tau) = \frac{\phi(\sqrt{\tau})}{\sqrt{\tau}} \text{ and } g(t) = 2\,t^{\alpha}\,\Psi\left(\frac{1}{t}\right).$$

Thus, the solution of (6.4.10) is

$$f(t) = \frac{1}{\Gamma(\alpha+1)}\, {}_0D_t^{\alpha}\, g(t) \tag{6.4.11}$$

so that the solution of the Poisson equation (6.4.9) is

$$\phi\left(\sqrt{t}\right) = \frac{2\sqrt{t}}{\Gamma(1+\alpha)}\, {}_0D_t^{\alpha}\, t^{\alpha+\frac{1}{2}} h\left(\sqrt{t}\right). \tag{6.4.12}$$

***Example 6.4.1***
Consider the *Abel integral equation*

$$g(t) = \int_0^t f'(t)(t-\tau)^{-\alpha}\, d\tau, \quad 0<\alpha<1. \tag{6.4.13}$$

Application of the Laplace transform gives

$$\bar{g}(s) = \mathscr{L}\{f'(t)\}\mathscr{L}\{t^{-a}\}.$$

Or,

$$\bar{f}(s) = \frac{f(0)}{s} + \frac{\bar{g}(s)}{\Gamma(1-\alpha)} s^{-a}.$$

Inverting, we find

$$f(t) = f(0) + \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t g(\tau)(t-\tau)^{\alpha-1}\, d\tau. \tag{6.4.14}$$

This is the required solution of Abel's equation. ▯

***Example 6.4.2***
Solve the Abel integral equation

$$g(t) = \int_0^t (t-x)^{-\alpha} f(x)dx, \quad 0 < \alpha < 1. \tag{6.4.15}$$

Clearly, it follows from (6.2.1) that

$$g(t) = \Gamma(1-\alpha) D^{\alpha-1} f(t).$$

Or,

$$D^{1-\alpha} g(t) = \Gamma(1-\alpha) f(t).$$

Hence,

$$\begin{aligned} f(t) &= \frac{1}{\Gamma(1-\alpha)} D \cdot D^{-\alpha} g(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{\Gamma(\alpha)} \cdot D \int_0^t (t-x)^{\alpha-1} g(x)dx \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \cdot \frac{d}{dt} \int_0^t (t-x)^{\alpha-1} g(x)dx. \end{aligned} \tag{6.4.16}$$

▯

***Example 6.4.3***
(*Abel's Problem of Tautochronous Motion*). The problem is to determine the form of a frictionless plane curve through the origin in a vertical plane along

which a particle of mass $m$ can fall in a time that does not depend on the initial position.

Suppose the particle is placed on a curve at the point $(\xi, \eta)$, where $\eta$ is measured positive upward. Let the particle be allowed to fall to the origin under the action of gravity. Suppose $(x, y)$ is any position of the particle during its descent as shown in Figure 6.1. According to the principle of conservation of energy, the sum of the kinetic and potential energies is constant, that is,

$$\frac{1}{2}mv^2 + mgy = mg\eta = \text{constant}. \tag{6.4.17}$$

This gives the velocity of the particle at any position $(x, y)$

$$v^2 = \left(\frac{ds}{dt}\right)^2 = 2g(\eta - y), \tag{6.4.18}$$

where $s$ is the length of the arc of the curve measured from the origin and $t$ is the time.

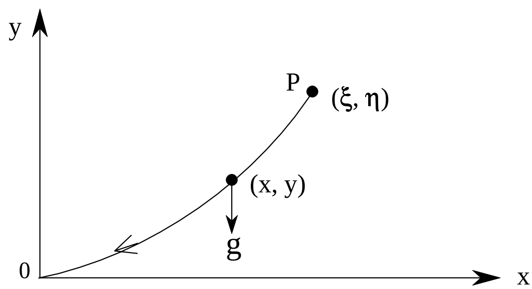


**Figure 6.1** Abel's problem.

Integrating (6.4.18) from $y = \eta$ to 0 gives

$$\int_0^{\tau} dt = -\frac{1}{\sqrt{2g}} \int_{\eta}^{0} \frac{ds}{\sqrt{\eta - y}} = \frac{1}{\sqrt{2g}} \int_0^{\eta} \frac{f'(y)\, dy}{\sqrt{\eta - y}},$$

where $s = f(y)$ represents the equation of the curve with $f(0) = 0$. Thus, we obtain

$$\sqrt{2g}\, T = \int_0^{\eta} \frac{f'(y)\, dy}{\sqrt{\eta - y}}. \tag{6.4.19}$$

This is the Abel integral equation (6.4.13) with $\alpha=\frac{1}{2}$ and $g(\eta)=T\sqrt{2g}=$ constant. Thus, the solution (6.4.14) becomes

$$f(\eta)=\frac{T\sqrt{2g}}{\pi}\int_0^{\eta}(\eta-y)^{\frac{1}{2}-1}\,dy,$$

which is, putting $y=\eta\sin^2\theta$ and $a=\dfrac{gT^2}{\pi^2}$,

$$f(\eta)=2\sqrt{2a\eta}\,. \tag{6.4.20}$$

If $\psi$ is the angle made by the tangent to the curve at a point $(x,y)$, then $\dfrac{dy}{ds}=\sin\psi$ and $\dfrac{dx}{ds}=\cos\psi$ so that

$$\operatorname{cosec}\psi=\frac{ds}{dy}=f'(y)=\sqrt{\frac{2a}{y}}.$$

Or,

$$s=f(y)=2\sqrt{2ay}=4a\sin\psi. \tag{6.4.21}$$

This is the equation of the curve and represents the *cycloid* with the vertex at the origin and the tangent at the vertex as the $x$-axis.

Alternatively, equation (6.4.19) can be expressed in terms of a half-order fractional derivative of one as given by (5.4.3) that was solved by using fractional derivatives with solution (5.4.4). The solution is the same as (6.4.21). ▯

***Example 6.4.4***

(*Abel's Equation and Fractional Derivatives in a Problem of Fluid Flow*). We consider the flow of water along the $x$ direction through a symmetric dam in a vertical $yz$ plane with the $y$-axis along the face of the dam. The problem is to determine the form $y=f(z)$ of the opening of the dam so that the quantity of water per unit time is proportional to a given power of the depth of the stream. It follows from Bernoulli's equation of fluid mechanics that the fluid velocity $v$ at a given height $z$ above the base of the dam is given by

$$v^2=2g(h-z). \tag{6.4.22}$$

The volume flux $dQ$ through an elementary cross section $dA$ of the opening of the dam is $dQ=vdA=2\sqrt{2g}(h-z)^{1/2}f(z)dz$ so that the total volume flux is

$$Q(h)=a\int_0^h(h-z)^{\frac{1}{2}}f(z)\,dz,\quad a=2\sqrt{2g}. \tag{6.4.23}$$

This is the *Abel integral equation* and hence can be written as the fractional integral

$$Q(h) = a\,\Gamma\left(\frac{3}{2}\right) D_h^{-3/2} f(h). \tag{6.4.24}$$

Multiplying this result by $D_h^{3/2}$ with a given $Q(h) = h^\beta$, we obtain, by (6.2.24),

$$f(h) = \frac{1}{a\,\Gamma\left(\frac{3}{2}\right)} D_h^{3/2}\, h^\beta = \frac{2a}{\sqrt{\pi}} \frac{\Gamma(\beta+1)}{\Gamma\left(\beta - \frac{1}{2}\right)} h^{\beta - 3/2}, \tag{6.4.25}$$

where $\beta > -1$. In particular, the shape $y = f(z)$ is either a parabola or a rectangle depending on whether, $\beta = \frac{7}{2}$ or $\frac{3}{2}$. There are also other shapes of the opening of the dam depending on the value of $\beta$. $\square$

## 6.5 Initial Value Problems for Fractional Differential Equations

(a) We consider the following fractional differential equation

$${}_0D_t^\alpha y(t) + \omega^2 y(t) = f(t), \quad t > 0 \tag{6.5.1}$$

with the initial conditions

$$\left[{}_0D_t^{\alpha-k} y(t)\right]_{t=0} = c_k, \quad k = 1, 2, ..., n, \tag{6.5.2}$$

where $n - 1 < \alpha < n$.

Application of the Laplace transform to (6.5.1)–(6.5.2) gives

$$\left(s^\alpha + \omega^2\right) \overline{y}(s) = \overline{f}(s) + \sum_{k=1}^{n} c_k s^{k-1}. \tag{6.5.3}$$

Thus, the Laplace transform solution is

$$\overline{y}(s) = \sum_{k=1}^{n} c_k \frac{s^{k-1}}{\left(s^\alpha + \omega^2\right)} + \frac{\overline{f}(s)}{s^\alpha + \omega^2}. \tag{6.5.4}$$

The inverse Laplace transform gives the solution of the initial value problem

$$y(t) = \sum_{k=1}^{n} c_k t^{\alpha-k} E_{\alpha,\alpha-k+1}\left(-\omega^2 t^\alpha\right) + \int_0^t f(t-\tau)\, \tau^{\alpha-1} E_{\alpha,\alpha}\left(-\omega^2 \tau^\alpha\right) d\tau, \tag{6.5.5}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler type function defined by the series

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \quad \alpha > 0, \quad \beta > 0, \tag{6.5.6}$$

and the inverse Laplace transform is

$$\mathscr{L}^{-1}\left\{\frac{m!\, s^{\alpha-\beta}}{(s^\alpha \mp a)^{m+1}}\right\} = t^{\alpha m+\beta-1} E_{\alpha,\beta}^{(m)}(\pm a t^\alpha), \tag{6.5.7}$$

with

$$E_{\alpha,\beta}^{(m)}(z) = \frac{d^m}{dz^m} E_{\alpha,\beta}(z). \tag{6.5.8}$$

(i) When $\alpha = 1$, $n = 1$ so that the initial condition is $y(0) = c_1$ and the solution (6.5.5) reduces to the form

$$y(t) = c_1 E_{1,1}\left(-\omega^2 t\right) + \int_0^t f(t-\tau) E_{1,1}\left(-\omega^2 \tau\right) d\tau, \tag{6.5.9}$$

where $E_{1,1}(z) = e^z$. Consequently, the solution assumes the standard form

$$y(t) = c_1 e^{-\omega^2 t} + \int_0^t f(t-\tau) e^{-\omega^2 \tau} d\tau. \tag{6.5.10}$$

(ii) When $\alpha = 2$, $n = 2$ so that the initial data are $y(0) = c_2$ and $y'(0) = c_1$. In this case, the solution (6.5.5) reduces to the form

$$\begin{aligned} y(t) &= c_1 t E_{2,2}\left(i^2\omega^2 t^2\right) + c_2 E_{2,1}\left(i^2\omega^2 t^2\right) \\ &\quad + \int_0^t f(t-\tau)\, \tau E_{2,2}\left(i^2\omega^2\tau^2\right) d\tau, \end{aligned} \tag{6.5.11}$$

where

$$\begin{aligned} E_{2,2}\left(i^2 z^2\right) &= \frac{\sinh(iz)}{iz} = \frac{\sin z}{iz}, \\ E_{2,1}\left(i^2 z^2\right) &= \cosh(iz) = \cos z. \end{aligned}$$

Consequently, the solution (6.5.11) reduces to the standard form

$$y(t) = \frac{c_1}{\omega} \sin \omega t + c_2 \cos \omega t + \frac{1}{\omega} \int_0^t f(t-\tau) \sin \omega \tau d\tau. \tag{6.5.12}$$

It is noted that equation (6.5.1) describes fractional relaxation when $0 < \alpha \leq 1$ and fractional oscillation when $1 < \alpha \leq 2$. It is easy to recognize the remarkable difference between the classical solutions for cases $\alpha = 1$ and $\alpha = 2$. On the other hand, the solution of the fractional equation (6.5.1) show remarkably different features. The classical solution corresponding to $\alpha = 1$ decays exponentially as $t \to \infty$, and the fractional solution $(0 < \alpha < 1)$ exhibits a faster decay as $t \to 0+$ and much slower decay (algebraic decay compared to exponential decay) as $t \to \infty$.

(b) *Fractional Simple Harmonic Oscillator*

The initial value problem for the fractional simple harmonic oscillator is given by

$$\frac{d^2y}{dt^2} + b\frac{d^\alpha y}{dt^\alpha} + \omega^2 y(t) = f(t), \quad t > 0, \quad 0 < \alpha < 2, \tag{6.5.13}$$

$$y(0) = c_0 \text{ and } y_0'(0) = c_1, \tag{6.5.14}$$

where $b$, $\omega$, $c_0$ and $c_1$ are constants and $\frac{d^\alpha y}{dt^\alpha}$ represents Caputo's fractional derivative (see Caputo, 1967).

Two special cases are of interest: (i) $0 < \alpha < 1$ and (ii) $1 < \alpha < 2$ and $\alpha = 1$.

Application of the Laplace transform gives the following transform solutions:

$$\text{(i)} \quad \overline{y}(s) = c_0 \overline{y}_0(s) + c_1 \overline{y}_\delta(s) + \overline{f}(s) . \overline{y}_\delta(s), \quad 0 < \alpha < 1 \tag{6.5.15}$$

$$\text{(ii)} \quad \overline{y}(s) = c_0 \overline{y}_0(s) + c_1 \frac{\overline{y}_0(s)}{s} + \overline{f}(s) . \overline{y}_\delta(s), \quad 1 < \alpha < 2 \tag{6.5.16}$$

where

$$\overline{y}_0(s) = \frac{\left(s + bs^{\alpha-1}\right)}{\overline{g}(s)}, \quad \overline{y}_\delta(s) = \frac{1}{\overline{g}(s)}, \quad 0 < \alpha < 2, \tag{6.5.17}$$

$$\overline{g}(s) = \left(s^2 + bs^\alpha + \omega^2\right), \quad \frac{\overline{y}_0(s)}{s} = \left(1 + bs^{\alpha-2}\right) \overline{y}_\delta(s). \tag{6.5.18}$$

Using the following properties of the Laplace transform

$$y_0(0) = \lim_{s \to \infty} s\, \overline{y}_0(s) = 1, \tag{6.5.19}$$

$$\overline{y}_\delta(s) = -\frac{1}{\omega^2}\left[s\, \overline{y}_0(s) - 1\right] = -\frac{1}{\omega^2} \mathscr{L}\left\{y_0'(t)\right\}, \tag{6.5.20}$$

$$\mathscr{L}\left\{\int_0^t y_0(\tau)\, d\tau\right\} = \frac{1}{s}\, \overline{y}_0(s), \tag{6.5.21}$$

the inverse Laplace transform of (6.5.15)–(6.5.16) yields the closed form solutions:

$$\text{(i)}\ y(t) = c_0 y_0(t) - \frac{c_1}{\omega^2} y_0'(t) + \int_0^t f(t-\tau)\, y_\delta(\tau)\, d\tau, \quad 0 < \alpha < 1 \tag{6.5.22}$$

$$\text{(ii)}\ y(t) = c_0 y_0(t) + c_1 \int_0^t y_0(\tau)\, d\tau + \int_0^t f(t-\tau)\, y_\delta(\tau)\, d\tau,\ 1 < \alpha < 2 \tag{6.5.23}$$

where $y_\delta(t) = -\frac{1}{\omega^2} y_0'(t)$ represents the *impulse response* solution of the equation (6.5.13) and this solution can be obtained from (6.5.23) by putting $c_0 = c_1 = 0$ and $f(t) = \delta(t)$.

In particular, when $\alpha = 1$, equation (6.5.13) represents the classical solution for the damped simple harmonic oscillator. In order to simplify the solution, we write $b = 2k$ and find

$$\overline{y}_0(s) = \frac{(s+2k)}{(s+k)^2 + (\omega^2 - k^2)}, \quad \overline{y}_\delta(s) = \frac{1}{(s+k)^2 + (\omega^2 - k^2)}. \tag{6.5.24}$$

The inverse Laplace transform yields the solution in three distinct cases: (i) $\omega > k$, (ii) $\omega = k$, and (iii) $\omega < k$. The final closed form solutions are given by

$$y(t) = c_0 e^{-kt}\left(\cos\sigma t + \frac{k}{\sigma}\sin\sigma t\right) + \frac{c_1}{\sigma}e^{-kt}\sin\sigma t + \frac{1}{\sigma}\int_0^t f(t-\tau)\,e^{-k\tau}\sin\sigma\tau d\tau, \qquad \omega > k \tag{6.5.25}$$

$$y(t) = c_0(1+kt)\,e^{-kt} + c_1 t\,e^{-kt} + \int_0^t f(t-\tau)\,\tau c^{-k\tau}d\tau, \ \omega = k \tag{6.5.26}$$

$$y(t) = c_0\,e^{-kt}\left(\cosh\,\mu t + \frac{k}{\mu}\sinh\,\mu t\right) + \frac{c_1}{\mu}\,e^{-kt}\sinh\,\mu t + \frac{1}{\mu}\int_0^t f(t-\tau)\,e^{-k\tau}\sinh\,\mu\tau\,d\tau, \qquad \omega < k, \tag{6.5.27}$$

where $\sigma^2 = \omega^2 - k^2$ and $\mu^2 = k^2 - \omega^2$.

As expected, all solutions exhibit an exponential decay as $t \to \infty$.

## 6.6 Green's Functions of Fractional Differential Equations

(a) We consider the linear system governed by the fractional order differential equation with constant coefficients and zero initial conditions

$${}_0D_t^\alpha\,y(t) = f(t). \tag{6.6.1}$$

Application of the Laplace transform gives

$$\overline{y}(s) = s^{-\alpha}\,\overline{f}(s). \tag{6.6.2}$$

So, the solution given by

$$y(t) = \mathscr{L}^{-1}\left\{s^{-\alpha}\,\overline{f}(s)\right\} = \int_0^t G(t-\tau)\,f(\tau)\,d\tau, \tag{6.6.3}$$

where

$$G(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\right\} = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \tag{6.6.4}$$

is called the *Green's function* of (6.6.1).
(b) We next consider a more general fractional order differential equation with constant coefficients and zero initial conditions in the form

$$_0D_t^\alpha\, y(t) + \omega^2 y(t) = f(t). \tag{6.6.5}$$

Application of the Laplace transform to (6.6.5) gives the solution

$$y(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha + \omega^2}\cdot \overline{f}(s)\right\} = \int_0^t G(t-\tau)\, f(\tau)\, d\tau, \tag{6.6.6}$$

where the Green's function $G(t)$ is given by

$$G(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha + \omega^2}\right\} = t^{\alpha-1} E_{\alpha,\alpha}\left(-\omega^2 t^\alpha\right). \tag{6.6.7}$$

Similarly, the Green's function for the $n$-term fractional-order differential equation with constant coefficients and zero initial conditions in the form

$$\left[a_n D^{\alpha_n} + a_{n-1} D^{\alpha_{n-1}} + ... + a_1 D^{\alpha_1} + a_0 D^{\alpha_0}\right] y(t) = f(t), \tag{6.6.8}$$

can be obtained from the inverse Laplace transform

$$G(t) = \mathscr{L}^{-1}\left[\left(\sum_{k=o}^{n} a_k s^{\alpha_n}\right)^{-1}\right]. \tag{6.6.9}$$

## 6.7 Fractional Partial Differential Equations

(a) The *Fractional Diffusion Equation* is given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad x \in R, \quad t > 0, \tag{6.7.1}$$

with the boundary and initial conditions

$$u(x,t) \to 0 \text{ as } |x| \to \infty, \tag{6.7.2}$$

$$\left[{}_0D_t^{\alpha-1} u(x,t)\right]_{t=0} = f(x) \quad \text{for } x \in R, \tag{6.7.3}$$

where $\kappa$ is a diffusivity constant and $0 < \alpha \le 1$.

Application of the Fourier transform to (6.7.1) with respect $x$ and using the boundary condition (6.7.2) yields

$$D_t^{\alpha}\,\tilde{u}\,(k,t) = -\kappa k^2 \tilde{u}, \tag{6.7.4}$$

$$\left[{}_0D_t^{\alpha-1}\tilde{u}\,(x,t)\right]_{t=0} = \tilde{f}\,(k)\,, \tag{6.7.5}$$

where $\tilde{u}\,(k,t)$ is the Fourier transform of $u\,(x,t)$ is defined by

$$\tilde{u}\,(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u\,(x,t)\,dx. \tag{6.7.6}$$

The Laplace transform solution of (6.7.4) and (6.7.5) is

$$\tilde{\bar{u}}\,(k,s) = \frac{\tilde{f}\,(k)}{(s^{\alpha} + \kappa\, k^2)}. \tag{6.7.7}$$

The inverse Laplace transform of (6.7.7) gives

$$\tilde{u}\,(k,t) = \tilde{f}\,(k)\,t^{\alpha-1} E_{\alpha,\alpha}\left(-\kappa k^2 t^{\alpha}\right), \tag{6.7.8}$$

where $E_{\alpha,\beta}$ is the Mittag-Leffler type function defined by (6.5.6).

Finally, the inverse Fourier transform leads to the solution of the diffusion problem as

$$u\,(x,t) = \int_{-\infty}^{\infty} G\,(x-\xi,t)\,f\,(\xi)\,d\xi, \tag{6.7.9}$$

where

$$G\,(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}\left(-\kappa k^2 t^{\alpha}\right) \cos\, kx\,dk. \tag{6.7.10}$$

This integral for $G\,(x,t)$ can be evaluated by using the Laplace transform of $G\,(x,t)$ as

$$\overline{G}\,(x,s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos\, kx\,dk}{s^{\alpha} + \kappa k^2} = \frac{1}{\sqrt{4\kappa}}\, s^{-\alpha/2} \exp\left(-\frac{|x|}{\sqrt{\kappa}}\, s^{\alpha/2}\right), \tag{6.7.11}$$

whence the inverse Laplace transform gives the explicit solution

$$G\,(x,t) = \frac{1}{\sqrt{4\kappa}}\, t^{\frac{\alpha}{2}-1}\, W\left(-\xi, -\frac{\alpha}{2}, \frac{\alpha}{2}\right), \tag{6.7.12}$$

where $\xi = \frac{|x|}{\sqrt{\kappa}\,t^{\alpha/2}}$, and $W\,(z,\alpha,\beta)$ is the Wright function defined by

$$W\,(z,\alpha,\beta) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma\,(\alpha n + \beta)}. \tag{6.7.13}$$

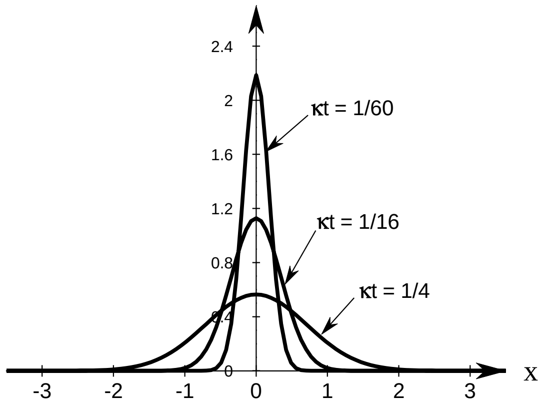


**Figure 6.2** Graphs of (6.7.14) for different $\kappa t$.

It is important to note that when $\alpha = 1$, the initial value problem (6.7.1)–(6.7.3) reduces to the classical diffusion problem and solution (6.7.9) reduces to the classical fundamental solution because

$$G(x,t) = \frac{1}{\sqrt{4\kappa t}} W\left(-\frac{x}{\sqrt{\kappa t}}, -\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right). \quad (6.7.14)$$

It is noted that the order $\alpha$ of the derivative with respect to time $t$ in equation (6.7.1) can be of arbitrary real order including $\alpha = 2$ so that it may be called the *fractional diffusion-wave equation.* For $\alpha = 2$, it becomes the classical wave equation. The equation (6.7.1) with $1 < \alpha \leq 2$ will be solved next in some detail.

(b) The *Nonhomogeneous Fractional Wave Equation* is given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} - c^2 \frac{\partial^2 u}{\partial x^2} = q(x,t), \quad x \in R, \quad t > 0 \quad (6.7.15)$$

with the initial condition

$$u(x,0) = f(x), \, u_t(x,0) = g(x), \, x \in R, \quad (6.7.16)$$

where $c$ is a constant and $1 < \alpha \leq 2$.

Application of the joint Laplace transform with respect to $t$ and Fourier transform with respect to $x$ gives the transform solution

$$\tilde{\bar{u}}(k,s) = \frac{\overline{f}(k)\, s^{\alpha-1}}{s^\alpha + c^2 k^2} + \frac{\tilde{g}(k)\, s^{\alpha-2}}{s^\alpha + c^2 k^2} + \frac{\tilde{\bar{q}}(k,s)}{s^\alpha + c^2 k^2}, \quad (6.7.17)$$

where $k$ is the Fourier transform variable and $s$ is the Laplace transform variable.

The inverse Laplace transform produces the following result

$$\tilde{u}(k,t) = \tilde{f}(k)\,\mathscr{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+c^2k^2}\right\} + \tilde{g}(k)\,\mathscr{L}^{-1}\left\{\frac{s^{\alpha-2}}{s^{\alpha}+c^2k^2}\right\} + \mathscr{L}^{-1}\left\{\frac{\tilde{\bar{q}}(k,s)}{s^{\alpha}+c^2k^2}\right\} \tag{6.7.18}$$

which is, by (6.5.7),

$$= \tilde{f}(k)\,E_{\alpha,1}\left(-c^2k^2t^{\alpha}\right) + \tilde{g}(k)\,tE_{\alpha,2}\left(-c^2k^2t^{\alpha}\right) + \int_0^t \tilde{q}(k,t-\tau)\,\tau^{\alpha-1}E_{\alpha,\alpha}\left(-c^2k^2\tau^{\alpha}\right)d\tau. \tag{6.7.19}$$

Finally, the inverse Fourier transform gives the formal solution

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(k)\,E_{\alpha,1}\left(-c^2k^2t^{\alpha}\right)e^{ikx}dk \\ &+ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t\,\tilde{g}(k)\,E_{\alpha,2}\left(-c^2k^2\tau^{\alpha}\right)e^{ikx}dk \\ &+ \frac{1}{\sqrt{2\pi}}\int_0^t \tau^{\alpha-1}d\tau\int_{-\infty}^{\infty}\tilde{q}(k,t-\tau)\,E_{\alpha,\alpha}\left(-c^2k^2t^{\alpha}\right)e^{ikx}dk. \end{aligned} \tag{6.7.20}$$

In particular, when $\alpha = 2$, the fractional wave equation (6.7.15) reduces to the classical wave equation. In this particular case, we obtain

$$E_{2,1}\left(-c^2k^2t^{\alpha}\right) = \cosh(ickt) = \cos(ckt), \tag{6.7.21}$$

$$tE_{2,2}\left(-c^2k^2t^{\alpha}\right) = t\cdot\frac{\sinh(ickt)}{ickt} = \frac{1}{ck}\sin(ckt). \tag{6.7.22}$$

Consequently, solution (6.7.20) reduces to the classical solution (see Debnath, 2005) of the wave equation (6.7.15) with $\alpha = 2$ in the form

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(k)\cos(ckt)\,e^{ikx}dk + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(k)\,\frac{\sin(ckt)}{ck}e^{ikx}dk \\ &+ \frac{1}{\sqrt{2\pi}\,c}\int_0^t d\tau\int_{-\infty}^{\infty}\tilde{q}(k,\tau)\,\frac{\sin ck(t-\tau)}{k}\,e^{ikx}dk \qquad (6.7.23) \\ &= \frac{1}{2}\left[f(x-ct)+f(x+ct)\right] \\ &+ \frac{1}{2c}\int_{x-ct}^{x+ct}g(\xi)\,d\xi + \frac{1}{2c}\int_0^t d\tau\int_{x-c(t-\tau)}^{x+c(t-\tau)}q(\xi,\tau)\,d\xi. \qquad (6.7.24) \end{aligned}$$

We now derive the solution of the *inhomogeneous fractional diffusion equation* (6.7.15) with $c^2 = \kappa$ and $g(x) \equiv 0$. In this case, the joint transform solutions (6.7.17) becomes

$$\tilde{\bar{u}}(k,s) = \frac{\tilde{f}(k)\, s^{\alpha-1}}{(s^\alpha + \kappa k^2)} + \frac{\tilde{\bar{q}}(k,s)}{(s^\alpha + \kappa k^2)}, \tag{6.7.25}$$

which is inverted by (6.5.7) to obtain

$$\tilde{u}(k,t) = \tilde{f}(k)\, E_{\alpha,1}\left(-\kappa k^2 t^\alpha\right) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\kappa k^2 (t-\tau)^\alpha\right) \tilde{q}(k,\tau)\, d\tau. \tag{6.7.26}$$

Finally, the inverse Fourier transform gives the exact solution for the temperature distribution

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, E_{\alpha,1}\left(-\kappa k^2 t^\alpha\right) e^{ikx} dk \\ &+ \frac{1}{\sqrt{2\pi}} \int_0^t d\tau \int_{-\infty}^{\infty} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\kappa k^2 (t-\tau)^\alpha\right) \\ &\qquad \times \tilde{q}(k,\tau)\, e^{ikx} dk. \end{aligned} \tag{6.7.27}$$

Application of the convolution theorem of the Fourier transform gives the final solution in the form

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} G_1(x-\xi,t)\, f(\xi)\, d\xi \\ &+ \int_0^t (t-\tau)^{\alpha-1} d\tau \int_{-\infty}^{\infty} G_2(x-\xi, t-\tau)\, q(\xi,\tau)\, d\xi, \end{aligned} \tag{6.7.28}$$

where

$$G_1(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} E_{\alpha,1}\left(-\kappa k^2 t^\alpha\right) dk, \tag{6.7.29}$$

and

$$G_2(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} E_{\alpha,\alpha}\left(-\kappa k^2 t^\alpha\right) dk. \tag{6.7.30}$$

In particular, when $\alpha = 1$, the classical solution of the nonhomogeneous diffusion equation is obtained in the form

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} G_1(x-\xi,t)\, f(\xi)\, d\xi \\ &+ \int_0^t d\tau \int_{-\infty}^{\infty} G_2(x-\xi,t-\tau) \cdot q(\xi,\tau)\, d\xi, \end{aligned} \tag{6.7.31}$$

where

$$G_1(x,t) = G_2(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right). \tag{6.7.32}$$

(c) We consider the fractional-order diffusion equation in a semi-infinite medium $x > 0$, when the boundary is kept at a temperature $u_0 f(t)$ and the initial temperature is zero in the whole medium. Thus, the initial-boundary value problem is described by the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0, \tag{6.7.33}$$

with

$$u(x, t = 0) = 0, \quad x > 0, \tag{6.7.34}$$

$$u(x = 0, t) = u_0 f(t), \quad t > 0 \quad \text{and} \quad u(x,t) \to 0 \text{ as } x \to \infty. \tag{6.7.35}$$

Application of the Laplace transform with respect to $t$ gives

$$\frac{d^2\overline{u}}{dx^2} + \left(\frac{s^\alpha}{\kappa}\right) \overline{u}(x,s), \quad x > 0, \tag{6.7.36}$$

$$\overline{u}(x = 0, s) = u_0 \overline{f}(s), \quad \overline{u}(x,s) \to 0 \quad \text{as } x \to \infty. \tag{6.7.37}$$

Evidently, the solution of this transformed boundary value problem is

$$\overline{u}(x,s) = u_0 \overline{f}(s) \exp(-ax), \qquad a = (s^\alpha/\kappa)^{\frac{1}{2}}. \tag{6.7.38}$$

Thus, the solution is given by

$$u(x,t) = u_0 \int_0^t f(t-\tau)\, g(x,\tau)\, d\tau = u_0 f(t) * g(x,t), \tag{6.7.39}$$

where

$$g(x,t) = \mathscr{L}^{-1}\{\exp(-ax)\}.$$

In this case, $\alpha = 1$ and $f(t) = 1$, solution (6.7.38) becomes

$$\overline{u}(x,s) = \left(\frac{u_0}{s}\right) \exp\left(-x\sqrt{\frac{s}{\kappa}}\right), \tag{6.7.40}$$

which yields the classical solution in terms of the complementary error function

$$u(x,t) = u_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right). \tag{6.7.41}$$

In the classical case $(\alpha = 1)$ and the more general solution is given by

$$u(x,t) = u_0 \int_0^t f(t-\tau)\, g(x,\tau)\, d\tau = u_0 f(t) * g(x,t), \tag{6.7.42}$$

where

$$g(x,t) = \mathscr{L}^{-1}\left\{\exp\left(-x\sqrt{\frac{s}{\kappa}}\right)\right\} = \frac{x}{2\sqrt{\pi\kappa t^3}}\exp\left(-\frac{x^2}{4\kappa t}\right). \quad (6.7.43)$$

(d) *The Fractional Stokes and Rayleigh Problems in Fluid Dynamics*
The classical Stokes problem deals with the unsteady boundary layer flows induced in a semi-infinite viscous fluid bounded by an infinite horizontal disk at $z=0$ due to non-torsional oscillations of the disk in its own plane with a given frequency $\omega$. When $\omega=0$, the Stokes problem reduces to the classical Rayleigh problem where the unsteady boundary layer flow is generated in the fluid from rest by moving the disk impulsively in its own plane with constant velocity $U$.

We consider the unsteady fractional boundary layer equation (see Debnath (2003)) for the fluid velocity $u(z,t)$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \nu\,\frac{\partial^2 u}{\partial t^2}, \quad 0<z<\infty, \quad t>0, \quad (6.7.44)$$

with the given boundary and initial conditions

$$u(0,t) = Uf(t), \quad u(z,t)\to 0 \quad \text{as} \quad z\to\infty, \quad t>0, \quad (6.7.45)$$

$$u(z,0)=0 \quad \text{for all} \quad z>0, \quad (6.7.46)$$

where $\nu$ is the kinematic viscosity, $U$ is a constant velocity and $f(t)$ is an arbitrary function of time $t$.

Application of the Laplace transform with respect to $t$ gives

$$s^\alpha\,\overline{u}(z,s) = \nu\,\frac{d^2\overline{u}}{dz^2}, \quad 0<z<\infty, \quad (6.7.47)$$

$$\overline{u}(0,s) = U\,\overline{f}(s), \quad \overline{u}(z,s)\to 0 \quad \text{as } z\to\infty. \quad (6.7.48)$$

Using the Fourier sine transform with respect to $z$ yields

$$\overline{U}_s(k,s) = \left(\sqrt{\frac{2}{\pi}}\,\nu\,U\right)\frac{k\,\overline{f}(s)}{(s^\alpha+\nu k^2)}. \quad (6.7.49)$$

The inverse Fourier sine transform of (6.7.49) leads to the solution

$$\overline{u}(z,s) = \left(\frac{2}{\pi}\,\nu U\right)\overline{f}(s)\int_0^\infty \frac{k\sin kz}{(s^\alpha+\nu k^2)}\,dk, \quad (6.7.50)$$

and the inverse Laplace transform gives the solution for the velocity field

$$u(z,t) = \left(\frac{2}{\pi}\,\nu\,U\right)\int_0^\infty k\sin kz\,dk \times \int_0^t f(t-\tau)\,\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\nu k^2\tau^\alpha\right)d\tau. \quad (6.7.51)$$

When $f(t) = \exp(i\omega t)$, the solution of the fractional Stokes problem is

$$u(z,t) = \left(\frac{2\nu U}{\pi}\right) e^{i\omega t} \int_0^\infty k \sin kz dk \times \int_0^t e^{-i\omega\tau} \tau^{\alpha-1} E_{\alpha,\alpha}\left(-\nu k^2 \tau^\alpha\right) d\tau. \tag{6.7.52}$$

When $\alpha = 1$, solution (6.7.52) reduces to the classical Stokes solution in the form

$$u(z,t) = \left(\frac{2\nu U}{\pi}\right) \int_0^\infty \left(1 - e^{-\nu t k^2}\right) \frac{k \sin kz}{(i\omega + \nu k^2)} dk. \tag{6.7.53}$$

For the fractional Rayleigh problem, $f(t) = 1$ and the solution follows from (6.7.51) in the form

$$u(z,t) = \left(\frac{2\nu U}{\pi}\right) \int_0^\infty k \sin kz dk \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}\left(-\nu k^2 \tau^\alpha\right) d\tau. \tag{6.7.54}$$

This solution reduces to the classical Rayleigh solution when $\alpha = 1$ as

$$\begin{aligned} u(z,t) &= \left(\frac{2\nu U}{\pi}\right) \int_0^\infty k \sin kz dk \int_0^t E_{1,1}\left(-\nu\tau k^2\right) d\tau \\ &= \left(\frac{2\nu U}{\pi}\right) \int_0^\infty k \sin kz dk \int_0^t \exp\left(-\nu\tau k^2\right) d\tau \\ &= \left(\frac{2U}{\pi}\right) \int_0^\infty \left(1 - e^{-\nu t k^2}\right) \frac{\sin kz}{k} dk, \end{aligned}$$

which is, by (2.15.10),

$$= \left(\frac{2U}{\pi}\right) \left[\frac{\pi}{2} - \frac{\pi}{2} \operatorname{erf}\left(\frac{z}{2\sqrt{\nu t}}\right)\right] = U \operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}}\right), \tag{6.7.55}$$

where $\operatorname{erfc}(x)$ is the complimentary error function.

(e) *The Fractional Unsteady Couette Flow*
We consider the unsteady viscous fluid flow between the plate at $z = 0$ at rest and the plate $z = h$ in motion parallel to itself with a variable velocity $U(t)$ in the $x$-direction. The fluid velocity $u(z,t)$ satisfies the fractional equation of motion (see Debnath(2003))

$$\frac{\partial^\alpha u}{\partial t^\alpha} = P(t) + \nu \frac{\partial^2 u}{\partial t^2}, \quad 0 \le z \le h, \quad t > 0, \tag{6.7.56}$$

with the boundary and initial conditions

$$u(0,t) = 0 \text{ and } u(h,t) = U(t), \quad t > 0, \tag{6.7.57}$$

$$u(z,t) = 0 \text{ at } t \le 0 \text{ for } 0 \le z \le h, \tag{6.7.58}$$

where $-\frac{1}{\rho}\,p_x = P(t)$ and $\nu$ is the kinematic viscosity of the fluid.

We apply the joint Laplace transform with respect to $t$ and the finite Fourier sine transform with respect to $z$ defined by

$$\bar{\tilde{u}}_s(n,s) = \int_0^\infty e^{-st}dt \int_0^h u(z,t)\sin\left(\frac{n\pi z}{h}\right)dz, \tag{6.7.59}$$

to the system (6.7.56) - (6.7.58) so that the transform solution is

$$\bar{\tilde{u}}_s(n,s) = \frac{\overline{P}(s)\,\frac{1}{a}\left[1-(-1)^n\right]}{(s^\alpha+\nu a^2)} + \frac{\nu a\,(-1)^{n+1}\,\overline{U}(s)}{(s^\alpha+\nu a^2)}, \tag{6.7.60}$$

where $a = \left(\frac{n\pi}{h}\right)$, $n$ is the finite Fourier sine transform variable.

Thus, the inverse Laplace transform yields

$$\begin{aligned}\tilde{u}_s(n,t) = \frac{1}{a}\left[1-(-1)^n\right]\int_0^t P(t-\tau)\,\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\nu a^2\tau^\alpha\right)d\tau \\ + \nu a\,(-1)^{n+1}\int_0^t U(t-\tau)\,\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\nu a^2\tau^\alpha\right)d\tau.\end{aligned} \tag{6.7.61}$$

Finally, the inverse finite Fourier sine transform leads to the solution

$$u(z,t) = \frac{2}{h}\sum_{n=1}^\infty \tilde{u}_s(n,t)\sin\left(\frac{n\pi z}{h}\right). \tag{6.7.62}$$

If, in particular, $P(t)$ =constant and $U(t)$ =constant, then solution (6.7.62) reduces to the solution of the generalized Couette flow.

(f) *Fractional Axisymmetric Wave-Diffusion Equation*
The fractional axisymmetric equation in an infinite domain

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \quad 0<r<\infty, \quad t>0, \tag{6.7.63}$$

is called the *diffusion* or *wave equation* according as $a=\kappa$ or $a=c^2$.

For the fractional diffusion equation, we prescribe the initial condition

$$u(r,0) = f(r), \quad 0<r<\infty. \tag{6.7.64}$$

Application of the joint Laplace transform with respect to $t$ and Hankel transform (7.4.3) of zero order (see Chapter 7) with respect to $r$ to (6.7.63)–(6.7.64) gives the transform solution

$$\bar{\tilde{u}}(k,s) = \frac{s^{\alpha-1}\tilde{f}(k)}{(s^\alpha+\kappa k^2)}, \tag{6.7.65}$$

where $k, s$ are the Hankel and Laplace transform variables, respectively.

The joint inverse transform leads to the solution

$$u(r,t) = \int_0^\infty r J_0(kr)\, \tilde{f}(k)\, E_{\alpha,1}\left(-\kappa k^2 t^\alpha\right) dk, \tag{6.7.66}$$

where $J_0(kr)$ is the Bessel function of the first kind of order zero and $\tilde{f}(k)$ is the Hankel transform of $f(r)$.

On the other hand, we can solve the wave equation (6.7.63) with $a = c^2$ and the initial conditions

$$u(r,0) = f(r), \quad u_t(r,0) = g(r) \quad \text{for } 0 < r < \infty, \tag{6.7.67}$$

provided the Hankel transforms of $f(r)$ and $g(r)$ exist.

Application of the joint Laplace and Hankel transform leads to the transform solution

$$\bar{\tilde{u}}(k,s) = \frac{s^{\alpha-1}\tilde{f}(k)}{(s^\alpha + c^2k^2)} + \frac{s^{\alpha-2}\tilde{g}(k)}{(s^\alpha + c^2k^2)}. \tag{6.7.68}$$

The joint inverse transformation gives the solution

$$\begin{aligned} u(r,t) = &\int_0^\infty k J_0(k,r)\, \tilde{f}(k)\, E_{\alpha,1}\left(-c^2k^2t^\alpha\right) dk \\ &+ \int_0^\infty k J_0(k,r)\, \tilde{g}(k)\, t E_{\alpha,2}\left(-c^2k^2t^\alpha\right) dk. \end{aligned} \tag{6.7.69}$$

When $\alpha = 2$, solution (6.7.69) is in total agreement with that of the classical axisymmetric wave equation (see Example 7.4.1 in Chapter 7).

In a finite domain $0 \le r \le a$, the fractional diffusion equation (6.7.63) can be solved by using the joint Laplace and finite Hankel transform with the boundary and initial data

$$u(r,t) = f(t) \text{ on } r = a, \quad t > 0, \tag{6.7.70}$$

$$u(r,0) = 0 \quad \text{for all } r \text{ in } (0,a). \tag{6.7.71}$$

Application of the joint Laplace and finite Hankel transform of zero order (see Chapter 13) yields the solution

$$u(r,t) = \frac{2}{a^2}\sum_{i=1}^\infty \tilde{u}(k_i,t)\,\frac{J_0(rk_i)}{J_1^2(ak_i)}, \tag{6.7.72}$$

where

$$\tilde{u}(k_i,t) = (a\kappa\, k_i)\, J_1(ak_i) \int_0^t f(t-\tau)\, \tau^{\alpha-1} E_{\alpha,\alpha}\left(-\kappa k_i^2 \tau^\alpha\right) d\tau. \tag{6.7.73}$$

Similarly, the fractional wave equation (6.7.63) with $a=c^2$ in a finite domain $0\le r\le a$ with the boundary and initial data

$$u(r,t)=0 \quad \text{on} \quad r=a, \quad t>0, \tag{6.7.74}$$

$$u(r,0)=f(r) \quad \text{and} \quad u_t(r,0)=g(r) \quad \text{for} \quad 0<r<a, \tag{6.7.75}$$

can be solved by means of the joint Laplace and the finite Hankel transform (13.2.8). The solution of this problem is

$$u(r,t) = \frac{2}{a^2}\sum_{i=1}^{\infty} \tilde{u}(k_i,t)\,\frac{J_0(rk_i)}{J_1^2(ak_i)}, \tag{6.7.76}$$

where

$$\tilde{u}(k_i,t) = \tilde{f}(k_i)\,E_{\alpha,1}\left(-c^2{k_i}^2 t^\alpha\right) + \tilde{g}(k_i)\,E_{\alpha,2}\left(-c^2k_i^2t^\alpha\right). \tag{6.7.77}$$

(g) *The Fractional Schrödinger Equation in Quantum Mechanics*
The one-dimensional fractional Schrödinger equation (see Debnath(2003)) for a free particle of mass $m$ is

$$i\hbar\,\frac{\partial^\alpha\psi}{\partial t^\alpha} = -\frac{\hbar^2}{2m}\,\frac{\partial^2\psi}{\partial x^2}, \qquad -\infty<x<\infty, \quad t>0, \tag{6.7.78}$$

$$\psi(x,0) = \psi_0(x), \qquad -\infty<x<\infty, \tag{6.7.79}$$

$$\psi(x,t)\to 0 \quad \text{as} \quad |x|\to\infty, \tag{6.7.80}$$

where $\psi(x,t)$ is the wave function, $h=2\pi\hbar=6.625\times 10^{-27}\,\text{erg sec}=4.14\times 10^{-21}\,\text{MeV sec}$ is the Planck constant and $\psi_0(x)$ is an arbitrary function.

Application of the joint Laplace and Fourier transform to (6.7.78)–(6.7.80) gives the solution in the transform space in the form

$$\tilde{\bar{\psi}}(k,s) = \frac{s^{\alpha-1}\tilde{\psi}_0(k)}{s^\alpha + ak^2}, \qquad \left(a=\frac{i\hbar}{2m}\right), \tag{6.7.81}$$

where $k, s$ represent the Fourier and the Laplace transforms variables.

The use of the joint inverse transform yields the solution

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\tilde{\psi}_0(k)\,E_{\alpha,1}\left(-ak^2t^\alpha\right)dk \tag{6.7.82}$$

$$= \mathscr{F}^{-1}\left\{\tilde{\psi}_0(k)\,E_{\alpha,1}\left(-ak^2t^\alpha\right)\right\}, \tag{6.7.83}$$

which is, by the convolution theorem of the Fourier transform,

$$= \int_{-\infty}^{\infty} G(x-\xi,\,t)\,\psi_0(\xi)\,d\xi. \tag{6.7.84}$$

where

$$G(x,t) = \frac{1}{\sqrt{2\pi}} \mathscr{F}^{-1}\left\{E_{\alpha,1}\left(-ak^2t^\alpha\right)\right\}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} E_{\alpha,1}\left(-ak^2t^\alpha\right) dk. \tag{6.7.85}$$

When $\alpha = 1$, the solution of the Schrödinger equation (6.7.78) becomes

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x-\xi,\, t)\, \psi_0(\xi)\, d\xi, \tag{6.7.86}$$

where the Green's function $G(x,t)$ is given by

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} E_{1,1}\left(-ak^2 t\right) dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(ikx - atk^2\right) = \frac{1}{\sqrt{4\pi a t}} \exp\left(-\frac{x^2}{4at}\right). \tag{6.7.87}$$

(h) *Linear Inhomogeneous Fractional Evolution Equation*
The fairly general linear inhomogeneous fractional evolution equation is given by (see Debnath and Bhatta (2004))

$$\frac{\partial^\alpha u}{\partial t^\alpha} + c\frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} + b_3 \frac{\partial^3 u}{\partial x^3} + .... + b_n \frac{\partial^n u}{\partial x^n} = q(x,t),\ x \in R,\ t > 0, \tag{6.7.88}$$

where $c$, $\nu$, and $b_3, ..., b_n$ are constants and $0 < \alpha \leq 1$. We solve this fractional evolution equation with the following initial and boundary conditions

$$u(x,0) = f(x) \qquad x \in R \tag{6.7.89}$$
$$u(x,t) \to 0 \qquad as \quad |x| \to \infty,\ t > 0. \tag{6.7.90}$$

Application of the joint Laplace transform with respect to $t$ and Fourier transform with respect to $x$ gives the transform solution

$$\tilde{\bar{u}}(k,s) = \frac{\overline{f}(k)\, s^{\alpha-1}}{s^\alpha + a^2} + \frac{\tilde{\bar{q}}(k,s)}{s^\alpha + a^2}, \tag{6.7.91}$$

where $k$ is the Fourier transform variable $s$ is the Laplace transform variable, and $a^2$ is given by

$$a^2 = ikc + k^2\nu + (ik)^3 b_3 + (ik)^4 b_4 + .... + (ik)^n b_n. \tag{6.7.92}$$

We use the following inverse Laplace transform formula

$$\mathscr{L}^{-1}\left\{\frac{m!\, s^{\alpha-\beta}}{(s^\alpha \mp a)^{m+1}}\right\} = t^{\alpha m+\beta-1} E_{\alpha,\beta}^{(m)}\left(\pm a t^\alpha\right), \tag{6.7.93}$$

where $E_{\alpha,\beta}(z)$ and $E_{\alpha,\beta}^{(m)}(z)$ are defined by (6.5.6) and (6.5.8) respectively. The inverse Laplace transform yields the following result

$$\tilde{u}(k,t) = \tilde{f}(k)\,\mathscr{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+a^2}\right\} + \mathscr{L}^{-1}\left\{\frac{\tilde{\bar{q}}(k,s)}{s^{\alpha}+a^2}\right\}, \qquad (6.7.94)$$

which can be written as

$$\tilde{u}(k,t) = \tilde{f}(k)\,E_{\alpha,1}\left(-a^2 t^{\alpha}\right) + \int_0^t \tilde{q}(k,t-\tau)\,\tau^{\alpha-1}E_{\alpha,\alpha}\left(-a^2\tau^{\alpha}\right)d\tau. \quad (6.7.95)$$

Finally, the inverse Fourier transform gives the formal solution

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(k)\,E_{\alpha,1}\left(-a^2t^{\alpha}\right)e^{ikx}dk \\ &+ \frac{1}{\sqrt{2\pi}}\int_0^t \tau^{\alpha-1}d\tau\int_{-\infty}^{\infty}\tilde{q}(k,t-\tau)\,E_{\alpha,\alpha}\left(-a^2t^{\alpha}\right)e^{ikx}dk, \quad (6.7.96) \end{aligned}$$

where $a^2$ is given by (6.7.92). The solution in equation (6.7.96) is fairly general and contains solutions of many special evolution equations including the fractional wave equation, Korteweg de Vries (KdV) equation, and KdV-Burgers equation (see Debnath and Bhatta, 2004).

(i) *Linear Inhomogeneous Fractional Telegraph Equation*
Here we solve the one-dimensional linear inhomogeneous fractional telegraph equation given by

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} - c^2\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial t} + bu = q(x,t), \quad x\in R, \quad t>0 \qquad (6.7.97)$$

where $a$, $b$, and $c$ are constants and $1<\alpha\le 2$.
We solve this fractional evolution equation with the following initial and boundary conditions

$$u(x,t) = f(x), \quad \frac{\partial u(x,t)}{\partial t} = g(x) \qquad at \quad t=0,\ x\in R \qquad (6.7.98)$$

$$u(x,t)\to 0 \qquad as \quad |x|\to\infty,\ t>0 \qquad (6.7.99)$$

Applying the joint transform of Laplace and Fourier and taking the inverse Laplace tansform, we have

$$\begin{aligned} \tilde{u}(k,t) &= \tilde{f}(k)\,E_{\alpha,1}\left(-\lambda^2t^{\alpha}\right) + t\tilde{g}(k)\,E_{\alpha,1}\left(-\lambda^2t^{\alpha}\right) \\ &\quad + \int_0^t \tilde{q}(k,t-\tau)\,\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda^2\tau^{\alpha}\right)d\tau, \quad (6.7.100) \end{aligned}$$

where

$$\lambda^2 = c^2k^2 + aik + b. \qquad (6.7.101)$$

Application of the inverse Fourier transform to equation (6.7.100) yields the solution $u(x,t)$ as

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, E_{\alpha,1}\left(-\lambda^2 t^\alpha\right) e^{ikx} dk \\
&+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t\tilde{g}(k)\, E_{\alpha,2}\left(-\lambda^2 t^\alpha\right) e^{ikx} dk \\
&+ \frac{1}{\sqrt{2\pi}} \int_0^t \tau^{\alpha-1} d\tau \int_{-\infty}^{\infty} \tilde{q}(k, t-\tau)\, E_{\alpha,\alpha}\left(-\lambda^2 t^\alpha\right) e^{ikx} dk. \quad (6.7.102)
\end{aligned}
$$

In the limit as $a \to 0$, the telegraph equation reduces to the Klein-Gordon equation and its solution is in perfect agreement with each other (see Debnath and Bhatta, 2004).

## 6.8 Exercises

1. Consider linear inhomogeneous fractional differential equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + c\,\frac{\partial u}{\partial x} = q(x,t), \quad x \in R, \quad t > 0$$

where $c$ is a constant, $0 < \alpha \leq 1$, and $q$, source term, is a function of $x$ and $t$.

Assuming the initial and boundary conditions

$$
\begin{aligned}
u(x,0) &= f(x), \qquad x \in R \\
u(x,t) &\to 0 \qquad as \quad |x| \to \infty,\ t > 0
\end{aligned}
$$

show that the solution $u(x,t)$ is given by

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, E_{\alpha,1}\left(-ickt^\alpha\right) e^{ikx} dk \\
&+ \frac{1}{\sqrt{2\pi}} \int_0^t \tau^{\alpha-1} d\tau \int_{-\infty}^{\infty} \tilde{q}(k, t-\tau)\, E_{\alpha,\alpha}\left(-ickt^\alpha\right) e^{ikx} dk.
\end{aligned}
$$

In particular, when $\alpha = 1$, the solution becomes

$$
\begin{aligned}
u(x,t) &= f(x-ct) + \frac{1}{\sqrt{2\pi}} \int_0^t d\tau \int_{-\infty}^{\infty} \tilde{q}(k,\tau)\, e^{ik\{x-c(t-\tau)\}} dk \\
&= f(x-ct) + \int_0^t q\{x - c(t-\tau),\ \tau\}\, d\tau.
\end{aligned}
$$

2. Consider linear inhomogeneous fractional Burgers equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+c\frac{\partial u}{\partial x}-\nu\frac{\partial^2 u}{\partial x^2}=q\left(x,t\right), \quad x\in R, \quad t>0,$$

where $c$ is a constant, $0<\alpha\leq 1$, $\nu$ is the kinematic viscosity and $q(x,t)$ is a source term.

Assuming the initial and boundary conditions

$$\begin{aligned} u\left(x,0\right) &= f\left(x\right), \qquad x\in R, \\ u\left(x,t\right) &\to 0 \qquad as \quad |x|\to\infty,\ t>0, \end{aligned}$$

show that the solution $u(x,t)$ is given by

$$\begin{aligned} u\left(x,t\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}\left(k\right)E_{\alpha,1}\left(-a^2t^\alpha\right)e^{ikx}dk \\ &+\frac{1}{\sqrt{2\pi}}\int_0^t \tau^{\alpha-1}d\tau\int_{-\infty}^{\infty}\tilde{q}\left(k,t-\tau\right)E_{\alpha,\alpha}\left(-a^2t^\alpha\right)e^{ikx}dk, \end{aligned}$$

where $a^2=\left(ick+\nu k^2\right)$.

3. Consider linear inhomogeneous fractional KdV equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+c\frac{\partial u}{\partial x}+b\frac{\partial^3 u}{\partial x^3}=q\left(x,t\right), \quad x\in R, \quad t>0$$

where $b$ and $c$ are constants, $0<\alpha\leq 1$.

Assuming the initial and boundary conditions

$$\begin{aligned} u\left(x,0\right) &= f\left(x\right), \qquad x\in R, \\ u\left(x,t\right) &\to 0 \qquad as \quad |x|\to\infty,\ t>0, \end{aligned}$$

show that the solution $u(x,t)$ is given by

$$\begin{aligned} u\left(x,t\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}\left(k\right)E_{\alpha,1}\left(-a^2t^\alpha\right)e^{ikx}dk \\ &+\frac{1}{\sqrt{2\pi}}\int_0^t \tau^{\alpha-1}d\tau\int_{-\infty}^{\infty}\tilde{q}\left(k,t-\tau\right)E_{\alpha,\alpha}\left(-a^2t^\alpha\right)e^{ikx}dk, \end{aligned}$$

where $a^2=\left(ick-ik^3b\right)$.

4. Consider linear inhomogeneous fractional KdV-Burgers equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+c\frac{\partial u}{\partial x}-\nu\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^3 u}{\partial x^3}=q\left(x,t\right), \quad x\in R, \quad t>0,$$

where $b$, $c$ and $\nu$ are constants, $0<\alpha\leq 1$.

Assuming the initial and boundary conditions

$$\begin{aligned} u(x,0) &= f(x), \qquad x\in R,\\ u(x,t) &\to 0 \qquad as \quad |x|\to\infty,\ t>0, \end{aligned}$$

show that the solution $u(x,t)$ is given by

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(k)\,E_{\alpha,1}\left(-a^2t^\alpha\right)e^{ikx}dk\\ &+\frac{1}{\sqrt{2\pi}}\int_0^t \tau^{\alpha-1}d\tau\int_{-\infty}^{\infty}\tilde{q}(k,t-\tau)\,E_{\alpha,\alpha}\left(-a^2t^\alpha\right)e^{ikx}dk, \end{aligned}$$

where $a^2=\left(ick+k^2\nu-ik^3b\right)$.

5. Consider linear inhomogeneous fractional Klein-Gordon equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}-c^2\frac{\partial^2 u}{\partial x^2}+d^2u=q(x,t), \quad x\in R, \quad t>0,$$

where $c$ and $d$ are constants, $1<\alpha\leq 2$.

Assuming the initial and boundary conditions

$$\begin{aligned} &u(x,t) = f(x), \quad \frac{\partial u(x,t)}{\partial t}=g(x) \qquad at \quad t=0,\ x\in R,\\ &u(x,t)\to 0, \qquad as \quad |x|\to\infty,\ t>0, \end{aligned}$$

show that the solution $u(x,t)$ is given by

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(k)\,E_{\alpha,1}\left(-a^2t^\alpha\right)e^{ikx}dk\\ &+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t\tilde{g}(k)\,E_{\alpha,2}\left(-a^2t^\alpha\right)e^{ikx}dk\\ &+\frac{1}{\sqrt{2\pi}}\int_0^t \tau^{\alpha-1}d\tau\int_{-\infty}^{\infty}\tilde{q}(k,t-\tau)\,E_{\alpha,\alpha}\left(-a^2t^\alpha\right)e^{ikx}dk, \end{aligned}$$

where $a^2=\left(c^2k^2+d^2\right)$.

# 7

# *Hankel Transforms and Their Applications*

> "In most sciences one generation tears down what another has built, and what one has established, another undoes. In mathematics alone each generation adds a new storey to the old structure."
>
> Hermann Hankel

> "I have always regarded mathematics as an object of amusement rather than of ambition, and I can assure you that I enjoy the works of others much more than my own."
>
> Joseph-Louis Lagrange

## 7.1 Introduction

Hermann Hankel (1839-1873), a German mathematician, is remembered for his numerous contributions to mathematical analysis including the Hankel transformation, which occurs in the study of functions which depend only on the distance from the origin. He also studied functions, now named Hankel functions or Bessel functions of the third kind. The Hankel transform involving Bessel functions as the kernel arises naturally in axisymmetric problems formulated in cylindrical polar coordinates. This chapter deals with the definition and basic operational properties of the Hankel transform. A large number of axisymmetric problems in cylindrical polar coordinates are solved with the aid of the Hankel transform. The use of the joint Laplace and Hankel transforms is illustrated by several examples of applications to partial differential equations.

## 7.2 The Hankel Transform and Examples

We introduce the definition of the *Hankel transform* from the two-dimensional Fourier transform and its inverse given by

$$\mathscr{F}\{f(x,y)\} = F(k,l) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-i(\boldsymbol{\kappa}\cdot\mathbf{r})\} f(x,y)\,dx\,dy, \quad (7.2.1)$$

$$\mathscr{F}^{-1}\{F(k,l)\} = f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i(\boldsymbol{\kappa}\cdot\mathbf{r})\} F(k,l)\,dk\,dl, \quad (7.2.2)$$

where $\mathbf{r} = (x,y)$ and $\boldsymbol{\kappa} = (k,l)$. Introducing polar coordinates $(x,y) = r(\cos\theta, \sin\theta)$ and $(k,l) = \kappa(\cos\phi, \sin\phi)$, we find $\boldsymbol{\kappa}\cdot\mathbf{r} = \kappa r\cos(\theta-\phi)$ and then

$$F(\kappa,\phi) = \frac{1}{2\pi}\int_0^{\infty} r\,dr \int_0^{2\pi} \exp[-i\kappa r\,\cos(\theta-\phi)] f(r,\theta)d\theta. \quad (7.2.3)$$

We next assume $f(r,\theta) = \exp(in\theta)f(r)$, which is not a very severe restriction, and make a change of variable $\theta - \phi = \alpha - \frac{\pi}{2}$ to reduce (7.2.3) to the form

$$\begin{aligned} F(\kappa,\phi) &= \frac{1}{2\pi}\int_0^{\infty} r f(r) dr \\ &\quad\times \int_{\phi_0}^{2\pi+\phi_0} \exp\left[in\left(\phi - \frac{\pi}{2}\right) + i(n\alpha - \kappa r\sin\alpha)\right] d\alpha, \end{aligned} \quad (7.2.4)$$

where $\phi_0 = \left(\frac{\pi}{2} - \phi\right)$.

Using the integral representation of the Bessel function of order $n$

$$J_n(\kappa r) = \frac{1}{2\pi}\int_{\phi_0}^{2\pi+\phi_0} \exp[i(n\alpha - \kappa r\sin\alpha)]d\alpha \quad (7.2.5)$$

integral (7.2.4) becomes

$$F(\kappa,\phi) = \exp\left[in\left(\phi - \frac{\pi}{2}\right)\right]\int_0^{\infty} rJ_n(\kappa r) f(r)dr \quad (7.2.6)$$

$$= \exp\left[in\left(\phi - \frac{\pi}{2}\right)\right]\tilde{f}_n(\kappa), \quad (7.2.7)$$

where $\tilde{f}_n(\kappa)$ is called the *Hankel transform* of $f(r)$ and is defined formally by

$$\mathscr{H}_n\{f(r)\} = \tilde{f}_n(\kappa) = \int_0^\infty r J_n(\kappa r) f(r)\, dr. \tag{7.2.8}$$

Similarly, in terms of the polar variables with the assumption $f(x,y) = f(r,\theta) = e^{in\theta} f(r)$ with (7.2.7), the inverse Fourier transform (7.2.2) becomes

$$\begin{aligned} e^{in\theta} f(r) &= \frac{1}{2\pi}\int_0^\infty \kappa\, d\kappa \int_0^{2\pi} \exp[i\kappa r\cos(\theta-\phi)] F(\kappa,\phi) d\phi \\ &= \frac{1}{2\pi}\int_0^\infty \kappa\, \tilde{f}_n(\kappa) d\kappa \int_0^{2\pi} \exp\left[in\left(\phi - \frac{\pi}{2}\right) + i\kappa r\cos(\theta-\phi)\right] d\phi, \end{aligned}$$

which is, by the change of variables $\theta - \phi = -\left(\alpha + \frac{\pi}{2}\right)$ and $\theta_0 = -\left(\theta + \frac{\pi}{2}\right)$,

$$\begin{aligned} &= \frac{1}{2\pi}\int_0^\infty \kappa \tilde{f}_n(\kappa) d\kappa \int_{\theta_0}^{2\pi+\theta_0} \exp[in(\theta+\alpha) - i\kappa r\sin\alpha] d\alpha \\ &= e^{in\theta}\int_0^\infty \kappa J_n(\kappa r)\tilde{f}_n(\kappa) d\kappa, \qquad \text{by (7.2.5).} \end{aligned} \tag{7.2.9}$$

Thus, the *inverse Hankel transform* is defined by

$$\mathscr{H}_n^{-1}\left[\tilde{f}_n(\kappa)\right] = f(r) = \int_0^\infty \kappa J_n(\kappa r)\tilde{f}_n(\kappa) d\kappa. \tag{7.2.10}$$

Instead of $\tilde{f}_n(\kappa)$, we often simply write $\tilde{f}(\kappa)$ for the Hankel transform specifying the order. Integrals (7.2.8) and (7.2.10) exist for certain large classes of functions, which usually occur in physical applications.

Alternatively, the famous Hankel integral formula (Watson, 1944, p. 453)

$$f(r) = \int_0^\infty \kappa\, J_n(\kappa r) d\kappa \int_0^\infty p J_n(\kappa p) f(p) dp, \tag{7.2.11}$$

can be used to define the Hankel transform (7.2.8) and its inverse (7.2.10).

In particular, the Hankel transforms of the zero order $(n=0)$ and of order one $(n=1)$ are often useful for the solution of problems involving Laplace's equation in an axisymmetric cylindrical geometry.

***Example 7.2.1***

Obtain the zero-order Hankel transforms of

(a) $r^{-1}\exp(-ar)$, (b) $\dfrac{\delta(r)}{r}$, (c) $H(a-r)$,

where $H(r)$ is the Heaviside unit step function.

We have

$$\text{(a)}\ \ \tilde{f}(\kappa) = \mathscr{H}_0\left\{\frac{1}{r}\exp(-ar)\right\} = \int_0^\infty \exp(-ar)\, J_0(\kappa r)dr = \frac{1}{\sqrt{\kappa^2+a^2}}.$$

$$\text{(b)}\ \ \tilde{f}(\kappa) = \mathscr{H}_0\left\{\frac{\delta(r)}{r}\right\} = \int_0^\infty \delta(r)J_0(\kappa r)dr = 1.$$

$$\text{(c)}\ \ \tilde{f}(\kappa) = \mathscr{H}_0\{H(a-r)\} = \int_0^a rJ_0(\kappa r)dr = \frac{1}{\kappa^2}\int_0^{a\kappa} pJ_0(p)dp$$
$$= \frac{1}{\kappa^2}[pJ_1(p)]_0^{a\kappa} = \frac{a}{\kappa}J_1(a\kappa).$$

□

***Example 7.2.2***

Find the first order Hankel transforms of

(a) $f(r) = e^{-ar}$, (b) $f(r) = \dfrac{1}{r}\,e^{-ar}$, (c) $f(r) = \dfrac{\sin ar}{r}$.

We can write

$$\text{(a)}\ \ \tilde{f}(\kappa) = \mathscr{H}_1\{e^{-ar}\} = \int_0^\infty re^{-ar}J_1(\kappa r)dr = \frac{\kappa}{(a^2+\kappa^2)^{\frac{3}{2}}}.$$

$$\text{(b)}\ \ \tilde{f}(\kappa) = \mathscr{H}_1\left\{\frac{a^{-ar}}{r}\right\} = \int_0^\infty e^{-ar}J_1(\kappa r)dr = \frac{1}{\kappa}\left[1 - a(\kappa^2+a^2)^{-\frac{1}{2}}\right].$$

$$\text{(c)}\ \ \tilde{f}(\kappa) = \mathscr{H}_1\left\{\frac{\sin ar}{r}\right\} = \int_0^\infty \sin ar\, J_1(\kappa r)\, dr = \frac{a\, H(\kappa-a)}{\kappa(\kappa^2-a^2)^{\frac{1}{2}}}.$$

□

***Example 7.2.3***

Find the $n$th $(n > -1)$ order Hankel transforms of

(a) $f(r) = r^n H(a-r)$, (b) $f(r) = r^n \exp(-ar^2)$.

Here we have, for $n > -1$,

$$\text{(a) } \tilde{\mathrm{f}}(\kappa) = \mathscr{H}_n\left[r^n H(a-r)\right] = \int_0^a r^{n+1} J_n(\kappa r)dr = \frac{a^{n+1}}{\kappa} J_{n+1}(a\kappa).$$

$$\text{(b) } \tilde{\mathrm{f}}(\kappa) = \mathscr{H}_n\left[r^n \exp(-ar^2)\right] = \int_0^\infty r^{n+1} J_n(\kappa r)\exp(-ar^2)dr$$

$$= \frac{\kappa^n}{(2a)^{n+1}} \exp\left(-\frac{\kappa^2}{4a}\right).$$

□

## 7.3 Operational Properties of the Hankel Transform

***THEOREM 7.3.1***
(*Scaling*). If $\mathscr{H}_n\{f(r)\} = \tilde{f}_n(\kappa)$, then

$$\mathscr{H}_n\{f(ar)\} = \frac{1}{a^2}\tilde{f}_n\left(\frac{\kappa}{a}\right), \qquad a > 0. \tag{7.3.1}$$

**PROOF** We have, by definition,

$$\mathscr{H}_n\{f(ar)\} = \int_0^\infty rJ_n(\kappa r)f(ar)dr$$

$$= \frac{1}{a^2}\int_0^\infty sJ_n\left(\frac{\kappa}{a}s\right)f(s)ds = \frac{1}{a^2}\tilde{f}_n\left(\frac{\kappa}{a}\right).$$

■

***THEOREM 7.3.2***
(*Parseval's Relation*). If $\tilde{f}(\kappa) = \mathscr{H}_n\{f(r)\}$ and $\tilde{g}(\kappa) = \mathscr{H}_n\{g(r)\}$, then

$$\int_0^\infty rf(r)g(r)dr = \int_0^\infty \kappa\tilde{f}(\kappa)\tilde{g}(\kappa)d\kappa. \tag{7.3.2}$$

**PROOF** We proceed formally to obtain

$$\int_0^\infty \kappa \tilde{f}(\kappa)\tilde{g}(\kappa)d\kappa = \int_0^\infty \kappa \tilde{f}(\kappa)d\kappa \int_0^\infty rJ_n(\kappa r)g(r)dr,$$

which is, interchanging the order of integration,

$$= \int_0^\infty rg(r)dr \int_0^\infty \kappa J_n(\kappa r)\tilde{f}(\kappa)d\kappa$$
$$= \int_0^\infty rg(r)f(r)dr.$$

∎

***THEOREM 7.3.3***
(*Hankel Transforms of Derivatives*) If $\tilde{f}_n(\kappa) = \mathscr{H}_n\{f(r)\}$, then

$$\mathscr{H}_n\{f'(r)\} = \frac{\kappa}{2n}\left[(n-1)\tilde{f}_{n+1}(\kappa) - (n+1)\tilde{f}_{n-1}(\kappa)\right], \quad n \geq 1, \tag{7.3.3}$$
$$\mathscr{H}_1\{f'(r)\} = -\kappa \tilde{f}_0(\kappa), \tag{7.3.4}$$

provided $[rf(r)]$ vanishes as $r \to 0$ and $r \to \infty$.

**PROOF** We have, by definition,

$$\mathscr{H}_n\{f'(r)\} = \int_0^\infty rJ_n(\kappa r)f'(r)dr$$

which is, integrating by parts,

$$= [rf(r)J_n(\kappa r)]_0^\infty - \int_0^\infty f(r)\frac{d}{dr}[rJ_n(\kappa r)]dr. \tag{7.3.5}$$

We now use the properties of the Bessel function

$$\frac{d}{dr}[rJ_n(\kappa r)] = J_n(\kappa r) + r\kappa J_n'(\kappa r) = J_n(\kappa r) + r\kappa J_{n-1}(\kappa r) - nJ_n(\kappa r)$$
$$= (1-n)J_n(\kappa r) + r\kappa J_{n-1}(\kappa r). \tag{7.3.6}$$

In view of the given condition, the first term of (7.3.5) vanishes as $r \to 0$ and $r \to \infty$, and the derivative within the integral in (7.3.5) can be replaced by (7.3.6) so that (7.3.5) becomes

$$\mathscr{H}_n\{f'(r)\} = (n-1)\int_0^\infty f(r)J_n(\kappa r)dr - \kappa\tilde{f}_{n-1}(\kappa). \tag{7.3.7}$$

We next use the standard recurrence relation for the Bessel function

$$J_n(\kappa r) = \frac{\kappa r}{2n}[J_{n-1}(\kappa r) + J_{n+1}(\kappa r)]. \tag{7.3.8}$$

Thus, (7.3.7) can be rewritten as

$$\begin{aligned}\mathscr{H}_n[f'(r)] &= -\kappa\tilde{f}_{n-1}(\kappa) + \kappa\left(\frac{n-1}{2n}\right)\left[\int_0^\infty rf(r)\{J_{n-1}(\kappa r) + J_{n+1}(\kappa r)\}dr\right] \\ &= -\kappa\tilde{f}_{n-1}(\kappa) + \kappa\left(\frac{n-1}{2n}\right)\left[\tilde{f}_{n-1}(\kappa) + \tilde{f}_{n+1}(\kappa)\right] \\ &= \left(\frac{\kappa}{2n}\right)\left[(n-1)\tilde{f}_{n+1}(\kappa) - (n+1)\tilde{f}_{n-1}(\kappa)\right].\end{aligned}$$

In particular, when $n=1$, (7.3.4) follows immediately.

Similarly, repeated applications of (7.3.3) lead to the following result

$$\begin{aligned}\mathscr{H}_n\{f''(r)\} &= \frac{\kappa}{2n}\left[(n-1)\mathscr{H}_{n+1}\{f'(r)\} - (n+1)\mathscr{H}_{n-1}\{f'(r)\}\right] \\ &= \frac{\kappa^2}{4}\left[\left(\frac{n+1}{n-1}\right)\tilde{f}_{n-2}(\kappa) - 2\left(\frac{n^2-3}{n^2-1}\right)\tilde{f}_n(\kappa)\right. \\ &\qquad\left. + \left(\frac{n-1}{n+1}\right)\tilde{f}_{n+2}(\kappa)\right].\end{aligned} \tag{7.3.9}$$

∎

***THEOREM 7.3.4***

If $\mathscr{H}_n\{f(r)\} = \tilde{f}_n(\kappa)$, then

$$\mathscr{H}_n\left\{\left(\nabla^2 - \frac{n^2}{r^2}\right)f(r)\right\} = \mathscr{H}_n\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2}{r^2}f(r)\right\} = -\kappa^2\tilde{f}_n(\kappa), \tag{7.3.10}$$

provided both $rf'(r)$ and $rf(r)$ vanish as $r\to 0$ and $r\to\infty$.

**PROOF** We have, by definition (7.2.8),

$$\mathscr{H}_n\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2}{r^2}f(r)\right\} = \int_0^\infty J_n(\kappa r)\left[\frac{d}{dr}\left(r\frac{df}{dr}\right)\right]dr - \int_0^\infty \frac{n^2}{r^2}[rJ_n(\kappa r)]f(r)dr,$$

which is, invoking integration by parts,

$$= \left[\left(r\frac{df}{dr}\right) J_n(\kappa r)\right]_0^\infty - \kappa \int_0^\infty r\frac{df}{dr} J_n'(\kappa r)dr - \int_0^\infty \frac{n^2}{r^2}[rJ_n(\kappa r)]f(r)dr,$$

which is, by replacing the first term with zero because of the given assumption, and by invoking integration by parts again,

$$= -\left[\kappa r f(r) J_n'(\kappa r)\right]_0^\infty + \int_0^\infty \frac{d}{dr}\left[\kappa r J_n'(\kappa r)\right] f(r)dr - \int_0^\infty \frac{n^2}{r^2}[rJ_n(\kappa r)]f(r)dr.$$

We use the given assumptions and Bessel's differential equation,

$$\frac{d}{dr}\left[\kappa r\, J_n'(\kappa r)\right] + r\left(\kappa^2 - \frac{n^2}{r^2}\right) J_n(\kappa r) = 0, \tag{7.3.11}$$

to obtain

$$\begin{aligned}\mathscr{H}_n\left\{\left(\nabla^2 - \frac{n^2}{r^2}\right) f(r)\right\} &= -\int_0^\infty \left(\kappa^2 - \frac{n^2}{r^2}\right) r f(r) J_n(\kappa r) dr \\ &\qquad - \int_0^\infty \frac{n^2}{r^2}[r f(r)]\, J_n(\kappa r)\, dr \\ &= -\kappa^2 \int_0^\infty r J_n(\kappa r) f(r) dr = -\kappa^2 \mathscr{H}_n[f(r)] = -\kappa^2 \tilde{f}_n(\kappa).\end{aligned}$$

This proves the theorem.

In particular, when $n=0$ and $n=1$, we obtain

$$\mathscr{H}_0\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right\} = -\kappa^2 \tilde{f}_0(\kappa), \tag{7.3.12}$$

$$\mathscr{H}_1\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{1}{r^2} f(r)\right\} = -\kappa^2 \tilde{f}_1(\kappa). \tag{7.3.13}$$

Results (7.3.10), (7.3.12), and (7.3.13) are widely used for finding solutions of partial differential equations in axisymmetric cylindrical configurations. We illustrate this point by considering several examples of applications. ∎

## 7.4 Applications of Hankel Transforms to Partial Differential Equations

The Hankel transforms are extremely useful in solving a variety of partial differential equations in cylindrical polar coordinates. The following examples illustrate applications of the Hankel transforms. The examples given here are only representative of a whole variety of physical problems that can be solved in a similar way.

***Example 7.4.1***

(*Free Vibration of a Large Circular Membrane*). Obtain the solution of the free vibration of a large circular elastic membrane governed by the initial value problem

$$c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial t^2}, \quad 0<r<\infty, \quad t>0, \tag{7.4.1}$$

$$u(r,0)=f(r), \quad u_t(r,0)=g(r), \quad \text{for } 0\le r<\infty, \tag{7.4.2ab}$$

where $c^2=(T/\rho)=$ constant, $T$ is the tension in the membrane, and $\rho$ is the surface density of the membrane.

Application of the zero-order Hankel transform with respect to $r$

$$\tilde{u}(\kappa,t)=\int_0^\infty r\,J_0(\kappa r)u(r,t)dr, \tag{7.4.3}$$

to (7.4.1)–(7.4.2ab) gives

$$\frac{d^2\tilde{u}}{dt^2}+c^2\kappa^2\tilde{u}=0, \tag{7.4.4}$$

$$\tilde{u}(\kappa,0)=\tilde{f}(\kappa), \qquad \tilde{u}_t(\kappa,0)=\tilde{g}(\kappa). \tag{7.4.5ab}$$

The general solution of this transformed system is

$$\tilde{u}(\kappa,t)=\tilde{f}(\kappa)\cos(c\kappa t)+(c\kappa)^{-1}\,\tilde{g}(\kappa)\sin(c\kappa t). \tag{7.4.6}$$

The inverse Hankel transform leads to the solution

$$\begin{aligned} u(r,t) &= \int_0^\infty \kappa\,\tilde{f}(\kappa)\cos(c\kappa t)J_0(\kappa r)d\kappa \\ &\quad +\frac{1}{c}\int_0^\infty \tilde{g}(\kappa)\sin(c\kappa t)J_0(\kappa r)d\kappa. \end{aligned} \tag{7.4.7}$$

In particular, we consider

$$u(r,0)=f(r)=Aa(r^2+a^2)^{-\frac{1}{2}}, \quad u_t(r,0)=g(r)=0, \tag{7.4.8ab}$$

so that $\tilde{g}(\kappa)\equiv 0$ and

$$\tilde{f}(\kappa)=Aa\int_0^\infty r(a^2+r^2)^{-\frac{1}{2}}J_0(\kappa r)dr=\frac{Aa}{\kappa}e^{-a\kappa}, \qquad \text{by Example 7.2.1(a).}$$

Thus, the formal solution (7.4.7) becomes

$$u(r,t) = Aa \int_0^\infty e^{-a\kappa} J_0(\kappa r)\cos(c\kappa t)\,d\kappa = Aa\,\mathrm{Re}\int_0^\infty \exp[-\kappa(a+ict)]J_0(\kappa r)\,d\kappa$$
$$= Aa\,\mathrm{Re}\left\{r^2 + (a+ict)^2\right\}^{-\frac{1}{2}}, \qquad \text{by Example 7.2.1(a).} \tag{7.4.9}$$

□

***Example 7.4.2***
(*Steady Temperature Distribution in a Semi-Infinite Solid with a Steady Heat Source*). Find the solution of the Laplace equation for the steady temperature distribution $u(r,z)$ with a steady and symmetric heat source $Q_0 q(r)$:

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = -Q_0 q(r), \quad 0<r<\infty, \quad 0<z<\infty, \tag{7.4.10}$$
$$u(r,0) = 0, \quad 0<r<\infty, \tag{7.4.11}$$

where $Q_0$ is a constant. This boundary condition represents zero temperature at the boundary $z=0$.

Application of the zero-order Hankel transform to (7.4.10) and (7.4.11) gives

$$\frac{d^2\tilde{u}}{dz^2} - \kappa^2\tilde{u} = -Q_0\tilde{q}(\kappa), \quad \tilde{u}(\kappa,0)=0.$$

The bounded general solution of this system is

$$\tilde{u}(\kappa,z) = A\exp(-\kappa z) + \frac{Q_0}{\kappa^2}\tilde{q}(\kappa),$$

where $A$ is a constant to be determined from the transformed boundary condition. In this case

$$A = -\frac{Q_0}{\kappa^2}\tilde{q}(\kappa).$$

Thus, the formal solution is

$$\tilde{u}(\kappa,z) = \frac{Q_0\,\tilde{q}(\kappa)}{\kappa^2}(1-e^{-\kappa z}). \tag{7.4.12}$$

The inverse Hankel transform yields the exact integral solution

$$u(r,z) = Q_0\int_0^\infty \frac{\tilde{q}(\kappa)}{\kappa}(1-e^{-\kappa z})J_0(\kappa r)\,d\kappa. \tag{7.4.13}$$

□

***Example 7.4.3***
(*Axisymmetric Diffusion Equation*). Find the solution of the axisymmetric diffusion equation

$$u_t = \kappa \left( u_{rr} + \frac{1}{r} u_r \right), \qquad 0 < r < \infty, \quad t > 0, \tag{7.4.14}$$

where $\kappa(>0)$ is a diffusivity constant and

$$u(r,0) = f(r), \qquad \text{for} \quad 0 < r < \infty. \tag{7.4.15}$$

We apply the zero-order Hankel transform defined by (7.4.3) to obtain

$$\frac{d\tilde{u}}{dt} + k^2 \kappa \tilde{u} = 0, \qquad \tilde{u}(k,0) = \tilde{f}(k),$$

where $k$ is the Hankel transform variable. The solution of this transformed system is

$$\tilde{u}(k,t) = \tilde{f}(k) \exp(-\kappa k^2 t). \tag{7.4.16}$$

Application of the inverse Hankel transform gives

$$u(r,t) = \int_0^\infty k \tilde{f}(k) J_0(kr) e^{-\kappa k^2 t} dk = \int_0^\infty k \left[ \int_0^\infty l J_0(kl) f(l) dl \right] e^{-\kappa k^2 t} J_0(kr) dk$$

which is, interchanging the order of integration,

$$= \int_0^\infty l\, f(l) dl \int_0^\infty k\, J_0(kl) J_0(kr) \exp(-\kappa k^2 t) dk. \tag{7.4.17}$$

Using a standard table of integrals involving Bessel functions, we state

$$\int_0^\infty k J_0(kl)\, J_0(kr) \exp(-k^2 \kappa t) dk = \frac{1}{2\kappa t} \exp\left[ -\frac{(r^2 + l^2)}{4\kappa t} \right] I_0 \left( \frac{rl}{2\kappa t} \right), \tag{7.4.18}$$

where $I_0(x)$ is the modified Bessel function and $I_0(0) = 1$. In particular, when $l = 0$, $J_0(0) = 1$ and integral (7.4.18) becomes

$$\int_0^\infty k J_0(kr) \exp(-k^2 \kappa t) dk = \frac{1}{2\kappa t} \exp\left( -\frac{r^2}{4\kappa t} \right). \tag{7.4.19}$$

We next use (7.4.18) to rewrite (7.4.17) as

$$u(r,t) = \frac{1}{2\kappa t} \int_0^\infty l f(l) I_0 \left( \frac{rl}{2\kappa t} \right) \exp\left[ -\frac{(r^2 + l^2)}{4\kappa t} \right] dl. \tag{7.4.20}$$

We now assume $f(r)$ to represent a heat source concentrated in a circle of radius $a$ and allow $a \to 0$ so that the heat source is concentrated at $r = 0$ and

$$\lim_{a\to 0} 2\pi \int_0^a r f(r) dr = 1.$$

Or, equivalently,

$$f(r) = \frac{1}{2\pi}\frac{\delta(r)}{r},$$

where $\delta(r)$ is the Dirac delta function.

Thus, the final solution due to the concentrated heat source at $r = 0$ is

$$\begin{aligned} u(r,t) &= \frac{1}{4\pi\kappa t}\int_0^\infty \delta(l) I_0\left(\frac{rl}{2\kappa t}\right)\exp\left[-\frac{r^2+l^2}{4\kappa t}\right] dl \\ &= \frac{1}{4\pi\kappa t}\exp\left(-\frac{r^2}{4\kappa t}\right). \end{aligned} \tag{7.4.21}$$

$\square$

***Example 7.4.4***

(*Axisymmetric Acoustic Radiation Problem*). Obtain the solution of the wave equation

$$c^2\left(u_{rr} + \frac{1}{r}u_r + u_{zz}\right) = u_{tt}, \quad 0 < r < \infty, \quad z > 0, \quad t > 0, \tag{7.4.22}$$

$$u_z = F(r,t) \quad \text{on } z = 0, \tag{7.4.23}$$

where $F(r,t)$ is a given function and $c$ is a constant. We also assume that the solution is bounded and behaves as outgoing spherical waves.

We seek a steady-state solution for the acoustic radiation potential $u = e^{i\omega t}\phi(r,z)$ with $F(r,t) = e^{i\omega t} f(r)$, so that $\phi$ satisfies the Helmholtz equation

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} + \left(\frac{\omega^2}{c^2}\right)\phi = 0, \quad 0 < r < \infty, \quad z > 0, \tag{7.4.24}$$

with the boundary condition

$$\phi_z = f(r) \qquad \text{on } z = 0, \tag{7.4.25}$$

where $f(r)$ is a given function of $r$.

Application of the Hankel transform $\mathscr{H}_0\{\phi(r,z)\} = \tilde{\phi}(k,z)$ to (7.4.24)-(7.4.25) gives

$$\tilde{\phi}_{zz} = \kappa^2\tilde{\phi}, \qquad z > 0,$$

$$\tilde{\phi}_z = \tilde{f}(k), \qquad \text{on } z = 0,$$

where

$$\kappa = \left(k^2 - \frac{\omega^2}{c^2}\right)^{\frac{1}{2}}.$$

The solution of this differential system is

$$\tilde{\phi}(k,z) = -\frac{1}{\kappa}\tilde{f}(k)\exp(-\kappa z), \tag{7.4.26}$$

where $\kappa$ is real and positive for $k > \omega/c$, and purely imaginary for $k < \omega/c$.

The inverse Hankel transform yields the formal solution

$$\phi(r,z) = -\int_0^\infty \frac{k}{\kappa}\tilde{f}(k)\, J_0(kr)\exp(-\kappa z)dk. \tag{7.4.27}$$

Since the exact evaluation of this integral is difficult for an arbitrary $\tilde{f}(k)$, we choose a simple form of $f(r)$ as

$$f(r) = AH(a-r), \tag{7.4.28}$$

where $A$ is a constant, and hence, $\tilde{f}(k) = \frac{Aa}{k}J_1(ak)$.

Thus, the solution (7.4.27) takes the form

$$\phi(r,z) = -Aa\int_0^\infty \frac{1}{\kappa}J_1(ak)J_0(kr)\exp(-\kappa z)dk. \tag{7.4.29}$$

For an asymptotic evaluation of this integral, it is convenient to express (7.4.29) in terms of $R$ which is the distance from the $z$-axis so that $R^2 = r^2 + z^2$ and $z = R\cos\theta$. Using the asymptotic result for the Bessel function

$$J_0(kr) \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}}\cos\left(kr - \frac{\pi}{4}\right) \qquad \text{as} \quad r\to\infty, \tag{7.4.30}$$

where $r = R\sin\theta$. Consequently, (7.4.29) combined with $u = \exp(i\omega t)\phi$ becomes

$$u \sim -\frac{Aa\sqrt{2}e^{i\omega t}}{\sqrt{\pi R\sin\theta}}\int_0^\infty \frac{1}{\kappa\sqrt{k}}J_1(ak)\cos\left(kR\sin\theta - \frac{\pi}{4}\right)\exp(-\kappa z)dk.$$

This integral can be evaluated asymptotically for $R\to\infty$ using the stationary phase approximation formula to obtain the final result

$$u \sim -\frac{Aac}{\omega R\sin\theta}J_1(ak_1)\exp\left[i\left(\omega t - \frac{\omega R}{c}\right)\right], \tag{7.4.31}$$

where $k_1 = \omega/(c\sin\theta)$ is the stationary point. Physically, this solution represents outgoing spherical waves with constant velocity $c$ and decaying amplitude as $R \to \infty$. □

***Example 7.4.5***

(*Axisymmetric Biharmonic Equation*). We solve the axisymmetric boundary value problem

$$\nabla^4 u(r,z) = 0, \qquad 0 \le r < \infty, \quad z > 0\,, \tag{7.4.32}$$

with the boundary data

$$u(r,0) = f(r), \quad 0 \le r < \infty, \tag{7.4.33}$$

$$\frac{\partial u}{\partial z} = 0 \qquad \text{on } z=0, \quad 0 \le r < \infty, \tag{7.4.34}$$

$$u(r,z) \to 0 \qquad \text{as } r \to \infty, \tag{7.4.35}$$

where the axisymmetric biharmonic operator is

$$\nabla^4 = \nabla^2(\nabla^2) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right). \tag{7.4.36}$$

The use of the Hankel transform $\mathscr{H}_0\{u(r,z)\} = \tilde{u}(k,z)$ to this problem gives

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 \tilde{u}(k,z) = 0, \quad z > 0, \tag{7.4.37}$$

$$\tilde{u}(k,0) = \tilde{f}(k), \quad \frac{d\tilde{u}}{dz} = 0 \qquad \text{on } z=0. \tag{7.4.38}$$

The bounded solution of (7.4.37) is

$$\tilde{u}(k,z) = (A + zB)\exp(-kz), \tag{7.4.39}$$

where $A$ and $B$ are integrating constants to be determined by (7.4.38) as $A = \tilde{f}(k)$ and $B = k\tilde{f}(k)$. Thus, solution (7.4.39) becomes

$$\tilde{u}(k,z) = (1 + kz)\tilde{f}(k)\exp(-kz). \tag{7.4.40}$$

The inverse Hankel transform gives the formal solution

$$u(r,z) = \int_0^\infty k(1+kz)\tilde{f}(k)J_0(kr)\exp(-kz)dk. \tag{7.4.41}$$

□

***Example 7.4.6***

(*The Axisymmetric Cauchy-Poisson Water Wave Problem*). We consider the initial value problem for an inviscid water of finite depth $h$ with a free horizontal surface at $z=0$, and the $z$-axis positive upward. We assume that the liquid has constant density $\rho$ with no surface tension. The surface waves are generated in water, which is initially at rest for $t<0$ by the prescribed free surface elevation. In cylindrical polar coordinates $(r,\theta,z)$, the axisymmetric water wave equations for the velocity potential $\phi(r,z,t)$ and the free surface elevation $\eta(r,t)$ are

$$\nabla^2\phi=\phi_{rr}+\frac{1}{r}\phi_r+\phi_{zz}=0, \quad 0\le r<\infty, \quad -h\le z\le 0, \quad t>0, \tag{7.4.42}$$

$$\left.\begin{aligned}\phi_z-\eta_t&=0\\ \phi_t+g\eta&=0\end{aligned}\right\} \quad \text{on } z=0, \quad t>0, \tag{7.4.43ab}$$

$$\phi_z=0 \quad \text{on } z=-h, \ \ t>0. \tag{7.4.44}$$

The initial conditions are

$$\phi(r,0,0)=0 \quad \text{and} \quad \eta(r,0)=\eta_0(r), \qquad \text{for } 0\le r<\infty, \tag{7.4.45}$$

where $g$ is the acceleration due to gravity and $\eta_0(r)$ is the given free surface elevation.

We apply the joint Laplace and the zero-order Hankel transform defined by

$$\bar{\tilde{\phi}}(k,z,s)=\int_0^\infty e^{-st}dt\int_0^\infty rJ_0(kr)\phi(r,z,t)dr, \tag{7.4.46}$$

to (7.4.42)–(7.4.44) so that these equations reduce to

$$\left(\frac{d^2}{dz^2}-k^2\right)\bar{\tilde{\phi}}=0,$$

$$\left.\begin{aligned}\frac{d\bar{\tilde{\phi}}}{dz}-s\,\bar{\tilde{\eta}}&=-\tilde{\eta}_0(k)\\ s\,\bar{\tilde{\phi}}+g\,\bar{\tilde{\eta}}&=0\end{aligned}\right\} \quad \text{on } z=0,$$

$$\bar{\tilde{\phi}}_z=0 \qquad\qquad \text{on } z=-h,$$

where $\tilde{\eta}_0(k)$ is the Hankel transform of $\eta_0(r)$ of order zero.

The solutions of this system are

$$\bar{\tilde{\phi}}(k,z,s)=-\frac{g\,\tilde{\eta}_0(k)}{(s^2+\omega^2)}\frac{\cosh k(z+h)}{\cosh kh}, \tag{7.4.47}$$

$$\bar{\tilde{\eta}}(k,s)=\frac{s\,\tilde{\eta}_0(k)}{(s^2+\omega^2)}, \tag{7.4.48}$$

where

$$\omega^2 = gk\ \tanh(kh), \tag{7.4.49}$$

is the famous *dispersion relation* between frequency $\omega$ and wavenumber $k$ for water waves in a liquid of depth $h$. Physically, this dispersion relation describes the interaction between the inertial and gravitational forces.

Application of the inverse transforms gives the integral solutions

$$\phi(r,z,t) = -g\int_0^\infty kJ_0(kr)\tilde{\eta}_0(k)\left(\frac{\sin\omega t}{\omega}\right)\frac{\cosh k(z+h)}{\cosh kh}dk, \tag{7.4.50}$$

$$\eta(r,t) = \int_0^\infty kJ_0(kr)\tilde{\eta}_0(k)\cos\omega t\, dk. \tag{7.4.51}$$

These wave integrals represent exact solutions for $\phi$ and $\eta$ at any $r$ and $t$, but the physical features of the wave motions cannot be described by them. In general, the exact evaluation of the integrals is almost a formidable task. In order to resolve this difficulty, it is necessary and useful to resort to asymptotic methods. It will be sufficient for the determination of the basic features of the wave motions to evaluate (7.4.50) or (7.4.51) asymptotically for a large time and distance with $(r/t)$ held fixed. We now replace $J_0(kr)$ by its asymptotic formula (7.4.30) for $kr\to\infty$, so that (7.4.51) gives

$$\begin{aligned}\eta(r,t) &\sim \left(\frac{2}{\pi r}\right)^{\frac{1}{2}}\int_0^\infty \sqrt{k}\,\tilde{\eta}_0(k)\cos\left(kr-\frac{\pi}{4}\right)\cos\omega t\, dk \\ &= (2\pi r)^{-\frac{1}{2}}\,\mathrm{Re}\int_0^\infty \sqrt{k}\,\tilde{\eta}_0(k)\exp\left[i\left(\omega t - kr + \frac{\pi}{4}\right)\right]\,dk. \end{aligned} \tag{7.4.52}$$

Application of the stationary phase method to (7.4.52) yields the solution

$$\eta(r,t)\sim\left[\frac{k_1}{rt|\omega''(k_1)|}\right]^{\frac{1}{2}}\tilde{\eta}_0(k_1)\cos[t\,\omega(k_1)-k_1 r], \tag{7.4.53}$$

where the stationary point $k_1 = \left(gt^2/4r^2\right)$ is the root of the equation

$$\omega'(k) = \frac{r}{t}. \tag{7.4.54}$$

For sufficiently deep water, $kh\to\infty$, the dispersion relation becomes

$$\omega^2 = gk. \tag{7.4.55}$$

The solution of the axisymmetric Cauchy-Poisson problem is based on a prescribed initial displacement of unit volume that is concentrated at the origin, which means that $\eta_0(r) = (a/2\pi r)\delta(r)$ so that $\tilde{\eta}_0(k) = \dfrac{a}{2\pi}$. Thus, the asymptotic solution is obtained from (7.4.53) in the form

$$\eta(r,t) \sim \frac{agt^2}{4\pi\sqrt{2}\,r^3}\cos\left(\frac{gt^2}{4r}\right), \quad gt^2 >> 4r. \tag{7.4.56}$$

It is noted that solution (7.4.53) is no longer valid when $\omega''(k_1) = 0$. This case can be handled by a modification of the asymptotic evaluation (see Debnath, 1994, p. 91).

A wide variety of other physical problems solved by the Hankel transform, and/or by the joint Hankel and Laplace transform are given in books by Sneddon (1951, 1972) and by Debnath (1994), and in research papers by Debnath (1969, 1983, 1989), Mohanti (1979), and Debnath and Rollins (1992) listed in the Bibliography. □

## 7.5 Exercises

1. Show that

(a) $\mathscr{H}_0\{(a^2 - r^2)H(a-r)\} = \dfrac{4a}{\kappa^3}J_1(\kappa a) - \dfrac{2a^2}{\kappa^2}J_0(a\kappa)$,

(b) $\mathscr{H}_n\{r^n e^{-ar}\} = \dfrac{a}{\sqrt{\pi}} \cdot 2^{n+1}\Gamma\left(n + \dfrac{3}{2}\right)\kappa^n (a^2 + \kappa^2)^{-\left(n+\frac{3}{2}\right)}$,

(c) $\mathscr{H}_n\left\{\dfrac{2n}{r}f(r)\right\} = k\,\mathscr{H}_{n-1}\{f(r)\} + k\,\mathscr{H}_{n+1}\{f(r)\}$.

2. (a) Show that the solution of the boundary value problem

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad 0 < z < \infty,$$
$$u(r,z) = \frac{1}{\sqrt{a^2 + r^2}} \quad \text{on z} = 0, \quad 0 < \text{r} < \infty,$$

is

$$u(r,z) = \int_0^\infty e^{-\kappa(z+a)} J_0(\kappa r)\, d\kappa = \frac{1}{\sqrt{(z+a)^2 + r^2}}.$$

(b) Obtain the solution of the equation in 2($a$) with $u(r,0) = f(r) = H(a-r), \quad 0 < r < \infty$.

3. (a) The axisymmetric initial value problem is governed by

$$u_t = \kappa\left(u_{rr} + \frac{1}{r}u_r\right) + \delta(t)f(r), \quad 0 < r < \infty, \quad t > 0,$$
$$u(r,0) = 0 \quad \text{for} \quad 0 < r < \infty.$$

Show that the formal solution of this problem is

$$u(r,t) = \int_0^\infty k J_0(kr)\tilde{f}(k)\exp(-k^2\kappa t)dk.$$

(b) For the special case when $f(r) = \left(\dfrac{Q}{\pi a^2}\right)H(a-r)$, show that the solution is

$$u(r,t) = \left(\frac{Q}{\pi a}\right)\int_0^\infty J_0(kr)J_1(ak)\exp(-k^2\kappa t)dk.$$

4. If $f(r) = A(a^2 + r^2)^{-\frac{1}{2}}$ where $A$ is a constant, show that the solution of the biharmonic equation described in Example 7.4.5 is

$$u(r,z) = A\frac{\{r^2 + (z+a)(2z+a)\}}{[r^2 + (z+a)^2]^{3/2}}.$$

5. Show that the solution of the boundary value problem

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \quad 0 \le r < \infty, \quad z > 0,$$
$$u(r,0) = u_0 \quad \text{for } 0 \le r \le a, \quad u_0 \text{ is a constant},$$
$$u(r,z) \to 0 \quad \text{as } z \to \infty,$$

is

$$u(r,z) = a\,u_0\int_0^\infty J_1(ak)J_0(kr)\exp(-kz)dk.$$

Find the solution of the problem when $u_0$ is replaced by an arbitrary function $f(r)$, and $a$ by infinity.

6. Solve the axisymmetric biharmonic equation for the small-amplitude free vibration of a thin elastic disk

$$b^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 u + u_{tt} = 0, \quad 0 < r < \infty, \quad t > 0,$$
$$u(r,0) = f(r), \quad u_t(r,0) = 0 \quad \text{for } 0 < r < \infty,$$

where $b^2 = \left(\dfrac{D}{2\sigma h}\right)$ is the ratio of the flexural rigidity of the disk and its mass $2h\sigma$ per unit area.

7. Show that the zero-order Hankel transform solution of the axisymmetric Laplace equation

$$u_{rr} + \frac{1}{r}\,u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad -\infty < z < \infty,$$

with the boundary data

$$\lim_{r\to 0}(r^2 u) = 0, \quad \lim_{t\to 0}(2\pi r)u_r = -f(z), \quad -\infty < z < \infty,$$

is

$$\tilde{u}(k,z) = \frac{1}{4\pi k}\int_{-\infty}^{\infty} \exp\{-k|z-\zeta|\} f(\zeta)d\zeta.$$

Hence, show that

$$u(r,z) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \left\{r^2 + (z-\zeta)^2\right\}^{-\frac{1}{2}} f(\zeta)d\zeta.$$

8. Solve the nonhomogeneous diffusion problem

$$u_t = \kappa\left(u_{rr} + \frac{1}{r}u_r\right) + Q(r,t), \quad 0 < r < \infty, \quad t > 0,$$
$$u(r,0) = f(r) \qquad \text{for } 0 < r < \infty,$$

where $\kappa$ is a constant.

9. Solve the problem of the electrified unit disk in the $x{-}y$ plane with center at the origin. The electric potential $u(r,z)$ is axisymmetric and satisfies the boundary value problem

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad 0 < z < \infty,$$
$$u(r,0) = u_0, \quad 0 \le r < a,$$
$$\frac{\partial u}{\partial z} = 0, \quad \text{on } z = 0 \quad \text{for } a < r < \infty,$$
$$u(r,z) \to 0 \quad \text{as } z \to \infty \ \text{ for all } r,$$

where $u_0$ is constant. Show that the solution is

$$u(r,z) = \left(\frac{2au_0}{\pi}\right)\int_0^{\infty} J_0(kr)\left(\frac{\sin ak}{k}\right)e^{-kz}dk.$$

10. Solve the axisymmetric surface wave problem in deep water due to an oscillatory surface pressure. The governing equations are

$$\nabla^2\phi = \phi_{rr} + \frac{1}{r}\,\phi_r + \phi_{zz} = 0, \quad 0 \le r < \infty, \quad -\infty < z \le 0,$$

$$\left.\begin{aligned} &\phi_t + g\eta = -\frac{P}{\rho}\,p(r)\exp(i\omega t) \\ &\phi_z - \eta_t = 0 \end{aligned}\right\} \text{on } z = 0, \quad t > 0,$$

$$\phi(r,z,0) = 0 = \eta(r,0), \quad \text{for } 0 \le r < \infty, \quad -\infty < z \le 0.$$

11. Solve the Neumann problem for the Laplace equation

$$u_{rr} + \frac{1}{r}\,u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad 0 < z < \infty$$
$$u_z(r,0) = -\frac{1}{\pi a^2}H(a-r), \quad 0 < r < \infty$$
$$u(r,z) \to 0 \quad \text{as } z \to \infty \quad \text{for } 0 < r < \infty.$$

Show that

$$\lim_{a\to 0} u(r,z) = \frac{1}{2\pi}(r^2 + z^2)^{-\frac{1}{2}}.$$

12. Solve the Cauchy problem for the wave equation in a dissipating medium

$$u_{tt} + 2\kappa u_t = c^2\left(u_{rr} + \frac{1}{r}\,u_r\right), \quad 0 < r < \infty, \quad t > 0,$$
$$u(r,0) = f(r), \quad u_t(r,0) = g(r) \quad \text{for } 0 < r < \infty,$$

where $\kappa$ is a constant.

13. Use the joint Laplace and Hankel transform to solve the initial-boundary value problem

$$c^2\left(u_{rr} + \frac{1}{r}u_r + u_{zz}\right) = u_{tt}, \quad 0 < r < \infty, \quad 0 < z < \infty, \quad t > 0,$$
$$u_z(r,0,t) = H(a-r)H(t), \quad 0 < r < \infty, \quad t > 0,$$
$$u(r,z,t) \to 0 \quad \text{as } r \to \infty \text{ and } u(r,z,t) \to 0 \quad \text{as } z \to \infty,$$
$$u(r,z,0) = 0 = u_t(r,z,0),$$

and show that

$$u_t(r,z,t) = -ac\,H\left(t - \frac{z}{c}\right)\int_0^\infty J_1(ak)\,J_0\left\{ck\sqrt{t^2 - \frac{z^2}{c^2}}\right\}J_0(kr)dk.$$

14. Find the steady temperature $u(r,z)$ in a beam $0 \le r < \infty$, $0 \le z \le a$ when the face $z = 0$ is kept at temperature $u(r,0) = 0$, and the face $z = a$ is insulated except that heat is supplied through a circular hole such that

$$u_z(r,a) = H(b-r).$$

The temperature $u(r,z)$ satisfies the axisymmetric equation

$$u_{rr} + \frac{1}{r}\,u_r + u_{zz} = 0, \quad 0 \le r < \infty, \quad 0 \le z \le a.$$

15. Find the integral solution of the initial-boundary value problem

$$u_{rr} + \frac{1}{r}\,u_r + u_{zz} = u_t, \quad 0 \le r < \infty, \quad 0 \le z < \infty, \quad t > 0,$$

$$u(r,z,0) = 0 \qquad \text{for all } r \text{ and } z,$$

$$\left(\frac{\partial u}{\partial r}\right)_{r=0} = 0, \qquad \text{for} \quad 0 \le z < \infty, \quad t > 0,$$

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = -\frac{H(a-r)}{\sqrt{a^2+r^2}}, \qquad \text{for} \quad 0 < r < \infty, \quad 0 < t < \infty,$$

$$u(r,z,t) \to 0 \quad \text{as } r \to \infty \quad \text{or} \quad z \to \infty.$$

16. Heat is supplied at a constant rate $Q$ per unit area per unit time over a circular area of radius $a$ in the plane $z = 0$ to an infinite solid of thermal conductivity $K$, the rest of the plane is kept at zero temperature. Solve for the steady temperature field $u(r,z)$ that satisfies the Laplace equation

$$u_{rr} + \frac{1}{r}\,u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad -\infty < z < \infty,$$

with the boundary conditions

$$u \to 0 \text{ as } r \to \infty, \quad u \to 0 \quad \text{as } |z| \to \infty,$$

$$-K u_z = \left(\frac{2Q}{\pi\, a^2}\right) H(a-r) \text{ when } z = 0.$$

17. The velocity potential $\phi(r,z)$ for the flow of an inviscid fluid through a circular aperture of unit radius in a plane rigid screen satisfies the Laplace equation

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \; 0 < r < \infty$$

with the boundary conditions

$$\left.\begin{array}{lll} \phi = 1 & \text{for} & 0 < r < 1 \\ \phi_z = 0 & \text{for} & r > 1 \end{array}\right\} \text{ on } z = 0.$$

Obtain the solution of this boundary value problem.

18. Solve the Cauchy-Poisson wave problem (Debnath, 1989) for a viscous liquid of finite or infinite depth governed by the equations, free surface, boundary, and initial conditions

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0,$$
$$\psi_t = \nu\left(\psi_{rr} + \frac{1}{r}\psi_r - \frac{1}{r^2}\psi + \psi_{zz}\right),$$

where $\phi(r,z,t)$ and $\psi(r,z,t)$ represent the potential and stream functions, respectively, $0 \le r < \infty$, $-h \le z \le 0$ (or $-\infty < z \le 0$) and $t > 0$.

The free surface conditions are

$$\left.\begin{aligned} &\eta_t - w = 0 \\ &\mu(u_z + w_r) = 0 \\ &\phi_t + g\eta + 2\nu\, w_z = 0 \end{aligned}\right\} \quad \text{on} \quad z = 0,\ t > 0$$

where $\eta = \eta(r,t)$ is the free surface elevation, $u = \phi_r + \psi_z$ and $w = \phi_z - \frac{\psi}{r} - \psi_r$ are the radial and vertical velocity components of liquid particles, $\mu = \rho\nu$ is the dynamic viscosity, $\rho$ is the density, and $\nu$ is the kinematic viscosity of the liquid.

The boundary conditions at the rigid bottom are

$$\left.\begin{aligned} &u = \phi_r + \psi_z = 0 \\ &w = \phi_z - \frac{1}{r}(r\psi)_r = 0 \end{aligned}\right\} \text{ on } z = -h.$$

The initial conditions are

$$\eta = a\,\frac{\delta(r)}{r}, \quad \phi = \psi = 0 \quad \text{at } t = 0,$$

where $a$ is a constant and $\delta(r)$ is the Dirac delta function.

If the liquid is of infinite depth, the bottom boundary conditions are

$$(\phi, \psi) \to (0,0) \qquad \text{as } z \to -\infty.$$

19. Use the joint Hankel and Laplace transform method to solve the initial-boundary value problem

$$u_{rr} + \frac{1}{r}\,u_r - u_{tt} - 2\varepsilon\, u_t = a\,\frac{\delta(r)}{r}\,\delta(t), \quad 0 < r < \infty, \quad t > 0,$$
$$u(r,t) \to 0 \qquad \text{as } r \to \infty,$$
$$u(0,t) \text{ is finite for } t > 0,$$
$$u(r,0) = 0 = u_t(r,0) \qquad \text{for } 0 < r < \infty.$$

20. Surface waves are generated in an inviscid liquid of infinite depth due to an explosion (Sen, 1963) above it, which generates the pressure field $p(r,t)$. The velocity potential $u=\phi(r,z,t)$ satisfies the Laplace equation

$$u_{rr}+\frac{1}{r}\,u_r+u_{zz}=0, \quad 0<r<\infty, \quad t>0,$$

and the free surface condition

$$u_{tt}+g\,u_z=\frac{1}{\rho}\left(\frac{\partial p}{\partial t}\right)[H(r)-H\{r,r_0(t)\}] \qquad \text{on } z=0,$$

where $\rho$ is the constant density of the liquid, $r_0(t)$ is the extent of the blast, and the liquid is initially at rest.

Solve this problem.

21. The electrostatic potential $u(r,z)$ generated in the space between two horizontal disks at $z=\pm a$ by a point charge $q$ at $r=z=0$ is described by a singular function at $r=z=0$ is

$$u(r,z)=\phi(r,z)+q(r^2+z^2)^{-\frac{1}{2}},$$

where $\phi(r,z)$ satisfies the Laplace equation

$$\phi_{rr}+\frac{1}{r}\,\phi_r+\phi_{zz}=0, \quad 0<r<\infty$$

with the boundary conditions

$$\phi(r,z)=-q(r^2+z^2)^{-\frac{1}{2}} \text{ at } z=\pm a.$$

Obtain the solution for $\phi(r,z)$ and then $u(r,z)$.

22. Show that

(a) $\mathscr{H}_n\left[e^{-ar}f(r)\right]=\mathscr{L}\left\{rf(r)J_n(kr)\right\},$

(b) $\mathscr{H}_0\left[e^{-ar^2}J_0(br)\right]=\dfrac{a}{2}\exp\left(\dfrac{k^2-b^2}{4a}\right)I_0\left(\dfrac{bk}{2a}\right),$

(c) $\mathscr{H}_n\left[r^{n-1}e^{-ar}\right]=\dfrac{(2k)^n(n-\frac{1}{2})!}{\sqrt{\pi}(k^2+a^2)^{n+\frac{1}{2}}},$

(d) $\mathscr{H}_n\left[\dfrac{f(r)}{r}\right]=\left(\dfrac{k}{2n}\right)\left[\tilde{f}_{n-1}(k)+\tilde{f}_{n+1}(k)\right],$

(e) $\mathscr{H}_n\left[r^{n-1}\dfrac{d}{dr}\left\{r^{1-n}f(r)\right\}\right]=-k\tilde{f}_{n-1}(k),$

(f) $\mathscr{H}_n\left[r^{-(n+1)}\dfrac{d}{dr}\left\{r^{n+1}f(r)\right\}\right]=k\tilde{f}_{n+1}(k).$

# 8

## *Mellin Transforms and Their Applications*

"One cannot understand ... the universality of laws of nature, the relationship of things, without an understanding of mathematics. There is no other way to do it."

Richard P. Feynman

"The research worker, in his efforts to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty. He should take simplicity into consideration in a subordinate way to beauty. ... It often happens that the requirements of simplicity and beauty are the same, but where they clash the latter must take precedence."

Paul Dirac

## 8.1 Introduction

This chapter deals with the theory and applications of the Mellin transform. We derive the Mellin transform and its inverse from the complex Fourier transform. This is followed by several examples and the basic operational properties of Mellin transforms. We discuss several applications of Mellin transforms to boundary value problems and to summation of infinite series. The Weyl transform and the Weyl fractional derivatives with examples are also included.

Historically, Riemann (1876) first recognized the *Mellin transform* in his famous memoir on prime numbers. Its explicit formulation was given by C-ahen (1894). Almost simultaneously, Mellin (1896, 1902) gave an elaborate discussion of the Mellin transform and its inversion formula.

## 8.2 Definition of the Mellin Transform and Examples

We derive the Mellin transform and its inverse from the complex Fourier transform and its inverse, which are defined respectively by

$$\mathscr{F}\{g(\xi)\} = G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\xi} g(\xi) d\xi, \tag{8.2.1}$$

$$\mathscr{F}^{-1}\{G(k)\} = g(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik\xi} G(k) dk. \tag{8.2.2}$$

Making the changes of variables $\exp(\xi) = x$ and $ik = c - p$, where $c$ is a constant, in results (8.2.1) and (8.2.2) we obtain

$$G(ip - ic) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x^{p-c-1} g(\log x) dx, \tag{8.2.3}$$

$$g(\log x) = \frac{1}{\sqrt{2\pi}} \int_{c-i\infty}^{c+i\infty} x^{c-p} G(ip - ic) dp. \tag{8.2.4}$$

We now write $\frac{1}{\sqrt{2\pi}} x^{-c} g(\log x) \equiv f(x)$ and $G(ip - ic) \equiv \tilde{f}(p)$ to define the *Mellin transform* of $f(x)$ and the *inverse Mellin transform* as

$$\mathscr{M}\{f(x)\} = \tilde{f}(p) = \int_{0}^{\infty} x^{p-1} f(x) dx, \tag{8.2.5}$$

$$\mathscr{M}^{-1}\{\tilde{f}(p)\} = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \tilde{f}(p) dp, \tag{8.2.6}$$

where $f(x)$ is a real valued function defined on $(0, \infty)$ and the Mellin transform variable $p$ is a complex number. Sometimes, the Mellin transform of $f(x)$ is denoted explicitly by $\tilde{f}(p) = \mathscr{M}[f(x), p]$. Obviously, $\mathscr{M}$ and $\mathscr{M}^{-1}$ are linear integral operators.

***Example 8.2.1***

(a) If $f(x) = e^{-nx}$, where $n > 0$, then

$$\mathscr{M}\{e^{-nx}\} = \tilde{f}(p) = \int_{0}^{\infty} x^{p-1} e^{-nx} dx,$$

which is, by putting $nx = t$,

$$= \frac{1}{n^p}\int_0^\infty t^{p-1}e^{-t}dt = \frac{\Gamma(p)}{n^p}. \tag{8.2.7}$$

(b) If $f(x) = \dfrac{1}{1+x}$, then

$$\mathscr{M}\left\{\frac{1}{1+x}\right\} = \tilde{f}(p) = \int_0^\infty x^{p-1}\cdot\frac{dx}{1+x},$$

which is, by substituting $x = \dfrac{t}{1-t}$ or $t = \dfrac{x}{1+x}$,

$$= \int_0^1 t^{p-1}(1-t)^{(1-p)-1}dt = B(p, 1-p) = \Gamma(p)\Gamma(1-p),$$

which is, by a well-known result for the gamma function,

$$= \pi \operatorname{cosec}(p\pi), \qquad 0 < \operatorname{Re}(p) < 1. \tag{8.2.8}$$

(c) If $f(x) = (e^x - 1)^{-1}$, then

$$\mathscr{M}\left\{\frac{1}{e^x-1}\right\} = \tilde{f}(p) = \int_0^\infty x^{p-1}\frac{1}{e^x-1}dx,$$

which is, by using $\displaystyle\sum_{n=0}^\infty e^{-nx} = \frac{1}{1-e^{-x}}$ and hence, $\displaystyle\sum_{n=1}^\infty e^{-nx} = \frac{1}{e^x-1}$,

$$= \sum_{n=1}^\infty \int_0^\infty x^{p-1}e^{-nx}dx = \sum_{n=1}^\infty \frac{\Gamma(p)}{n^p} = \Gamma(p)\zeta(p), \tag{8.2.9}$$

where $\zeta(p) = \displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$, $(\operatorname{Re} p > 1)$ is the famous *Riemann zeta function.*

(d) If $f(x) = \dfrac{2}{e^{2x}-1}$, then

$$\begin{aligned}\mathscr{M}\left\{\frac{2}{e^{2x}-1}\right\} &= \tilde{f}(p) = 2\int_0^\infty x^{p-1}\frac{dx}{e^{2x}-1} = 2\sum_{n=1}^\infty \int_0^\infty x^{p-1}e^{-2nx}dx \\ &= 2\sum_{n=1}^\infty \frac{\Gamma(p)}{(2n)^p} = 2^{1-p}\,\Gamma(p)\sum_{n=1}^\infty \frac{1}{n^p} = 2^{1-p}\,\Gamma(p)\zeta(p). \quad (8.2.10)\end{aligned}$$

(e) If $f(x) = \dfrac{1}{e^x+1}$, then

$$\mathscr{M}\left\{\frac{1}{e^x+1}\right\} = (1-2^{1-p})\Gamma(p)\,\zeta(p). \tag{8.2.11}$$

This follows from the result

$$\left[\frac{1}{e^x-1} - \frac{1}{e^x+1}\right] = \frac{2}{e^{2x}-1}$$

combined with (8.2.9) and (8.2.10).

(f) If $f(x) = \dfrac{1}{(1+x)^n}$, then

$$\mathscr{M}\left\{\frac{1}{(1+x)^n}\right\} = \int_0^\infty x^{p-1}(1+x)^{-n}\,dx,$$

which is, by putting $x = \dfrac{t}{1-t}$ or $t = \dfrac{x}{1+x}$,

$$= \int_0^1 t^{p-1}(1-t)^{n-p-1}\,dt$$

$$= B(p, n-p) = \frac{\Gamma(p)\Gamma(n-p)}{\Gamma(n)}, \tag{8.2.12}$$

where $B(p,q)$ is the standard beta function.

Hence,

$$\mathscr{M}^{-1}\{\Gamma(p)\Gamma(n-p)\} = \frac{\Gamma(n)}{(1+x)^n}.$$

(g) Find the Mellin transform of $\cos kx$ and $\sin kx$.

It follows from Example 8.2.1(a) that

$$\mathscr{M}\left[e^{-ikx}\right] = \frac{\Gamma(p)}{(ik)^p} = \frac{\Gamma(p)}{k^p}\left(\cos\frac{p\pi}{2} - i\sin\frac{p\pi}{2}\right).$$

Separating real and imaginary parts, we find

$$\mathscr{M}\left[\cos kx\right] = k^{-p}\,\Gamma(p)\cos\left(\frac{\pi p}{2}\right), \tag{8.2.13}$$

$$\mathscr{M}\left[\sin kx\right] = k^{-p}\,\Gamma(p)\sin\left(\frac{\pi p}{2}\right). \tag{8.2.14}$$

These results can be used to calculate the Fourier cosine and Fourier sine transforms of $x^{p-1}$. Result (8.2.13) can be written as

$$\int_0^\infty x^{p-1}\cos kx\,dx = \frac{\Gamma(p)}{k^p}\cos\left(\frac{\pi p}{2}\right).$$

Or, equivalently,

$$\mathscr{F}_c\left\{\sqrt{\frac{\pi}{2}}\,x^{p-1}\right\} = \frac{\Gamma(p)}{k^p}\cos\left(\frac{\pi p}{2}\right).$$

Or,

$$\mathscr{F}_c\left\{x^{p-1}\right\} = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(p)}{k^p}\cos\left(\frac{\pi p}{2}\right). \tag{8.2.15}$$

Similarly,

$$\mathscr{F}_s\left\{x^{p-1}\right\} = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(p)}{k^p}\sin\left(\frac{\pi p}{2}\right). \tag{8.2.16}$$

□

## 8.3 Basic Operational Properties of Mellin Transforms

If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$, then the following operational properties hold:

(a) (*Scaling Property*).

$$\mathscr{M}\left\{f(ax)\right\} = a^{-p}\tilde{f}(p), \;\; a>0. \tag{8.3.1}$$

**PROOF** By definition, we have,

$$\mathscr{M}\left\{f(ax)\right\} = \int_0^\infty x^{p-1} f(ax)dx,$$

which is, by substituting $ax = t$,

$$= \frac{1}{a^p}\int_0^\infty t^{p-1} f(t)dt = \frac{\tilde{f}(p)}{a^p}.$$

∎

(b) (*Shifting Property*).

$$\mathscr{M}\left[x^a f(x)\right] = \tilde{f}(p+a). \tag{8.3.2}$$

Its proof follows from the definition.

(c)

$$\mathscr{M}\left\{f(x^a)\right\} = \frac{1}{a}\tilde{f}\left(\frac{p}{a}\right), \tag{8.3.3}$$

$$\mathscr{M}\left\{\frac{1}{x}\,f\left(\frac{1}{x}\right)\right\}=\tilde{f}(1-p), \tag{8.3.4}$$

$$\mathscr{M}\left\{(\log x)^n\,f(x)\right\}=\frac{d^n}{dp^n}\tilde{f}(p), \quad n=1,2,3,\ldots. \tag{8.3.5}$$

The proofs of (8.3.3) and (8.3.4) are easy and hence, left to the reader.

Result (8.3.5) can easily be proved by using the result

$$\frac{d}{dp}x^{p-1}=(\log x)x^{p-1}. \tag{8.3.6}$$

(d) (*Mellin Transforms of Derivatives*).

$$\mathscr{M}\left[f'(x)\right]=-(p-1)\tilde{f}(p-1), \tag{8.3.7}$$

provided $[x^{p-1}f(x)]$ vanishes as $x\to 0$ and as $x\to\infty$.

$$\mathscr{M}\left[f''(x)\right]=(p-1)(p-2)\tilde{f}(p-2). \tag{8.3.8}$$

More generally,

$$\begin{aligned}\mathscr{M}\left[f^{(n)}(x)\right]&=(-1)^n\frac{\Gamma(p)}{\Gamma(p-n)}\tilde{f}(p-n)\\&=(-1)^n\frac{\Gamma(p)}{\Gamma(p-n)}\,\mathscr{M}\left[f(x),p-n\right],\end{aligned} \tag{8.3.9}$$

provided $x^{p-r-1}f^{(r)}(x)=0$ as $x\to 0$ for $r=0,1,2,\ldots,(n-1)$.

**PROOF** We have, by definition,

$$\mathscr{M}\left[f'(x)\right]=\int_0^\infty x^{p-1}f'(x)\,dx,$$

which is, integrating by parts,

$$\begin{aligned}&=\left[x^{p-1}f(x)\right]_0^\infty-(p-1)\int_0^\infty x^{p-2}f(x)\,dx\\&=-(p-1)\tilde{f}(p-1).\end{aligned}$$

∎

The proofs of (8.3.8) and (8.3.9) are similar and left to the reader.

(e) If $\mathscr{M}\{f(x)\}=\tilde{f}(p)$, then

$$\mathscr{M}\left\{xf'(x)\right\}=-p\tilde{f}(p), \tag{8.3.10}$$

provided $x^p f(x)$ vanishes at $x=0$ and as $x\to\infty$.

$$\mathscr{M}\{x^2 f''(x)\} = (-1)^2 p(p+1)\tilde{f}(p). \tag{8.3.11}$$

More generally,

$$\mathscr{M}\{x^n f^{(n)}(x)\} = (-1)^n \frac{\Gamma(p+n)}{\Gamma(p)}\tilde{f}(p). \tag{8.3.12}$$

**PROOF** We have, by definition,

$$\mathscr{M}\{xf'(x)\} = \int_0^\infty x^p f'(x)dx,$$

which is, integrating by parts,

$$= [x^p f(x)]_0^\infty - p\int_0^\infty x^{p-1} f(x)dx = -p\tilde{f}(p).$$

▮

Similar arguments can be used to prove results (8.3.11) and (8.3.12).

(f) (*Mellin Transforms of Differential Operators*).
If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$, then

$$\mathscr{M}\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] = \mathscr{M}[x^2 f''(x) + xf'(x)] = (-1)^2 p^2 \tilde{f}(p), \tag{8.3.13}$$

and more generally,

$$\mathscr{M}\left[\left(x\frac{d}{dx}\right)^n f(x)\right] = (-1)^n p^n \tilde{f}(p). \tag{8.3.14}$$

**PROOF** We have, by definition,

$$\begin{aligned}\mathscr{M}\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] &= \mathscr{M}[x^2 f''(x) + x\,f'(x)] \\ &= \mathscr{M}[x^2 f''(x)] + \mathscr{M}[x\,f'(x)] \\ &= -p\tilde{f}(p) + p(p+1)\tilde{f}(p) \qquad \text{by (8.3.10) and (8.3.11)} \\ &= (-1)^2\,p^2\,\tilde{f}(p).\end{aligned}$$

▮

Similar arguments can be used to prove the general result (8.3.14).

(g) (*Mellin Transforms of Integrals*).

$$\mathscr{M}\left\{\int_0^x f(t)dt\right\} = -\frac{1}{p}\,\tilde{f}(p+1). \tag{8.3.15}$$

In general,

$$\mathscr{M}\{I_n\, f(x)\} = \mathscr{M}\left\{\int_0^x I_{n-1} f(t)dt\right\} = (-1)^n \frac{\Gamma(p)}{\Gamma(p+n)}\tilde{f}(p+n), \tag{8.3.16}$$

where $I_n\, f(x)$ is the $n$th repeated integral of $f(x)$ defined by

$$I_n f(x) = \int_0^x I_{n-1} f(t)dt. \tag{8.3.17}$$

**PROOF** We write

$$F(x) = \int_0^x f(t)dt$$

so that $F'(x) = f(x)$ with $F(0) = 0$. Application of (8.3.7) with $F(x)$ as defined gives

$$\mathscr{M}\,\{f(x) = F'(x), p\} = -(p-1)\mathscr{M}\left\{\int_0^x f(t)dt,\, p-1\right\},$$

which is, replacing $p$ by $p+1$,

$$\mathscr{M}\left\{\int_0^x f(t)\,dt,\, p\right\} = -\frac{1}{p}\mathscr{M}\,\{f(x), p+1\} = -\frac{1}{p}\,\tilde{f}(p+1).$$

An argument similar to this can be used to prove (8.3.16). ∎

(h) (*Convolution Type Theorems*).
If $\mathscr{M}\,\{f(x)\} = \tilde{f}(p)$ and $\mathscr{M}\{g(x)\} = \tilde{g}(p)$, then

$$\mathscr{M}\,[f(x) * g(x)] = \mathscr{M}\left[\int_0^\infty f(\xi)\, g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right] = \tilde{f}(p)\tilde{g}(p), \tag{8.3.18}$$

$$\mathscr{M}\,[f(x) \circ g(x)] = \mathscr{M}\left[\int_0^\infty f(x\xi)\, g(\xi)d\xi\right] = \tilde{f}(p)\tilde{g}(1-p). \tag{8.3.19}$$

**PROOF** We have, by definition,

$$\begin{aligned}
\mathscr{M}\left[f(x)*g(x)\right] &= \mathscr{M}\left[\int_0^\infty f(\xi)\,g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right] \\
&= \int_0^\infty x^{p-1}dx\int_0^\infty f(\xi)\,g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi} \\
&= \int_0^\infty f(\xi)\,\frac{d\xi}{\xi}\int_0^\infty x^{p-1}g\left(\frac{x}{\xi}\right)\,dx, \quad \left(\frac{x}{\xi}=\eta\right), \\
&= \int_0^\infty f(\xi)\,\frac{d\xi}{\xi}\int_0^\infty (\xi\eta)^{p-1}g(\eta)\,\xi\,d\eta \\
&= \int_0^\infty \xi^{p-1}f(\xi)d\xi\int_0^\infty \eta^{p-1}g(\eta)d\eta = \tilde{f}(p)\tilde{g}(p).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathscr{M}\left[f(x)\circ g(x)\right] &= \mathscr{M}\left[\int_0^\infty f(x\xi)\,g(\xi)d\xi\right] \\
&= \int_0^\infty x^{p-1}dx\int_0^\infty f(x\xi)\,g(\xi)d\xi, \quad (x\xi=\eta), \\
&= \int_0^\infty g(\xi)d\xi\int_0^\infty \eta^{p-1}\xi^{1-p}f(\eta)\frac{d\eta}{\xi} \\
&= \int_0^\infty \xi^{1-p-1}g(\xi)d\xi\int_0^\infty \eta^{p-1}f(\eta)d\eta = \tilde{g}(1-p)\tilde{f}(p).
\end{aligned}$$

∎

Note that, in this case, the operation $\circ$ is not commutative.
Clearly, putting $x=s$,

$$\mathscr{M}^{-1}\{\tilde{f}(1-p)\tilde{g}(p)\} = \int_0^\infty g(st)f(t)dt.$$

Putting $g(t)=e^{-t}$ and $\tilde{g}(p)=\Gamma(p)$, we obtain the Laplace transform of $f(t)$

$$\mathscr{M}^{-1}\left\{\tilde{f}(1-p)\Gamma(p)\right\} = \int_0^\infty e^{-st}f(t)dt = \mathscr{L}\left\{f(t)\right\} = \bar{f}(s). \tag{8.3.20}$$

(i) (*Parseval's Type Property*).
If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$ and $\mathscr{M}\{g(x)\} = \tilde{g}(p)$, then

$$\mathscr{M}[f(x)g(x)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds. \tag{8.3.21}$$

Or, equivalently,

$$\int_0^\infty x^{p-1} f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds. \tag{8.3.22}$$

In particular, when $p=1$, we obtain the *Parseval formula* for the Mellin transform,

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(1-s)ds. \tag{8.3.23}$$

**PROOF** By definition, we have

$$\begin{aligned}
\mathscr{M}[f(x)g(x)] &= \int_0^\infty x^{p-1} f(x)g(x)dx \\
&= \frac{1}{2\pi i}\int_0^\infty x^{p-1} g(x)dx \int_{c-i\infty}^{c+i\infty} x^{-s}\tilde{f}(s)ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)ds \int_0^\infty x^{p-s-1} g(x)dx \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds.
\end{aligned}$$

When $p=1$, the above result becomes (8.3.23). ∎

## 8.4 Applications of Mellin Transforms

***Example 8.4.1***

Obtain the solution of the boundary value problem

$$x^2 u_{xx} + x u_x + u_{yy} = 0, \qquad 0 \le x < \infty, \qquad 0 < y < 1 \tag{8.4.1}$$

$$u(x,0) = 0, \quad u(x,1) = \left\{\begin{matrix} A, & 0 \le x \le 1 \\ 0, & x > 1 \end{matrix}\right\}, \tag{8.4.2}$$

where $A$ is a constant.

We apply the Mellin transform of $u(x,\, y)$ with respect to $x$ defined by

$$\tilde{u}(p,y) = \int_0^\infty x^{p-1} u(x,y)\, dx$$

to reduce the given system into the form

$$\tilde{u}_{yy} + p^2 \tilde{u} = 0, \quad 0 < y < 1$$

$$\tilde{u}(p,0) = 0, \quad \tilde{u}(p,1) = A \int_0^1 x^{p-1} dx = \frac{A}{p}.$$

The solution of the transformed problem is

$$\tilde{u}(p,y) = \frac{A}{p} \frac{\sin\, py}{\sin\, p}, \qquad 0 < \operatorname{Re} p < 1.$$

The inverse Mellin transform gives

$$u(x,y) = \frac{A}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{-p}}{p} \frac{\sin\, py}{\sin\, p}\, dp, \tag{8.4.3}$$

where $\tilde{u}(p,y)$ is analytic in the vertical strip $0 < \operatorname{Re}(p) = c < \pi$. The integrand of (8.4.3) has simple poles at $p = n\pi,\ n = 1,2,3,\dots$ which lie inside a semicircular contour in the right half plane. Evaluating (8.4.3) by theory of residues gives the solution for $x > 1$ as

$$u(x,y) = \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n\, x^{-n\pi} \sin\, n\pi y. \tag{8.4.4}$$

□

***Example 8.4.2***
(*Potential in an Infinite Wedge*). Find the potential $\phi(r,\theta)$ that satisfies the Laplace equation

$$r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0 \tag{8.4.5}$$

in an infinite wedge $0 < r < \infty$, $-\alpha < \theta < \alpha$ as shown in Figure 8.1 with the boundary conditions

$$\phi(r,\alpha) = f(r), \;\; \phi(r,-\alpha) = g(r) \quad 0 \le r < \infty, \tag{8.4.6ab}$$

$$\phi(r,\theta) \to 0 \;\text{ as }\; r \to \infty \;\text{ for all }\; \theta \;\text{ in }\; -\alpha < \theta < \alpha. \tag{8.4.7}$$

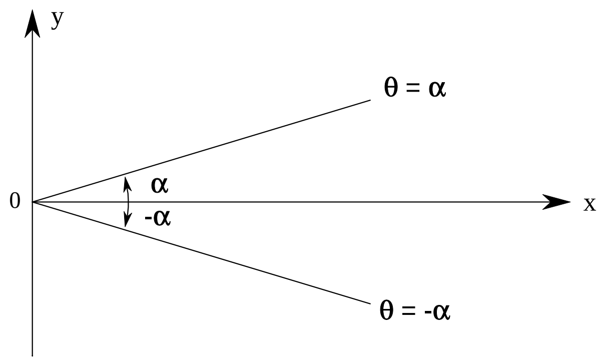


**Figure 8.1** An infinite wedge.

We apply the Mellin transform of the potential $\phi(r,\theta)$ defined by

$$\mathscr{M}\,[\phi(r,\theta)] = \tilde{\phi}(p,\theta) = \int_0^\infty r^{p-1}\phi(r,\theta)\,dr$$

to the differential system (8.4.5)–(8.4.7) to obtain

$$\frac{d^2\tilde{\phi}}{d\theta^2} + p^2\tilde{\phi} = 0, \tag{8.4.8}$$

$$\tilde{\phi}(p,\alpha) = \tilde{f}(p), \qquad \tilde{\phi}(p,-\alpha) = \tilde{g}(p). \tag{8.4.9ab}$$

The general solution of the transformed equation is

$$\tilde{\phi}(p,\theta) = A\cos p\theta + B\,\sin\,p\theta, \tag{8.4.10}$$

where $A$ and $B$ are functions of $p$ and $\alpha$. The boundary conditions (8.4.9ab) determine $A$ and $B$, which satisfy

$$A \cos p\alpha + B \sin p\alpha = \tilde{f}(p),$$
$$A \cos p\alpha - B \sin p\alpha = \tilde{g}(p).$$

These give $\quad A = \dfrac{\tilde{f}(p) + \tilde{g}(p)}{2\cos p\alpha}, \quad B = \dfrac{\tilde{f}(p) - \tilde{g}(p)}{2\sin p\alpha}.$

Thus, solution (8.4.10) becomes

$$\begin{aligned}\tilde{\phi}(p,\theta) &= \tilde{f}(p).\frac{\sin p(\alpha+\theta)}{\sin(2\,p\alpha)} + \tilde{g}(p)\frac{\sin p(\alpha-\theta)}{\sin(2\,p\alpha)} \\ &= \tilde{f}(p)\tilde{h}(p,\alpha+\theta) + \tilde{g}(p)\tilde{h}(p,\alpha-\theta),\end{aligned} \tag{8.4.11}$$

where

$$\tilde{h}(p,\theta) = \frac{\sin p\theta}{\sin(2\,p\alpha)}.$$

Or, equivalently,

$$h(r,\theta) = \mathscr{M}^{-1}\left\{\frac{\sin p\theta}{\sin 2\,p\alpha}\right\} = \left(\frac{1}{2\,\alpha}\right)\frac{r^n \sin n\theta}{(1 + 2\,r^n \cos n\theta + r^{2n})}, \tag{8.4.12}$$

where

$$n = \frac{\pi}{2\alpha} \quad \text{or,} \quad 2\alpha = \frac{\pi}{n}.$$

Application of the inverse Mellin transform to (8.4.11) gives

$$\phi(r,\theta) = \mathscr{M}^{-1}\left\{\tilde{f}(p)\tilde{h}(p,\alpha+\theta)\right\} + \mathscr{M}^{-1}\left\{\tilde{g}(p)\tilde{h}(p,\alpha-\theta)\right\},$$

which is, by the convolution property (8.3.18),

$$\begin{aligned}\phi(r,\theta) = \frac{r^n \cos n\theta}{2\alpha}&\left[\int_0^\infty \frac{\xi^{n-1} f(\xi)d\xi}{\xi^{2n} - 2(r\xi)^n \sin n\theta + r^{2n}}\right. \\ &\left. + \int_0^\infty \frac{\xi^{n-1} g(\xi)d\xi}{\xi^{2n} + 2(r\xi)^n \sin n\theta + r^{2n}}\right], \quad |\alpha| < \frac{\pi}{2n}.\end{aligned} \tag{8.4.13}$$

This is the formal solution of the problem.

In particular, when $f(r) = g(r)$, solution (8.4.11) becomes

$$\tilde{\phi}(p,\theta) = \tilde{f}(p)\,\frac{\cos p\theta}{\cos p\alpha} = \tilde{f}(p)\tilde{h}(p,\theta), \tag{8.4.14}$$

where

$$\tilde{h}(p,\theta) = \frac{\cos p\theta}{\cos p\alpha} = \mathscr{M}\{h(r,\theta)\}.$$

Application of the inverse Mellin transform to (8.4.14) combined with the convolution property (8.3.18) yields the solution

$$\phi(r,\theta) = \int_0^\infty f(\xi)\, h\left(\frac{r}{\xi}, \theta\right) \frac{d\xi}{\xi}, \tag{8.4.15}$$

where

$$h(r,\theta) = \mathscr{M}^{-1}\left\{\frac{\cos p\theta}{\cos p\alpha}\right\} = \left(\frac{r^n}{\alpha}\right) \frac{(1+r^{2n})\cos(n\theta)}{(1+2r^{2n}\cos 2n\theta + r^{2n})}, \tag{8.4.16}$$

and $n = \dfrac{\pi}{2\alpha}$. ▯

Some applications of the Mellin transform to boundary value problems are given by Sneddon (1951) and Tranter (1966).

***Example 8.4.3***
Solve the integral equation

$$\int_0^\infty f(\xi)\, k(x\xi) d\xi = g(x), \qquad x > 0. \tag{8.4.17}$$

Application of the Mellin transform with respect to $x$ to equation (8.4.17) combined with (8.3.19) gives

$$\tilde{f}(1-p)\tilde{k}(p) = \tilde{g}(p),$$

which gives, replacing $p$ by $1-p$,

$$\tilde{f}(p) = \tilde{g}(1-p)\tilde{h}(p),$$

where

$$\tilde{h}(p) = \frac{1}{\tilde{k}(1-p)}.$$

The inverse Mellin transform combined with (8.3.19) leads to the solution

$$f(x) = \mathscr{M}^{-1}\left\{\tilde{g}(1-p)\tilde{h}(p)\right\} = \int_0^\infty g(\xi) h(x\xi) d\xi, \tag{8.4.18}$$

provided $h(x) = \mathscr{M}^{-1}\left\{\tilde{h}(p)\right\}$ exists. Thus, the problem is formally solved.

If, in particular, $\tilde{h}(p) = \tilde{k}(p)$, then the solution of (8.4.18) becomes

$$f(x) = \int_0^\infty g(\xi)\, k(x\xi) d\xi, \tag{8.4.19}$$

provided $\tilde{k}(p)\tilde{k}(1-p) = 1$. ▯

***Example 8.4.4***
Solve the integral equation

$$\int_0^\infty f(\xi)\, g\left(\frac{x}{\xi}\right) \frac{d\xi}{\xi} = h(x), \tag{8.4.20}$$

where $f(x)$ is unknown and $g(x)$ and $h(x)$ are given functions.

Applications of the Mellin transform with respect to $x$ gives

$$\tilde{f}(p) = \tilde{h}(p)\tilde{k}(p), \qquad \tilde{k}(p) = \frac{1}{\tilde{g}(p)}.$$

Inversion, by the convolution property (8.3.18), gives the solution

$$f(x) = \mathscr{M}^{-1}\left\{\tilde{h}(p)\tilde{k}(p)\right\} = \int_0^\infty h(\xi)\, k\left(\frac{x}{\xi}\right) \frac{d\xi}{\xi}. \tag{8.4.21}$$

▯

## 8.5 Mellin Transforms of the Weyl Fractional Integral and the Weyl Fractional Derivative

**DEFINITION 8.5.1** *The Mellin transform of the Weyl fractional integral of $f(x)$ is defined by*

$$W^{-\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt, \quad 0 < \operatorname{Re}\alpha < 1, \quad x > 0. \tag{8.5.1}$$

*Often ${}_xW_\infty^{-\alpha}$ is used instead of $W^{-\alpha}$ to indicate the limits to integration. Result (8.5.1) can be interpreted as the Weyl transform of $f(t)$, defined by*

$$W^{-\alpha}[f(t)] = F(x,\alpha) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt. \tag{8.5.2}$$

We first give some simple examples of the Weyl transform.

If $f(t)=\exp(-at)$, Re $a>0$, then the Weyl transform of $f(t)$ is given by

$$W^{-\alpha}[\exp(-at)]=\frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}\exp(-at)dt,$$

which is, by the change of variable $t-x=y$,

$$=\frac{e^{-ax}}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}\exp(-ay)dy$$

which is, by letting $ay=t$,

$$W^{-\alpha}[f(t)]=\frac{e^{-ax}}{a^\alpha}\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}dt=\frac{e^{-ax}}{a^\alpha}. \tag{8.5.3}$$

Similarly, it can be shown that

$$W^{-\alpha}[t^{-\mu}]=\frac{\Gamma(\mu-\alpha)}{\Gamma(\mu)}x^{\alpha-\mu}, \qquad 0<\operatorname{Re}\alpha<\operatorname{Re}\mu. \tag{8.5.4}$$

Making reference to Gradshteyn and Ryzhik (2000, p. 424), we obtain

$$W^{-\alpha}[\sin at] = a^{-\alpha}\sin\left(ax+\frac{\pi\alpha}{2}\right), \tag{8.5.5}$$

$$W^{-\alpha}[\cos at] = a^{-\alpha}\cos\left(ax+\frac{\pi\alpha}{2}\right), \tag{8.5.6}$$

where $0<\operatorname{Re}\alpha<1$ and $a>0$.

It can be shown that, for any two positive numbers $\alpha$ and $\beta$, the Weyl fractional integral satisfies the laws of exponents

$$W^{-\alpha}[W^{-\beta}f(x)]=W^{-(\beta+\alpha)}[f(x)]=W^{-\beta}[W^{-\alpha}f(x)]. \tag{8.5.7}$$

Invoking a change of variable $t-x=y$ in (8.5.1), we obtain

$$W^{-\alpha}[f(x)]=\frac{1}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}f(x+y)dy. \tag{8.5.8}$$

We next differentiate (8.5.8) to obtain, $D=\dfrac{d}{dx}$,

$$\begin{aligned} D[W^{-\alpha}f(x)] &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\frac{\partial}{\partial x}f(x+t)dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}Df(x+t)dt \\ &= W^{-\alpha}[Df(x)]. \end{aligned} \tag{8.5.9}$$

A similar argument leads to a more general result

$$D^n[W^{-\alpha}f(x)] = W^{-\alpha}[D^n f(x)], \tag{8.5.10}$$

where $n$ is a positive integer.

Or, symbolically,

$$D^n\, W^{-\alpha} = W^{-\alpha} D^n. \tag{8.5.11}$$

We now calculate the Mellin transform of the Weyl fractional integral by putting $h(t) = t^\alpha f(t)$ and $g\left(\frac{x}{t}\right) = \frac{1}{\Gamma(\alpha)}\left(1-\frac{x}{t}\right)^{\alpha-1} H\left(1-\frac{x}{t}\right)$, where $H\left(1-\frac{x}{t}\right)$ is the Heaviside unit step function so that (8.5.1) becomes

$$F(x,\alpha) = \int_0^\infty h(t)\, g\left(\frac{x}{t}\right)\frac{dt}{t}, \tag{8.5.12}$$

which is, by the convolution property (8.3.18),

$$\tilde{F}(p,\alpha) = \tilde{h}(p)\tilde{g}(p),$$

where

$$\tilde{h}(p) = \mathscr{M}\{x^\alpha f(x)\} = \tilde{f}(p+\alpha),$$

and

$$\begin{aligned}\tilde{g}(p) &= \mathscr{M}\left\{\frac{1}{\Gamma(\alpha)}(1-x)^{\alpha-1}H(1-x)\right\}\\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 x^{p-1}(1-x)^{\alpha-1}dx = \frac{B(p,\alpha)}{\Gamma(\alpha)} = \frac{\Gamma(p)}{\Gamma(p+\alpha)}.\end{aligned}$$

Consequently,

$$\tilde{F}(p,\alpha) = \mathscr{M}\left[W^{-\alpha}f(x), p\right] = \frac{\Gamma(p)}{\Gamma(p+\alpha)}\,\tilde{f}(p+\alpha). \tag{8.5.13}$$

It is important to note that this result is an obvious extension of result 7(b) in Exercise 8.8

***DEFINITION 8.5.2*** *If $\beta$ is a positive number and $n$ is the smallest integer greater than $\beta$ such that $n-\beta=\alpha>0$, the Weyl fractional derivative of a function $f(x)$ is defined by*

$$\begin{aligned}W^\beta[f(x)] &= E^n\, W^{-(n-\beta)}[f(x)]\\ &= \frac{(-1)^n}{\Gamma(n-\beta)}\frac{d^n}{dx^n}\int_x^\infty (t-x)^{n-\beta-1}f(t)dt,\end{aligned} \tag{8.5.14}$$

where $E = -D$.

Or, symbolically,

$$W^{\beta} = E^n W^{-\alpha} = E^n W^{-(n-\beta)}. \tag{8.5.15}$$

It can be shown that, for any $\beta$,

$$W^{-\beta} W^{\beta} = I = W^{\beta} W^{-\beta}. \tag{8.5.16}$$

And, for any $\beta$ and $\gamma$, the Weyl fractional derivative satisfies the laws of exponents

$$W^{\beta}[W^{\gamma} f(x)] = W^{\beta+\gamma}[f(x)] = W^{\gamma}[W^{\beta} f(x)]. \tag{8.5.17}$$

We now calculate the Weyl fractional derivative of some elementary functions.

If $f(x) = \exp(-ax)$, $a > 0$, then the definition (8.5.14) gives

$$W^{\beta} e^{-ax} = E^n [W^{-(n-\beta)} e^{-ax}]. \tag{8.5.18}$$

Writing $n - \beta = \alpha > 0$ and using (8.5.3) yields

$$\begin{aligned} W^{\beta} e^{-ax} &= E^n [W^{-\alpha} e^{-ax}] = E^n [a^{-\alpha} e^{-ax}] \\ &= a^{-\alpha}(a^n e^{-ax}) = a^{\beta} e^{-ax}. \end{aligned} \tag{8.5.19}$$

Replacing $\beta$ by $-\alpha$ in (8.5.19) leads to result (8.5.3) as expected.

Similarly, we obtain

$$W^{\beta} x^{-\mu} = \frac{\Gamma(\beta+\mu)}{\Gamma(\mu)}\, x^{-(\beta+\mu)}. \tag{8.5.20}$$

It is easy to see that

$$W^{\beta}(\cos ax) = E[W^{-(1-\beta)} \cos ax],$$

which is, by (8.5.6),

$$= a^{\beta} \cos\left(ax - \frac{1}{2}\pi\beta\right). \tag{8.5.21}$$

Similarly,

$$W^{\beta}(\sin ax) = a^{\beta} \sin\left(ax - \frac{1}{2}\pi\beta\right), \tag{8.5.22}$$

provided $\alpha$ and $\beta$ lie between 0 and 1.

If $\beta$ is replaced by $-\alpha$, result (8.5.20)–(8.5.22) reduce to (8.5.4)–(8.5.6) respectively.

Finally, we calculate the Mellin transform of the Weyl fractional derivative with the help of (8.3.9) and find

$$\begin{aligned} \mathscr{M}[W^{\beta} f(x)] &= \mathscr{M}[E^n W^{-(n-\beta)} f(x)] = (-1)^n \mathscr{M}[D^n W^{-(n-\beta)} f(x)] \\ &= \frac{\Gamma(p)}{\Gamma(p-n)} \mathscr{M}[W^{-(n-\beta)} f(x), p-n], \end{aligned}$$

which is, by result (8.5.13),

$$\begin{aligned}
&= \frac{\Gamma(p)}{\Gamma(p-n)} \cdot \frac{\Gamma(p-n)}{\Gamma(p-\beta)} \tilde{f}(p-\beta) \\
&= \frac{\Gamma(p)}{\Gamma(p-\beta)} \mathscr{M}[f(x), p-\beta] \\
&= \frac{\Gamma(p)}{\Gamma(p-\beta)} \tilde{f}(p-\beta). \qquad (8.5.23)
\end{aligned}$$

***Example 8.5.1***

(*The Fourier Transform of the Weyl Fractional Integral*).

$$\mathscr{F}\{W^{-\alpha} f(x)\} = \exp\left(-\frac{\pi i \alpha}{2}\right) k^{-\alpha} \mathscr{F}\{f(x)\}. \qquad (8.5.24)$$

We have, by definition,

$$\begin{aligned}
\mathscr{F}\{W^{-\alpha} f(x)\} &= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-ikx} dx \int_{x}^{\infty} (t-x)^{\alpha-1} f(t) dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) dt \cdot \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} \exp(-ikx)(t-x)^{\alpha-1} dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathscr{F}\{W^{-\alpha} f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikt} f(t) dt \cdot \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} e^{ik\tau} \tau^{\alpha-1} d\tau, \quad (t-x=\tau) \\
&= \mathscr{F}\{f(x)\} \frac{1}{\Gamma(\alpha)} \mathscr{M}\{e^{ik\tau}\} \\
&= \exp\left(-\frac{\pi i \alpha}{2}\right) k^{-\alpha} \mathscr{F}\{f(x)\}.
\end{aligned}$$

In the limit as $\alpha \to 0$

$$\lim_{\alpha \to 0} \mathscr{F}\{W^{-\alpha} f(x)\} = \mathscr{F}\{f(x)\}.$$

This implies that

$$W^0\{f(x)\} = f(x).$$

We conclude this section by proving a general property of the Riemann-Liouville fractional integral operator $D^{-\alpha}$, and the Weyl fractional integral operator $W^{-\alpha}$. It follows from the definition (6.2.1) that $D^{-\alpha}f(t)$ can be expressed as the convolution

$$D^{-\alpha}f(x) = g_\alpha(t) * f(t), \tag{8.5.25}$$

where

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \; t > 0.$$

Similarly, $W^{-\alpha}f(x)$ can also be written in terms of the convolution

$$W^{-\alpha}f(x) = g_\alpha(-x) * f(x). \tag{8.5.26}$$

Then, under suitable conditions,

$$\mathscr{M}\left[D^{-\alpha}f(x)\right] = \frac{\Gamma(1-\alpha-p)}{\Gamma(1-p)}\,\tilde{f}(p+\alpha), \tag{8.5.27}$$

$$\mathscr{M}\left[W^{-\alpha}f(x)\right] = \frac{\Gamma(p)}{\Gamma(\alpha+p)}\,\tilde{f}(p+\alpha). \tag{8.5.28}$$

Finally, a formal computation gives

$$\begin{aligned}
\int_0^\infty \{D^{-\alpha}f(x)\}g(x)dx &= \frac{1}{\Gamma(\alpha)}\int_0^\infty g(x)dx \int_0^x (x-t)^{\alpha-1}f(t)dt \\
&= \int_0^\infty f(t)dt \cdot \frac{1}{\Gamma(\alpha)}\int_t^\infty (x-t)^{\alpha-1}g(x)dx \\
&= \int_0^\infty f(t)\left[W^{-\alpha}g(t)\right]dt,
\end{aligned}$$

which is, using the inner product notation,

$$\langle D^{-\alpha}f,\, g\rangle = \langle f,\, W^{-\alpha}g\rangle. \tag{8.5.29}$$

This show that $D^{-\alpha}$ and $W^{-\alpha}$ behave like adjoint operators. Obviously, this result can be used to define fractional integrals of distributions. This result is taken from Debnath and Grum (1988). ▯

## 8.6 Application of Mellin Transforms to Summation of Series

In this section we discuss a method of summation of series that is particularly associated with the work of Macfarlane (1949).

***THEOREM 8.6.1***
If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$, then

$$\sum_{n=0}^{\infty} f(n+a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p)\,\xi(p,a)dp, \tag{8.6.1}$$

where $\xi(p,a)$ is the *Hurwitz zeta function* defined by

$$\xi(p,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^p}, \qquad 0 \le a \le 1, \ \mathrm{Re}(p) > 1. \tag{8.6.2}$$

**PROOF** If follows from the inverse Mellin transform that

$$f(n+a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p)(n+a)^{-p}\,dp. \tag{8.6.3}$$

Summing this over all $n$ gives

$$\sum_{n=0}^{\infty} f(n+a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p)\,\xi(p,a)\,dp.$$

This completes the proof.

Similarly, the scaling property (8.3.1) gives

$$f(nx) = \mathscr{M}^{-1}\{n^{-p}\,\tilde{f}(p)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p}\,n^{-p}\tilde{f}(p)dp.$$

Thus,

$$\sum_{n=1}^{\infty} f(nx) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p}\tilde{f}(p)\,\zeta(p)dp = \mathscr{M}^{-1}\{\tilde{f}(p)\,\zeta(p)\}, \tag{8.6.4}$$

where $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$ is the *Riemann zeta function.*

When $x = 1$, result (8.6.4) reduces to

$$\sum_{n=1}^{\infty} f(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p)\,\zeta(p)dp. \tag{8.6.5}$$

This can be obtained from (8.6.1) when $a = 0$. ∎

***Example 8.6.1***
Show that

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-p} = (1-2^{1-p})\,\zeta(p). \tag{8.6.6}$$

Using Example 8.2.1(a), we can write the left-hand side of (8.6.6) multiplied by $t^n$ as

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-p}t^n &= \sum_{n=1}^{\infty}(-1)^{n-1}t^n\cdot\frac{1}{\Gamma(p)}\int_0^{\infty}x^{p-1}e^{-nx}dx\\ &= \frac{1}{\Gamma(p)}\int_0^{\infty}x^{p-1}dx\sum_{n=1}^{\infty}(-1)^{n-1}t^{nx}e^{-nx}\\ &= \frac{1}{\Gamma(p)}\int_0^{\infty}x^{p-1}\cdot\frac{te^{-x}}{1+te^{-x}}\cdot dx\\ &= \frac{1}{\Gamma(p)}\int_0^{\infty}x^{p-1}\cdot\frac{t}{e^x+t}\,dx.\end{aligned}$$

In the limit as $t\to 1$, the above result gives

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-p} &= \frac{1}{\Gamma(p)}\int_0^{\infty}x^{p-1}\frac{1}{e^x+1}\,dx\\ &= \frac{1}{\Gamma(p)}\mathscr{M}\left\{\frac{1}{e^x+1}\right\} = (1-2^{1-p})\,\zeta(p),\end{aligned}$$

in which result (8.2.11) is used. ▯

***Example 8.6.2***
Show that

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right) = \frac{1}{2}(\pi-a), \qquad 0<a<2\pi. \tag{8.6.7}$$

The Mellin transform of $f(x)=\left(\dfrac{\sin ax}{x}\right)$ gives

$$\begin{aligned}\mathscr{M}\left[\frac{\sin ax}{x}\right] &= \int_0^{\infty}x^{p-2}\sin ax\,dx\\ &= \mathscr{F}_s\left\{\sqrt{\frac{\pi}{2}}\,x^{p-2}\right\}\\ &= -\frac{\Gamma(p-1)}{a^{p-1}}\cos\left(\frac{\pi p}{2}\right).\end{aligned}$$

Substituting this result into (8.6.5) gives

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right)=-\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{\Gamma(p-1)}{a^{p-1}}\,\zeta(p)\,\cos\left(\frac{\pi p}{2}\right)\,dp. \tag{8.6.8}$$

We next use the well-known functional equation for the zeta function

$$(2\pi)^p\,\zeta(1-p)=2\Gamma(p)\,\zeta(p)\cos\left(\frac{\pi p}{2}\right) \tag{8.6.9}$$

in the integrand of (8.6.8) to obtain

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right)=-\frac{a}{2}\cdot\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\left(\frac{2\pi}{a}\right)^p\frac{\zeta(1-p)}{p-1}\,dp.$$

The integral has two simple poles at $p=0$ and $p=1$ with residues 1 and $-\pi/a$, respectively, and the complex integral is evaluated by calculating the residues at these poles. Thus, the sum of the series is

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right)=\frac{1}{2}(\pi-a).$$

□

## 8.7 Generalized Mellin Transforms

In order to extend the applicability of the classical Mellin transform, Naylor (1963) generalized the method of Mellin integral transforms. This generalized Mellin transform is useful for finding solutions of boundary value problems in regions bounded by the natural coordinate surfaces of a spherical or cylindrical coordinate system. They can be used to solve boundary value problems in finite regions or in infinite regions bounded internally.

The *generalized Mellin transform* of a function $f(r)$ defined in $a<r<\infty$ is introduced by the integral

$$\mathscr{M}_-\{f(r)\}=F_-(p)=\int\limits_a^{\infty}\left(r^{p-1}-\frac{a^{2p}}{r^{p+1}}\right)f(r)\,dr. \tag{8.7.1}$$

The inverse transform is given by

$$\mathscr{M}_-^{-1}\{F_-(p)\}=f(r)=\frac{1}{2\pi i}\int\limits_L r^{-p}\,F(p)\,dp,\quad r>a, \tag{8.7.2}$$

where $L$ is the line Re $p = c$, and $F(p)$ is analytic in the strip $|\mathrm{Re}(p)| = |c| < \gamma$.

By integrating by parts, we can show that

$$\mathscr{M}_{-}\left[r^2\,\frac{\partial^2 f}{\partial r^2} + r\,\frac{\partial f}{\partial r}\right] = p^2\,F_{-}(p) + 2\,p\,a^p f(a), \tag{8.7.3}$$

provided $f(r)$ is appropriately behaved at infinity. More precisely,

$$\lim_{r\to\infty}\left[(r^p - a^{2p}\,r^{-p})r f_r - p(r^p + a^{2p} r^{-p})f\right] = 0. \tag{8.7.4}$$

Obviously, this generalized transform seems to be very useful for finding the solution of boundary value problems in which $f(r)$ is prescribed on the internal boundary at $r = a$.

On the other hand, if the derivative of $f(r)$ is prescribed at $r = a$, it is convenient to define the associated integral transform by

$$\mathscr{M}_{+}[f(r)] = F_{+}(p) = \int_a^\infty \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)\,dr, \quad |\mathrm{Re}(p)| < r, \tag{8.7.5}$$

and its inverse given by

$$\mathscr{M}_{+}^{-1}[f(p)] = f(r) = \frac{1}{2\pi i}\int_L r^{-p}\,F_{+}(p)dp, \quad r > a. \tag{8.7.6}$$

In this case, we can show by integration by parts that

$$\mathscr{M}_{+}\left[r^2\,\frac{\partial^2 f}{\partial r^2} + r\,\frac{\partial f}{\partial r}\right] = p^2 F_{+}(p) - 2\,a^{p+1} f'(a), \tag{8.7.7}$$

where $f'(r)$ exists at $r = a$.

***THEOREM 8.7.1***

(*Convolution*). If $\mathscr{M}_{+}\{f(r)\} = F_{+}(p)$, and $\mathscr{M}_{+}\{g(r)\} = G_{+}(p)$, then

$$\mathscr{M}_{+}\{f(r)\,g(r)\} = \frac{1}{2\pi i}\int_L F_{+}(\xi)G_{+}(p-\xi)\,d\xi. \tag{8.7.8}$$

Or, equivalently,

$$f(r)g(r) = \mathscr{M}_{+}^{-1}\left[\frac{1}{2\pi i}\int_L F_{+}(\xi)\,G_{+}(p-\xi)d\xi\right]. \tag{8.7.9}$$

**PROOF** We assume that $F_+(p)$ and $G_+(p)$ are analytic in some strip $|\mathrm{Re}(p)| < \gamma$. Then

$$\begin{aligned}\mathscr{M}_+\{f(r)\,g(r)\} &= \int_a^\infty \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)g(r)dr \\ &= \int_a^\infty r^{p-1} f(r)g(r)dr + \int_a^\infty \frac{a^{2p}}{r^{p+1}} f(r)g(r)dr. \quad (8.7.10) \\ &= \frac{1}{2\pi i}\int_L F_+(\xi)d\xi \int_a^\infty r^{p-\xi-1} g(r)dr \\ &\quad + \frac{1}{2\pi}\int_a^\infty \frac{a^{2p}}{r^{p+1}}\, g(r)dr \int_L r^{-\xi} F_+(\xi)\, d\xi. \quad (8.7.11)\end{aligned}$$

Replacing $\xi$ by $-\xi$ in the first integral term and using $F_+(\xi) = a^{2\xi} F_+(-\xi)$, which follows from the definition (8.7.5), we obtain

$$\int_L r^{-\xi}\, F_+(\xi)d\xi = \int_L r^{\xi}\, a^{-2\xi}\, F_+(\xi)d\xi. \quad (8.7.12)$$

The path of integration $L$, $\mathrm{Re}(\xi) = c$, becomes $\mathrm{Re}(\xi) = -c$, but these paths can be reconciled if $F(\xi)$ tends to zero for large $\mathrm{Im}(\xi)$.

In view of (8.7.11), we have rewritten

$$\int_a^\infty \frac{a^{2p}}{r^{p+1}} f(r)\, g(r)dr = \frac{1}{2\pi i}\int_L F_+(\xi)d\xi \int_a^\infty \frac{a^{2p-2\xi}}{r^{p-\xi+1}}\, g(r)\, dr. \quad (8.7.13)$$

This result is used to rewrite (8.7.10) as

$$\begin{aligned}\mathscr{M}_+\{f(r)g(r)\} &= \int_a^\infty \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)g(r)dr \\ &= \int_a^\infty r^{p-1} f(r)\, g(r)\, dr + \int_a^\infty \frac{a^{2p}}{r^{p+1}} f(r)\, g(r)\, dr \\ &= \frac{1}{2\pi i}\int_L F_+(\xi)\, d\xi \int_a^\infty r^{p-\xi-1} g(r)\, dr \\ &\quad + \frac{1}{2\pi i}\int_L F_+(\xi)d\xi \int_a^\infty \frac{a^{2p-2\xi}}{r^{p-\xi+1}}\, g(r)\, dr \\ &= \frac{1}{2\pi i}\int_L F_+(\xi)\, G_+(p-\xi)\, d\xi.\end{aligned}$$

This completes the proof. ∎

If the range of integration is finite, then we define the *generalized finite Mellin transform* by

$$\mathscr{M}_{-}^{a}\{f(r)\} = F_{-}^{a}(p) = \int_{0}^{a}\left(r^{p-1} - \frac{a^{2p}}{r^{p+1}}\right) f(r)dr, \tag{8.7.14}$$

where $\operatorname{Re} p < \gamma$.

The corresponding inverse transform is given by

$$f(r) = -\frac{1}{2\pi i}\int_{L}\left(\frac{r}{a^2}\right)^{p} F_{-}^{a}(p)dp, \quad 0 < r < a,$$

which is, by replacing $p$ by $-p$ and using $F_{-}^{a}(-p) = -a^{-2p}\,F_{-}^{a}(p)$,

$$= \frac{1}{2\pi i}\int_{L} r^{-p}\,F_{-}^{a}(p)dp, \quad 0 < r < a, \tag{8.7.15}$$

where the path $L$ is Re $p = -c$ with $|c| < \gamma$.

It is easy to verify the result

$$\begin{aligned}\mathscr{M}_{-}^{a}\{r^2 f_{rr} + r f_{-r}\} &= \int_{0}^{a}\left(r^{p-1} - \frac{a^{2p}}{r^{p+1}}\right)\{r^2 f_{rr} + r f_r\}dr \\ &= p^2\,F_{-}^{a}(p) - 2\,p\,a^p\,f(a).\end{aligned} \tag{8.7.16}$$

This is a useful result for applications.

Similarly, we define the generalized finite Mellin transform-pair by

$$\mathscr{M}_{+}^{a}\{f(r)\} = F_{+}^{a}(p) = \int_{0}^{a}\left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)\,dr, \tag{8.7.17}$$

$$f(r) = \left(\mathscr{M}_{+}^{a}\right)^{-1}\left[F_{+}^{a}(p)\right] = \frac{1}{2\pi i}\int_{L} r^{-p} F_{+}^{a}(p)\,dp, \tag{8.7.18}$$

where $|\operatorname{Re} p| < \gamma$.

For this finite transform, we can also prove

$$\begin{aligned}\mathscr{M}_{+}^{a}\left[r^2 f_{rr} + r\ f_r\right] &= \int_{0}^{a}\left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right)\left(r^2 f_{rr} + r\ f_r\right)dr \\ &= p^2 F_{+}^{a}(p) + 2\,a^{p-1}\ f'(a).\end{aligned} \tag{8.7.19}$$

This result also seems to be useful for applications. The reader is referred to Naylor (1963) for applications of the above results to boundary value problems.

## 8.8 Exercises

1. Find the Mellin transform of each of the following functions:

(a) $f(x) = H(a - x),\ a > 0,$

(b) $f(x) = x^m e^{-nx}, \quad m, n > 0,$

(c) $f(x) = \dfrac{1}{1 + x^2},$

(d) $f(x) = J_0^2(x),$

(e) $f(x) = x^z H(x - x_0),$

(f) $f(x) = [H(x - x_0) - H(x)] x^z,$

(g) $f(x) = Ei(x),$

(h) $f(x) = e^x Ei(x),$

where the exponential integral is defined by

$$Ei(x) = \int_x^\infty t^{-1} e^{-t}\, dt = \int_1^\infty \xi^{-1} e^{-\xi x}\, d\xi.$$

2. Derive the Mellin transform-pairs from the bilateral Laplace transform and its inverse given by

$$\bar{g}(p) = \int_{-\infty}^{\infty} e^{-pt} g(t) dt, \qquad g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt}\, \bar{g}(p) dp.$$

3. Show that

$$\mathscr{M}\left[\frac{1}{e^x + e^{-x}}\right] = \Gamma(p)\, L(p),$$

where $L(p) = \dfrac{1}{1^p} - \dfrac{1}{3^p} + \dfrac{1}{5^p} - \cdots$ is the *Dirichlet L-function.*

4. Show that

$$\mathscr{M}\left\{\frac{1}{(1 + ax)^n}\right\} = \frac{\Gamma(p)\Gamma(n - p)}{a^p\, \Gamma(n)}.$$

5. Show that

$$\mathscr{M}\left\{x^{-n} J_n(ax)\right\} = \frac{1}{2}\left(\frac{a}{2}\right)^{n-p} \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(n - \frac{p}{2} + 1\right)}, \quad a > 0,\ n > -\frac{1}{2}.$$

6. Show that

(a) $\mathscr{M}^{-1}\left[\cos\left(\dfrac{\pi p}{2}\right) \Gamma(p)\, \tilde{f}(1 - p)\right] = \mathscr{F}_c\left\{\sqrt{\dfrac{\pi}{2}}\, f(x)\right\},$

(b) $\mathscr{M}^{-1}\left[\sin\left(\frac{\pi p}{2}\right)\Gamma(p)\,\tilde{f}(1-p)\right] = \mathscr{F}_s\left\{\sqrt{\frac{\pi}{2}}\,f(x)\right\}$.

7. If $I_n^\infty f(x)$ denotes the $n$th repeated integral of $f(x)$ defined by

$$I_n^\infty f(x) = \int_x^\infty I_{n-1}^\infty f(t)dt,$$

show that

(a) $\mathscr{M}\left[\int_x^\infty f(t)dt,\, p\right] = \frac{1}{p}\,\tilde{f}(p+1)$,

(b) $\mathscr{M}\left[I_n^\infty\ f(x)\right] = \frac{\Gamma(p)}{\Gamma(p+n)}\ \tilde{f}(p+n)$.

8. Show that the integral equation

$$f(x) = h(x) + \int_0^\infty g(x\xi)\ f(\xi)\ d\xi$$

has the formal solution

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{\tilde{h}(p) + \tilde{g}(p)\,\tilde{h}(1-p)}{1-\tilde{g}(p)\,\tilde{g}(1-p)}\right] x^{-p}\,dp.$$

9. Find the solution of the Laplace integral equation

$$\int_0^\infty e^{-x\xi}\ f(\xi)\ d\xi = \frac{1}{(1+x)^n}.$$

10. Show that the integral equation

$$f(x) = h(x) + \int_0^\infty\ f(\xi)\ g\left(\frac{x}{\xi}\right)\ \frac{d\xi}{\xi}$$

has the formal solution

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{-p}\,\tilde{h}(p)}{1-\tilde{g}(p)}\,dp.$$

11. Show that the solution of the integral equation

$$f(x) = e^{-ax} + \int_0^\infty \exp\left(-\frac{x}{\xi}\right) \; f(\xi)\, \frac{d\xi}{\xi}$$

is

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (ax)^{-p} \left\{\frac{\Gamma(p)}{1-\Gamma(p)}\right\} dp.$$

12. Assuming (see Harrington, 1967)

$$\mathscr{M}\left[f(re^{i\theta})\right] = \int_0^\infty r^{p-1} f(re^{i\theta})\; dr, \quad p \text{ is real},$$

and putting $re^{i\theta} = \xi$, $\mathscr{M}\{f(\xi)\} = F(p)$ show that

(a) $\mathscr{M}\,[f(re^{i\theta});\; r \to p] = \exp(-ip\theta)\, F(p)$.

Hence, deduce

(b) $\mathscr{M}^{-1}\{F(p)\,\cos\; p\theta\} = \mathrm{Re}[f(re^{i\theta})]$,
(c) $\mathscr{M}^{-1}\{F(p)\,\sin\; p\theta\} = -\mathrm{Im}[f(re^{i\theta})]$.

13. (a) If $\mathscr{M}\,[\exp(-r)] = \Gamma(p)$, show that

$$\mathscr{M}\left[\exp(-re^{i\theta})\right] = \Gamma(p)\, e^{-i\,p\theta},$$

(b) If $\mathscr{M}\,[\log(1+r)] = \dfrac{\pi}{p\sin\pi p}$, then show that

$$\mathscr{M}\left[\mathrm{Re}\;\log\left(1+re^{i\theta}\right)\right] = \frac{\pi\cos\; p\theta}{p\sin\pi p}.$$

14. Use $\mathscr{M}^{-1}\left\{\dfrac{\pi}{\sin p\pi}\right\} = \dfrac{1}{1+x} = f(x)$, and Exercises 12(b) and 12(c), respectively, to show that

(a) $\mathscr{M}^{-1}\left\{\dfrac{\pi\cos p\theta}{\sin p\pi};\; p \to r\right\} = \dfrac{1+r\cos\theta}{1+2r\cos\theta+r^2}$,

(b) $\mathscr{M}^{-1}\left\{\dfrac{\pi\sin p\theta}{\sin p\pi};\; p \to r\right\} = \dfrac{r\sin\theta}{1+2r\cos\theta+r^2}$.

15. Find the inverse Mellin transforms of

(a) $\Gamma(p)\cos p\theta$, where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, (b) $\Gamma(p)\sin p\theta$.

16. Obtain the solution of Example 8.4.2 with the boundary data

    (a) $\phi(r,\,\alpha)=\phi(r,\,-\alpha)=H(a-r)$.

    (b) Solve equation (8.4.5) in $0<r<\infty$, $0<\theta<\alpha$ with the boundary conditions $\phi(r,\,0)=0$ and $\phi(r,\,\alpha)=f(r)$.

17. Show that

    (a) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos kn}{n^2}=\left[\frac{k^2}{4}-\frac{\pi k}{2}+\frac{\pi^2}{6}\right]$, and (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

18. If $f(x)=\sum_{n=1}^{\infty} a_n e^{-nx}$, show that

$$\mathscr{M}\{f(x)\}=\tilde{f}(p)=\Gamma(p)\,g(p),$$

where $g(p)=\displaystyle\sum_{n=1}^{\infty} a_n\; n^{-p}$ is the Dirichlet series.
If $a_n=1$ for all $n$, derive

$$\tilde{f}(p)=\Gamma(p)\,\zeta(p).$$

Show that

$$\mathscr{M}\left\{\frac{\exp(-ax)}{1-e^{-x}}\right\}=\Gamma(p)\,\xi(p,a).$$

19. Show that

    (a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}=(1-2^{1-p})\,\zeta(p)$.

    Hence, deduce

    (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$, (c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4}=\left(\frac{7}{8}\right)\frac{\pi^4}{90}$.

20. Find the sum of the following series

    (a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\cos kn$, (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin kn$.

21. Show that the solution of the boundary value problem

$$r^2\phi_{rr}+r\phi_r+\phi_{\theta\theta}=0,\quad 0<r<\infty,\ 0<\theta<\pi$$

$$\phi(r,0)=\phi(r,\pi)=f(r),$$

is

$$\phi(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-p}\,\frac{\tilde{f}(p)\ \cos\left\{p\left(\theta - \frac{\pi}{2}\right)\right\}\,dp}{\cos\left(\frac{\pi p}{2}\right)}.$$

22. Evaluate

$$\sum_{n=1}^{\infty}\frac{\cos an}{n^3} = \frac{1}{12}(a^3 - 3\pi a^2 + 2\pi^2 a).$$

23. Prove the following results:

(a) $\mathscr{M}\left[\int_0^\infty \xi^n f(x\xi)\, g(\xi) d\xi\right] = \tilde{f}(p)\,\tilde{g}(1+n-p),$

(b) $\mathscr{M}\left[\int_0^\infty \xi^n f\left(\frac{x}{\xi}\right) g(\xi) d\xi\right] = \tilde{f}(p)\,\tilde{g}(p+n+1).$

24. Show that

(a) $W^{-\alpha}[e^{-x}] = e^{-x}, \quad \alpha > 0,$

(b) $W^{\frac{1}{2}}\left[\frac{1}{\sqrt{x}}\exp\left(-\sqrt{x}\right)\right] = \frac{K_1(\sqrt{x})}{\sqrt{\pi x}}, \quad x > 0,$

where $K_1(x)$ is the modified Bessel function of the second kind and order one.

25. (a) Show that the integral (Wong, 1989, pp. 186–187)

$$I(x) = \int_0^{\pi/2} J_\nu^2(x\cos\theta)\, d\theta, \qquad \nu > -\frac{1}{2},$$

can be written as a Mellin convolution

$$I(x) = \int_0^\infty f(x\xi)\, g(\xi)\, d\xi,$$

where

$$f(\xi) = J_\nu^2(\xi) \ \text{ and } \ \mathrm{g}(\xi) = \begin{Bmatrix} (1-\xi^2)^{-\frac{1}{2}}, & 0 < \xi < 1 \\ 0, & \xi \ge 1 \end{Bmatrix}.$$

(b) Prove that the integration contour in the Parseval identity

$$I(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}\ \tilde{f}(p)\,\tilde{g}(1-p)\ dp, \qquad -2\nu < c < 1,$$

cannot be shifted to the right beyond the vertical line $\operatorname{Re} p = 2$.

26. If $f(x)=\int_0^\infty \exp(-x^2t^2)\cdot\frac{\sin t}{t^2}J_1(t)dt$, show that

$$\mathscr{M}\{f(x)\}=\frac{\Gamma\left(p+\frac{3}{2}\right)\Gamma\left(\frac{1-p}{2}\right)}{p\,\Gamma(p+3)}.$$

27. Prove the following relations to the Laplace and the Fourier transforms:

(a) $\mathscr{M}[f(x),p]=\mathscr{L}[f(e^{-t}),p]$,

(b) $\mathscr{M}[f(x);a+i\omega]=\mathscr{F}[f(e^{-t})e^{-at};\omega]$,

where $\mathscr{L}$ is the two-sided Laplace transform and $\mathscr{F}$ is the Fourier transform without the factor $(2\pi)^{-\frac{1}{2}}$.

28. Prove the following properties of convolution:

(a) $f*g=g*f$,

(b) $(f*g)*h=f*(g*h)$,

(c) $f(x)*\delta(x-1)=f(x)$,

(d) $\delta(x-a)*f(x)=a^{-1}f\left(\frac{x}{a}\right)$,

(e) $\delta^n(n-1)*f(x)=\left(\frac{d}{dx}\right)^n(x^nf(x))$,

(f) $\left(x\frac{d}{dx}\right)^n(f*g)=\left[\left(x\frac{d}{dx}\right)^n f\right]*g=f*\left[\left(x\frac{d}{dx}\right)^n g\right]$.

29. If $\mathscr{M}\{f(r,\theta)\}=\tilde{f}(p,\theta)$ and $\nabla^2f(r,\theta)=f_{rr}+\frac{1}{r}f_r+\frac{1}{r^2}f_{\theta\theta}$, show that

$$\mathscr{M}\left\{\nabla^2f(r,\theta)\right\}=\left[\frac{d^2}{d\theta^2}+(p-2)^2\right]\tilde{f}(p-2,\theta).$$

# 9

# *Hilbert and Stieltjes Transforms*

> "The organic unity of mathematics is inherent in the nature of this science, for mathematics is the foundation of all exact knowledge of natural phenomena."
>
> David Hilbert

> "Mathematics knows no races or geographic boundaries; for mathematics the cultural world is one country."
>
> David Hilbert

## 9.1 Introduction

In his 1912 famous paper on integral equations, David Hilbert (1862−1943) introduced an integral transformation, which is now known as the *Hilbert transform.* Although it was named after Hilbert, the Hilbert transform and its basic properties were developed mainly by G.H. Hardy (1924) and simultaneously by E.C. Titchmarsh during 1925-1930. On the other hand, T.J. Stieltjes (1856−1894) introduced the Stieltjes transform in his studies on continued fractions. This transform was also involved in Stieltjes' moment problems.

Both the *Hilbert and Stieltjes transforms* arise in many problems in applied mathematics, mathematical physics, and engineering science. The former plays an important role in fluid mechanics, aerodynamics, signal processing, and electronics, while the latter arises in the moment problem. This chapter deals with definitions of Hilbert and Stieltjes transforms with examples. This is followed by a discussion of basic operational properties of these transforms. Finally, examples of applications of Hilbert and Stieltjes transforms to physical problems are discussed.

## 9.2 Definition of the Hilbert Transform and Examples

If $f(t)$ is defined on the real line $-\infty < t < \infty$, its *Hilbert transform*, denoted by $\hat{f}_{\boldsymbol{H}}(x)$, is defined by

$$\boldsymbol{H}\{f(t)\} = \hat{f}_{\boldsymbol{H}}(x) = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(t)}{t-x} dt, \tag{9.2.1}$$

where $x$ is real and the integral is treated as a Cauchy principal value, that is,

$$\oint_{-\infty}^{\infty} \frac{f(t)dt}{t-x} = \lim_{\varepsilon \to 0} \left[ \int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty} \right] \frac{f(t)dt}{t-x}. \tag{9.2.2}$$

To derive the *inverse Hilbert transform,* we rewrite (9.2.1) as

$$\hat{f}_{\boldsymbol{H}}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt = (f * g)(x), \tag{9.2.3}$$

where $g(x) = \sqrt{\frac{2}{\pi}} \left(-\frac{1}{x}\right)$. Application of the Fourier transform with respect to $x$ gives

$$F(k) = \frac{\hat{F}_{\boldsymbol{H}}(k)}{G(k)}, \quad G(k) = i\, sgn\, k. \tag{9.2.4}$$

Taking the inverse Fourier transform, we obtain the solution for $f(x)$ as

$$f(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (i\, sgn\, k) \hat{F}_{\boldsymbol{H}}(k) \exp(ikx) dk$$

which is, by the Convolution Theorem 2.5.5,

$$= \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{\hat{f}_{\boldsymbol{H}}(\xi)}{x-\xi} d\xi = -\boldsymbol{H} \left\{ \hat{f}_{\boldsymbol{H}}(\xi) \right\}. \tag{9.2.5}$$

Obviously, $-\boldsymbol{H}^2\{f(t)\} = -\boldsymbol{H}\left[\boldsymbol{H}\{f(t)\}\right] = f(x)$ and hence, $\boldsymbol{H}^{-1} = -\boldsymbol{H}$. Thus, the *inverse Hilbert transform* is given by

$$f(t) = \boldsymbol{H}^{-1} \left\{ \hat{f}_{\boldsymbol{H}}(x) \right\} = -\boldsymbol{H} \left\{ \hat{f}_{\boldsymbol{H}}(x) \right\} = -\frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{\hat{f}_{\boldsymbol{H}}(x) dx}{x-t}. \tag{9.2.6}$$

***Example 9.2.1***

Find the Hilbert transform of a rectangular pulse given by

$$f(t) = \begin{Bmatrix} 1, & \text{for } |t| < a \\ 0, & \text{for } |t| > a \end{Bmatrix}. \tag{9.2.7}$$

We have, by definition,

$$\hat{f}_{\boldsymbol{H}}(x) = \frac{1}{\pi} \int_{-a}^{a} \frac{dt}{t-x}.$$

If $|x| < a$, the integrand has a singularity at $t = x$, and hence,

$$\begin{aligned}
\hat{f}_{\boldsymbol{H}}(x) &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \left[ \int_{-a}^{x-\varepsilon} \frac{dt}{t-x} + \int_{x+\varepsilon}^{a} \frac{dt}{t-x} \right] \\
&= \frac{1}{\pi} \lim_{\varepsilon \to 0} \left\{ [\log|t-x|]_{-a}^{x-\varepsilon} + [\log|t-x|]_{x+\varepsilon}^{a} \right\} \\
&= \frac{1}{\pi} \lim_{\varepsilon \to 0} \{ \log|\varepsilon| - \log|a+x| + \log|a-x| - \log|\varepsilon| \} \\
&= \frac{1}{\pi} \log \left| \frac{a-x}{a+x} \right| \quad \text{for } |x| < a.
\end{aligned}$$

On the other hand, if $|x| > a$, the integrand has no singularity in $-a < t < a$, and hence,

$$\hat{f}_{\boldsymbol{H}}(x) = \frac{1}{\pi} \int_{-a}^{a} \frac{dt}{t-x} = \frac{1}{\pi} [\log|t-x|]_{-a}^{a} = \frac{1}{\pi} \log \left| \frac{a-x}{a+x} \right| \quad \text{for } |x| > a.$$

Finally, we obtain the Hilbert transform of $f(t)$ defined by (9.2.7) as

$$\hat{f}_{\boldsymbol{H}}(x) = \frac{1}{\pi} \log \left| \frac{a-x}{a+x} \right|. \tag{9.2.8}$$

□

***Example 9.2.2***

Find the Hilbert transform of

$$f(t) = \frac{t}{(t^2 + a^2)}, \quad a > 0. \tag{9.2.9}$$

We have, by definition,

$$\hat{f}_{\boldsymbol{H}}(x) = \frac{1}{\pi}\oint_{-\infty}^{\infty} \frac{t\ dt}{(t^2+a^2)(t-x)}$$

$$= \frac{1}{\pi(a^2+x^2)}\oint_{-\infty}^{\infty}\left[\frac{a^2}{t^2+a^2}+\frac{x}{t-x}-\frac{xt}{t^2+a^2}\right]dt$$

$$= \frac{1}{\pi}\frac{1}{(a^2+x^2)}\left[a^2\int_{-\infty}^{\infty}\frac{dt}{t^2+a^2}+x\oint_{-\infty}^{\infty}\frac{dt}{(t-x)}-x\int_{-\infty}^{\infty}\frac{t\ dt}{(t^2+a^2)}\right].$$

The second and third integrals as the Cauchy principal value vanish and hence, only the first integral makes a non-zero contribution. Thus, we obtain

$$\hat{f}_{\boldsymbol{H}}(x) = \frac{1}{\pi}\frac{1}{(a^2+x^2)}\cdot(a\pi) = \frac{a}{(a^2+x^2)}. \tag{9.2.10}$$

▯

***Example 9.2.3***
Find the Hilbert transform of

(a) $f(t) = \cos\omega t$ and (*b*) $f(t) = \sin\omega t$.

It follows from the definition of the Hilbert transform that

$$\hat{f}_{\boldsymbol{H}}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\omega t}{(t-x)}dt = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\{\omega(t-x)+\omega x\}dt}{(t-x)}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}(t-x)^{-1}[\cos\omega(t-x)\cos\omega x - \sin\omega(t-x)\sin\omega x]dt$$

$$= \frac{\cos\omega x}{\pi}\int_{-\infty}^{\infty}\frac{\cos\omega(t-x)}{t-x}dt - \frac{\sin\omega x}{\pi}\int_{-\infty}^{\infty}\frac{\sin\omega(t-x)dt}{t-x},$$

which is, in terms of the new variable $T = t - x$,

$$= \frac{\cos\omega x}{\pi}\int_{-\infty}^{\infty}\frac{\cos\omega T}{T}dT - \frac{\sin\omega x}{\pi}\int_{-\infty}^{\infty}\frac{\sin\omega T}{T}dT. \tag{9.2.11}$$

Obviously, the first integral vanishes because its integrand is an odd function of $T$. On the other hand, the second integral makes a non-zero contribution so that (9.2.11) gives

$$\boldsymbol{H}\{\cos\omega t\} = \hat{f}_{\boldsymbol{H}}(x) = -\frac{\sin\omega x}{\pi}\cdot\pi = -\sin\omega x. \tag{9.2.12}$$

Similarly, it can be shown that

$$\boldsymbol{H}\{\sin\omega t\} = \cos\omega x. \tag{9.2.13}$$

□

## 9.3 Basic Properties of Hilbert Transforms

***THEOREM 9.3.1***
If $\boldsymbol{H}\{f(t)\} = \hat{f}_{\boldsymbol{H}}(x)$, then the following properties hold:

(a) $\boldsymbol{H}\{f(t+a)\} = \hat{f}_{\boldsymbol{H}}(x+a),$ (9.3.1)

(b) $\boldsymbol{H}\{f(at)\} = \hat{f}_{\boldsymbol{H}}(ax), \quad a > 0,$ (9.3.2)

(c) $\boldsymbol{H}\{f(-at)\} = -\hat{f}_{\boldsymbol{H}}(-ax),$ (9.3.3)

(d) $\boldsymbol{H}\{f'(t)\} = \dfrac{d}{dx}\hat{f}_{\boldsymbol{H}}(x),$ (9.3.4)

(e) $\boldsymbol{H}\{t\ f(t)\} = x\ \hat{f}_{\boldsymbol{H}}(x) + \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(t)dt,$ (9.3.5)

(f) $\mathscr{F}\,[\boldsymbol{H}\{f(t)\}] = (-i\,sgn\,k)\mathscr{F}\,\{f(x)\},$ (9.3.6)

(g) $\|\boldsymbol{H}\{f(t)\}\| = \|f(t)\|,$ (9.3.7)

where $\|f\| = \sqrt{<f, f>}$ denotes the norm in $L^2(\mathbb{R})$,

(h) $\boldsymbol{H}[f](x) = \hat{f}_{\boldsymbol{H}}(x), \quad \boldsymbol{H}[\hat{f}_{\boldsymbol{H}}](x) = -f$ (Reciprocity relations), (9.3.8)

(i) $<f, \boldsymbol{H}g> = <-\boldsymbol{H}f,\ g>$ and $<\boldsymbol{H}f,\ g> = <f, -\boldsymbol{H}g>,$ (9.3.9)

(Parseval's formulas).

**PROOF** (a) We have, by definition,

$$\boldsymbol{H}\{f(t+a)\} = \frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{f(t+a)dt}{t-x} \qquad (t+a=u)$$

$$= \frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{f(u)du}{u-(x+a)} = \hat{f}_{\boldsymbol{H}}(x+a).$$

(b) $$\boldsymbol{H}\{f(at)\} = \frac{1}{\pi}\oint_{-\infty}^{\infty} \frac{f(at)dt}{t-x} \qquad (at = u,\ a > 0)$$

$$= \frac{1}{\pi}\oint_{-\infty}^{\infty} \frac{f(u)du}{u-ax} = \hat{f}_{\boldsymbol{H}}(ax).$$

Similarly, result (c) can be proved.

(d) $$\boldsymbol{H}\{f'(t)\} = \frac{1}{\pi}\oint_{-\infty}^{\infty} \frac{f'(t)dt}{t-x}$$

which is, integrating by parts,

$$= \frac{1}{\pi}\left[\frac{f(t)}{t-x}\right]_{-\infty}^{\infty} + \frac{1}{\pi}\oint_{-\infty}^{\infty} \frac{f(t)\,dt}{(t-x)^2} = \frac{d}{dx}\hat{f}_{\boldsymbol{H}}(x).$$

Proofs of (e)–(i) are similar and hence, are left to the reader. ∎

***THEOREM 9.3.2***
If $f(t)$ is an even function of $t$, then, an alternative form of the Hilbert transform is

$$\hat{f}_{\boldsymbol{H}}(x) = \frac{x}{\pi}\oint_{-\infty}^{\infty} \frac{f(t)-f(x)}{(t^2-x^2)}dt. \tag{9.3.10}$$

**PROOF** As the Cauchy principal value, we have

$$\oint_{-\infty}^{\infty} \frac{dt}{t-x} = 0.$$

Consequently,

$$\hat{f}_{\boldsymbol{H}}(x) = \frac{1}{\pi}\oint_{-\infty}^{\infty} \frac{f(t)-f(x)}{t-x}dt = \frac{1}{\pi}\oint_{-\infty}^{\infty} \frac{(t+x)\{f(t)-f(x)\}}{(t^2-x^2)}dt$$

$$= \frac{1}{\pi}\oint_{-\infty}^{\infty} \frac{t\{f(t)-f(x)\}}{(t^2-x^2)}dx + \frac{x}{\pi}\oint_{-\infty}^{\infty} \frac{\{f(t)-f(x)\}dt}{(t^2-x^2)}. \tag{9.3.11}$$

Since $f(t)$ is an even function, the integrand of the first integral of (9.3.11) is an odd function; hence, the first integral vanishes, and (9.3.11) gives (9.3.10). Since result (9.2.3) reveals that the Hilbert transform can be written as a convolution transform, we state the following. ∎

***THEOREM 9.3.3***

If $f$ and $g \in L^1(\mathbb{R})$ are such that their Hilbert transforms are also in $L^1(\mathbb{R})$, then

$$\boldsymbol{H}(f*g)(x) = (\boldsymbol{H}f*g)(x) = (f*\boldsymbol{H}g)(x) \tag{9.3.12}$$

and

$$(f*g)(x) = -(\boldsymbol{H}f*\boldsymbol{H}g)(x). \tag{9.3.13}$$

**PROOF** We have, by definition

$$\begin{aligned}\boldsymbol{H}(f*g)(x) &= \frac{1}{\pi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{dt}{t-x}\left[\int_{-\infty}^{\infty}f(y)g(t-y)dy\right], \quad (t-y=\xi),\\ &= \frac{1}{\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)dy\left[\int_{-\infty}^{\infty}\frac{g(\xi)d\xi}{\xi-(x-y)}\right],\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)(\boldsymbol{H}g)(x-y)dy = (f*\boldsymbol{H}g)(x).\end{aligned}$$

Similarly, we can prove the second result in (9.3.12).

To prove (9.3.13), we replace $g$ by $\boldsymbol{H}g$ in (9.3.12) and then use $\boldsymbol{H}^2 g = -g$.

∎

Another version of the Hilbert transform and its inversion formula is stated in the following theorem:

***THEOREM 9.3.4***

If $f \in L^2(\mathbb{R})$, then the Hilbert transform $(\boldsymbol{H}f)(x) \in L^2(\mathbb{R})$ is given by

$$(\boldsymbol{H}f)(x) = -\frac{1}{\pi}\frac{d}{dx}\int_{-\infty}^{\infty}f(t)\ln\left(1-\frac{x}{t}\right)dt \tag{9.3.14}$$

almost everywhere. Further, the following inversion formula

$$f(t) = \frac{1}{\pi}\frac{d}{dt}\int_{-\infty}^{\infty}(\boldsymbol{H}f)(x)\ln\left(1-\frac{t}{x}\right)dx, \tag{9.3.15}$$

holds everywhere with $\|f\| = \|\boldsymbol{H}f\|$, that is,

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx = \int_{-\infty}^{\infty}|(\boldsymbol{H}f)(x)|^2\,dx. \tag{9.3.16}$$

If the differentiation is performed under the integral signs in (9.3.14) and (9.3.15), we obtain the Hilbert transform pair (9.2.1) and (9.2.6).

We close this section by adding a comment. A more rigorous mathematical treatment of classical Hilbert transforms can be found in a treatise by Titchmarsh (1959). Further results and references of related work on Hilbert transforms and their applications are given by Kober (1943a,b), Gakhov (1966), Newcomb (1962), and Muskhelishvili (1953).

Several authors including Okikiolu (1965) and Kober (1967) introduced the *modified Hilbert transform* of a function $f(t)$, which is defined by

$$\boldsymbol{H}_\alpha[f(t)] = \hat{f}_{\boldsymbol{H}\alpha}(x) = \frac{\operatorname{cosec}\left(\frac{\pi\alpha}{2}\right)}{2\,\Gamma(\alpha)} \oint_{-\infty}^{\infty} (t-x)^{\alpha-1} f(t)dt, \tag{9.3.17}$$

where $x$ is real and $0 < \alpha < 1$, and the integral is treated as the Cauchy principal value. Obviously, $\boldsymbol{H}_\alpha[f(t)]$ is closely related to the Weyl fractional integral $W^{-\alpha}$ so that

$$2\sin\left(\frac{\pi\alpha}{2}\right)\boldsymbol{H}_\alpha[f(t)] = W^{-\alpha}[f(t), x] - W^{\alpha}[f(-t), -x]. \tag{9.3.18}$$

Several properties of $\boldsymbol{H}_\alpha[f(t)]$ and $W^{-\alpha}[f(t)]$ are investigated by Kober (1967). He also proved the following results, which is stated below without proof.

***THEOREM 9.3.5***

(*Parseval's Relation*). If $\boldsymbol{H}_\alpha[f(t)] = \hat{f}_{\boldsymbol{H}_\alpha}(x)$, then $<\boldsymbol{H}_\alpha f,\, g> = -<f,\, \boldsymbol{H}_\alpha g>$. Or, equivalently,

$$\int_{-\infty}^{\infty} \boldsymbol{H}_\alpha[f(t), x]\; g(x)dx = -\int_{-\infty}^{\infty} \boldsymbol{H}_\alpha[g(t), x]\; f(x)dx. \tag{9.3.19}$$

## 9.4 Hilbert Transforms in the Complex Plane

In communication and coherence problems in electrical engineering (see, for example, Tuttle, 1958), the Hilbert transform in the complex plane plays an important role. In order to define such a transform, we first consider the function $f_0(z)$ of a complex variable $z = x + iy$ given by

$$f_0(z) = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(t)dt}{t-z}, \qquad y > 0. \tag{9.4.1}$$

Application of the Fourier transform defined by (2.3.1) and its inverse by (2.3.2) to (9.2.1) and (9.4.1) gives

$$\hat{F}_{\boldsymbol{H}}(\omega) = i\, sgn(\omega)\ F(\omega), \tag{9.4.2}$$

$$F_0(\omega) = 2i\ \exp(-\omega y)\ \boldsymbol{H}(\omega)\ F(\omega). \tag{9.4.3}$$

In view of (9.4.2), (9.4.3) can be written as

$$F_0(\omega) = 2\exp(-\omega y)\ \boldsymbol{H}(\omega)\ \hat{F}_{\boldsymbol{H}}(\omega). \tag{9.4.4}$$

Taking the inverse Fourier transform, we obtain

$$f_0(z) = \frac{i}{\pi} \oint_{-\infty}^{\infty} \frac{\hat{f}_{\boldsymbol{H}}(t)}{t-z} dt.$$

Or,

$$\oint_{-\infty}^{\infty} \frac{f(t)dt}{t-z} = i \oint_{-\infty}^{\infty} \frac{\hat{f}_{\boldsymbol{H}}(t)dt}{t-z}, \qquad \mathrm{Im\,(z) = y > 0.} \tag{9.4.5}$$

Since,

$$\lim_{y\to 0+} \frac{1}{t-z} = \frac{1}{t-x} + \pi i\, \delta(t-x),$$

we have, from (9.4.5),

$$f_0(z) = \lim_{y\to 0+} \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(t)dt}{t-z} = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(t)dt}{t-x} + i\ f(x) = \hat{f}_{\boldsymbol{H}}(x) + i\ f(x).$$

This gives a relation between $f_0(z)$ and Hilbert transforms.

We now define a *complex analytic signal* $f_c(x)$ from a real signal $f(x)$ by

$$f_c(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(\omega)\ \boldsymbol{H}(\omega)\ \exp(i\omega x) d\omega. \tag{9.4.6}$$

Since

$$F(\omega)\boldsymbol{H}(\omega) = \frac{1}{2}[F(\omega) + sgn(\omega)\ F(\omega)] = \frac{1}{2}\left[F(\omega) - i\,\hat{F}_{\boldsymbol{H}}(\omega)\right],$$

$$f_c(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[F(\omega) - i\,\hat{F}_{\boldsymbol{H}}(\omega)\right] \exp(i\omega x) d\omega = f(x) - i\,\hat{f}_{\boldsymbol{H}}(x). \tag{9.4.7}$$

Since $f(x)$ is real, $\mathrm{Re}\{f_c(x)\} = f(x)$ and

$$\mathrm{Im}\ \{f_c(x)\} = -\hat{f}_{\boldsymbol{H}}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{x-t}.$$

Thus, it follows from the inverse Hilbert transform that

$$f(t) = -\frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{\text{Im}\,\{f_c(x)\}dx}{x-t} = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f_{\boldsymbol{H}}(x)dx}{x-t}. \tag{9.4.8}$$

## 9.5 Applications of Hilbert Transforms

***Example 9.5.1***
(*Boundary Value Problems*). Solve the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0, \tag{9.5.1}$$

with the boundary conditions

$$u_x(x,y) = f(x) \qquad \text{on } y = 0, \text{ for } \quad -\infty < x < \infty, \tag{9.5.2}$$

$$u(x,y) \to 0 \qquad \text{as } r = (x^2 + y^2)^{\frac{1}{2}} \to \infty. \tag{9.5.3}$$

Application of the Fourier transform defined by (2.3.1) with respect to $x$ gives the solution for $U(k,y)$ as

$$U(k,y) = \frac{F(k)}{ik} \exp(-|k|y) = F(k)\,G(k), \tag{9.5.4}$$

where $G(k) = (ik)^{-1}\exp(-|k|y)$ so that $g(x) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{y}\right)$.

Using the Convolution Theorem 2.5.5 gives the formal solution

$$\begin{aligned} u(x,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\tan^{-1}\left(\frac{x-t}{y}\right)dt. \end{aligned} \tag{9.5.5}$$

Obviously, it follows from (9.5.5) that

$$u_y(x,0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)\,dt}{t-x} = \boldsymbol{H}\{f(t)\}. \tag{9.5.6}$$

Thus, the Hilbert transform of the tangential derivative $u_x(x,0) = f(x)$ is the normal derivative $u_y(x,0)$ on the boundary at $y = 0$. ▯

***Example 9.5.2***
(*Nonlinear Internal Waves*). We consider a linear homogeneous partial differential equation with constant coefficients in the form

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) u(\mathbf{x}, t) = 0, \tag{9.5.7}$$

where $P$ is a polynomial in partial derivatives, and $\mathbf{x} = (x, y, z)$ and time $t > 0$. We seek a three-dimensional plane wave solution of (9.5.7) in the form

$$u(\mathbf{x}, t) = a \exp[i(\boldsymbol{\kappa} \cdot \mathbf{x} - \omega t)], \tag{9.5.8}$$

where $a$ is the amplitude, $\boldsymbol{\kappa} = (k, \ell, m)$ is the wavenumber vector, and $\omega$ is the frequency. If this solution (9.5.8) is substituted into (9.5.7), partial derivatives $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$ will be replaced by $-i\omega$, $ik$, $i\ell$, and $im$ respectively.

Hence, the solution of (9.5.7) exists provided the algebraic equation

$$P(-i\omega, ik, i\ell, im) = 0 \tag{9.5.9}$$

is satisfied. This relation is universally known as the *dispersion relation*. Physically, this gives the frequency $\omega$ in terms of wavenumbers $k, \ell$, and $m$. Further, the above analysis shows that there is a direct correspondence between the governing equation (9.5.7) and the dispersion relation (9.5.9) given by

$$\frac{\partial}{\partial t} \leftrightarrow -i\,\omega, \qquad \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \leftrightarrow (ik, i\ell, im). \tag{9.5.10}$$

Clearly, the dispersion relation can be derived from the governing equation and vice versa by using (9.5.10).

In many physical problems, the dispersion relation can be written explicitly in terms of the wavenumbers as

$$\omega = W(k, \ell, m). \tag{9.5.11}$$

The phase and the group velocities of the waves are defined by

$$C_p(\boldsymbol{\kappa}) = \frac{\omega}{\kappa}\hat{\kappa}, \qquad C_g(\boldsymbol{\kappa}) = \nabla_\kappa \omega, \tag{9.5.12ab}$$

where $\hat{\kappa}$ is the unit vector in the direction of the wavevector $\boldsymbol{\kappa}$. In the one-dimensional problem, (9.5.11)–(9.5.12ab) reduce to

$$\omega = W(k), \qquad C_p = \frac{\omega}{k}, \qquad C_g = \frac{d\omega}{dk}. \tag{9.5.13abc}$$

Thus, the one-dimensional waves given by (9.5.8) are called *dispersive* if the group velocity $C_g = \omega'(k)$ is not constant, that is, $\omega''(k) \neq 0$. Physically, as time progresses, the different waves disperse in the medium with the result that a single hump breaks into a series of wavetrains.

We consider a simple model of *internal solitary waves* in an inviscid, stably stratified two-fluid system between rigid horizontal planes at $z=h_1$ and $z=h_2$. The upper fluid of depth $h_1$ and density $\rho_1$ lies over the heavier lower fluid of depth $h_2$ and density $\rho_2(>\rho_1)$. Both fluids are subjected to a vertical gravitational force $g$, and the effects of surface tension are neglected. With $z=\eta(x,t)$ as the internal wave displacement field, the linear dispersion relation for the two-fluid system is

$$\omega^2 = \frac{gk(\rho_2-\rho_1)}{(\rho_1 \coth\, kh_1 + \rho_2 \coth\, kh_2)}, \tag{9.5.14}$$

where $\omega(k)$ and $k$ are frequency and wavenumber for a small amplitude sinusoidal disturbance at the interface of the two fluids. Several important limiting cases of (9.5.14) are of interest.

***Case (i)***: *Deep-Water Theory* (Benjamin, 1967; Ono, 1975).

In this case, the depth of the lower fluid is assumed to be infinite ($h_2\to\infty$), and waves are long compared with the depth $h_1$ of the upper fluid. This leads to the double limit in the form

$$\lim_{k\to 0}\lim_{h_2\to\infty} \omega^2 = c_0^2k^2 - 2\alpha c_0 k^3(sgn\, k+\cdots), \tag{9.5.15}$$

where $k\to 0$ is used with fixed $h_1$, and the limit $h_2\to\infty$ is taken with $k$ and $h_1$ fixed, and

$$c_0^2 = \left(\frac{\rho_2-\rho_1}{\rho_1}\right) gh_1 \quad \text{and} \quad \alpha = \left(\frac{\rho_2}{\rho_1}\right)\left(\frac{h_1 c_0}{2}\right). \tag{9.5.16ab}$$

We consider internal waves propagating only in one direction and retain the first dispersive term so that the associated dispersion relation becomes

$$\omega = c_0\, k - \alpha\, k|k|. \tag{9.5.17}$$

This enables us to define the appropriate space and time scales associated with this limiting case as

$$\xi = \beta(x-c_0t), \qquad \tau = \beta^2 t, \tag{9.5.18ab}$$

where $\beta(<<1)$ is the long wave parameter defined as the ratio of the waveguide scale to the wavelength.

The linear evolution equation associated with (9.5.17) is

$$\eta_t + c_o\eta_x + \alpha\, \boldsymbol{H}\{\eta_{xx}\} = \beta^2[\eta_\tau + \alpha\, \boldsymbol{H}\{\eta_{\xi\xi}\}] = 0, \tag{9.5.19}$$

where $\boldsymbol{H}\{\eta(x',t)\}$ is the Hilbert transform of $\eta(x',t)$ defined by

$$\boldsymbol{H}\{\eta(x',t)\} = \frac{1}{\pi}\oint_{-\infty}^{\infty} \frac{\eta(x',t)dx'}{(x'-x)}. \tag{9.5.20}$$

Equation (9.5.19) is often called the *linear Benjamin–Ono equation*. Benjamin (1967) and Ono (1975) investigated nonlinear internal wave motion and discovered the following nonlinear equation

$$(\eta_t + c_o\eta_x) + c_1\,\eta\eta_x + \alpha\,\boldsymbol{H}\{\eta_{xx}\} = 0, \tag{9.5.21}$$

where $c_1$ and $\alpha$ are constants, which are the characteristics of specific flows. This equation is usually known as the *Benjamin–Ono equation*.

The solitary wave solution of (9.5.21) has the form (Benjamin, 1967)

$$\eta(x-ct) = \frac{a\lambda^2}{(x-ct)^2+\lambda^2}, \tag{9.5.22}$$

where $c = c_0 + \frac{1}{2}a\,c_1$ and $a\lambda = -\frac{4\alpha}{c_1}$.

It is noted here that the Benjamin–Ono equation is one of the model nonlinear evolution equations and it arises in a large variety of physical wave systems.

***Case (ii)***: *Shallow-Water Theory* (Benjamin, 1966).

In this case, long wave ($k \to 0$) disturbances with the length scale $h = (h_1 + h_2)$ fixed lead to the result

$$\lim_{k\to 0}(\omega^2) = c_0^2k^2 - 2c_0\gamma k^4, \tag{9.5.23}$$

where

$$c_0^2 = \frac{g(\rho_2-\rho_1)h_1h_2}{(\rho_1h_2+\rho_2h_1)} \quad \text{and} \quad \gamma = c_0h_1h_2\left(\frac{\rho_1h_1+\rho_2h_2}{\rho_1h_2+\rho_2h_1}\right). \tag{9.5.24ab}$$

If we retain only the first dispersive term in (9.5.23) and assume that the wave propagates only to the right, it turns out that

$$\omega(k) = c_0k - \gamma k^3. \tag{9.5.25}$$

The evolution equation associated with (9.6.25) is the well-known linear KdV equation

$$\eta_t + c_0\,\eta_x + \gamma\,\eta_{xxx} = 0. \tag{9.5.26}$$

In terms of a slow time scale $\tau$ and a slow spatial modulation $\xi$ in a coordinate system moving at the linear wave velocity defined by $\xi = \beta(x-c_0t)$ and $\tau = \beta^2 t$, the above equation (9.5.26) reduces to the linear KdV equation

$$\eta_\tau + \gamma\,\eta_{\xi\xi\xi} = 0. \tag{9.5.27}$$

The standard nonlinear *KdV equation* is given by

$$\eta_t + c_0\eta_x + \alpha\,\eta\eta_x + \gamma\,\eta_{xxx} = 0. \tag{9.5.28}$$

It is well known that this equation admits the *soliton solution* in the form

$$\eta(x-ct) = a\,\text{sech}^2\left(\frac{x-ct}{\lambda}\right), \tag{9.5.29}$$

where $c = c_0 + \dfrac{a\alpha}{3}$ and $a\lambda^2 = \dfrac{12\gamma}{\alpha}$.

A similar argument can be employed to determine the integrodifferential nonlinear evolution equation associated with an arbitary dispersion relation $\omega(k) = k\ c(k)$ in the form

$$\frac{\partial\eta}{\partial t} + c_1\ \eta\,\eta_x + \int_{-\infty}^{\infty} K(x-\zeta)\left(\frac{\partial\eta}{\partial\zeta}\right)d\zeta = 0, \tag{9.5.30}$$

where the kernel $K(x)$ is a given function. The linearized version of (9.5.30) admits the plane wavelike solution

$$\eta(x,t) = A\exp[i(kx-\omega t)], \tag{9.5.31}$$

provided the following dispersion relation holds,

$$(-i\omega)\exp(ikx) + i\int_{-\infty}^{\infty} K(x-\zeta)\,k\exp(ik\zeta)d\zeta = 0.$$

Substituting $x - \zeta = \xi$, this can be rewritten in the form

$$\omega = k\int_{-\infty}^{\infty} K(\xi)\ \exp(-ik\xi)\,d\xi = k\ c(k), \tag{9.5.32}$$

where $c(k)$ is the Fourier transform of the given kernel $K(x)$ so that $K(x) = \mathscr{F}^{-1}\{c(k)\}$. This means that any phase velocity $c(k) = \mathscr{F}\{K(x)\}$ can be obtained by choosing the kernel $K(x)$.

In particular, if

$$K(x) = c_0\delta(x) + \gamma\ \delta''(x), \quad c(k) = c_0 + \gamma\ k^2, \tag{9.5.33}$$

equation (9.5.30) reduces to the linear KdV equation

$$\eta_t + c_0\,\eta_x + \gamma\,\eta_{xxx} = 0. \tag{9.5.34}$$

Combining the general dispersion relation of the integral form with typical nonlinearity, we obtain

$$\eta_t + c_0\,\eta_x + \alpha\,\eta\,\eta_x + \int_{-\infty}^{\infty} K(x-\zeta)\left(\frac{\partial\eta}{\partial\zeta}\right)d\zeta = 0. \tag{9.5.35}$$

Using (9.5.33) in (9.5.35), we can derive the KdV equation (9.5.28). On the other hand, if $c(k) = c_0(1 - \alpha|k|)$, we can deduce the Benjamin−Ono equation (9.5.21) from (9.5.30).

***Case (iii)***: *Finite-Depth Water Wave Theory* (Kubota et al., 1978).

In this case, $h_2 >> h_1$, that is, $(h_1/h_2) = O(\beta)$, but $k\,h_1 = O(\beta)$ and $k\,h_2 = O(1)$. This dispersion relation appropriate for this case is

$$\omega = c_0 k - \frac{1}{2}\left(\frac{\rho_2}{\rho_1}\right) c_0\, h_1\, k^2 \coth(k\, h_2), \tag{9.5.36}$$

where

$$c_0^2 = \left(\frac{\rho_2 - \rho_1}{\rho_1}\right) g h_1. \tag{9.5.37}$$

We can use (9.5.18ab) for the appropriate space and time scales to investigate this case. Thus, the finite-depth evolution equation can be derived from (9.5.36) and has the form (see Kubota et al., 1978)

$$\eta_t + c_0\,\eta_x + c_1\,\eta\,\eta_x + c_2\,\frac{\partial^2}{\partial x^2}\left[\int_{-\infty}^{\infty} \eta(x', t) \times \left\{\coth\frac{\pi(x - x')}{2h} - sgn\left(\frac{x - x'}{h}\right)\right\}\right] dx'. \tag{9.5.38}$$

The solitary wave solution of this equation was obtained by Joseph and Adams (1981).

It is noted that the finite-depth equation reduces to the Benjamin−Ono equation and the KdV equation in the deep- and shallow-water limits, respectively.

Finally, all of the above theories can be formulated in the framework of a generalized evolution equation usually known as the *Whitham equation* (Whitham, 1967) in the form

$$\frac{\partial \eta}{\partial t} + c_1\,\eta\,\eta_x + \frac{\partial}{\partial x}\left[\int_{-\infty}^{\infty} \eta(x', t)\, dx' \times \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{ik(x - x')\}\, c(k) dk\right] = 0. \tag{9.5.39}$$

Subsequently, Maslowe and Redekopp (1980) have generalized the theory of long nonlinear waves in stratified shear flows. They have obtained the governing nonlinear evolution equation, which involves the Hilbert transform. In their analysis, the evolution equation contains a damping term describing energy loss by radiation, which can be used to determine the persistence of solitary waves or nonlinear wave packets in physically realistic situations. ▯

***Example 9.5.3***

(*Airfoil Design*). An example of application arises in the design of airfoil which is a symmetric body designed to produce a desired lifting force when it moves in an air medium. A typical example of an airfoil is an airplane wing.

We denote the $x$-coordinates of the leading and trailing edges by $x_L = a$ and $x_T = b$, respectively. The normal component of the induced velocity at a point $(\xi,\, \eta)$ on the surface of the airfoil is given by

$$(v_i)_n = \frac{1}{2\pi} \int_a^b \frac{f(x)dx}{x-\xi}, \tag{9.5.40}$$

for some function $f$, which depends on the curl of the velocity vector. Evidently, $(v_i)_n(\xi)$ is the finite Hilbert transform of $f$ in $[a, b]$. The normal component of the uniform stream velocity is found to be

$$(v_\infty)_n = v_\infty \sin\left[\alpha - \tan^{-1}\left(\frac{dz}{dx}\right)\right],$$

where $\alpha$ is the angle made by the uniform stream with the $x$-axis and $\left(\frac{dz}{dx}\right)$ is the slope of the tangent line to the mean camber line at $(\xi,\, \eta)$. Since the sum of the normal components is zero, we have

$$\frac{1}{2\pi} \int_a^b \frac{f(x)dx}{\xi - x} = v_\infty \sin\left[\alpha - \tan^{-1}\left(\frac{dz}{dx}\right)\right] \qquad \text{at } x = \xi, \tag{9.5.41}$$

together with the boundary condition $f(x_T = b) = 0$ which is known as the *Kutta boundary condition*.

The major problem of a thin airfoil is to solve the integral equation for $f$. For small $\alpha$ and $\left(\frac{dz}{dx}\right)$, equation (9.5.41) becomes

$$\frac{1}{2\pi} \int_a^b \frac{f(x)dx}{\xi - x} = v_\infty \left(\alpha - \frac{dz}{dx}\right)_{x=\xi}, \tag{9.5.42}$$

which, for a symmetrical airfoil with $z$=constant, reduces to

$$\frac{1}{2\pi} \int_a^b \frac{f(x)dx}{\xi - x} = \alpha\, v_\infty. \tag{9.5.43}$$

We can solve (9.5.43) explicitly. Without loss of generality, we set $b = 0$ with $b - a = c$ as the length of the main chord of the airfoil. We also assume

$$x = \frac{1}{2} c\,(1 - \cos\,\theta) \qquad \text{and} \qquad \xi = \frac{1}{2} c\,(1 - \cos\,\theta_0)$$

so that (9.5.43) can be transformed into the form

$$\frac{1}{2\pi}\int_0^{\pi}\frac{f(\theta)\sin\theta\,d\theta}{(\cos\theta-\cos\theta_0)}=\alpha v_\infty,\quad f(\pi)=0. \tag{9.5.44}$$

In view of the fact

$$\int_0^{\pi}\frac{\cos n\theta\,d\theta}{(\cos\theta-\cos\theta_0)}=\pi\left(\frac{\sin n\theta_0}{\sin\theta_0}\right), \tag{9.5.45}$$

the solution of (9.5.44) is

$$f(\theta)=2\alpha\,v_\infty\left(\frac{1+\cos\theta}{\sin\theta}\right). \tag{9.5.46}$$

The lift per unit span is given by

$$L=\int_0^{C}\rho v_\infty f(\theta)\,dx=\rho v_\infty\int_0^{\pi}2\alpha\,v_\infty\left(\frac{1+\cos\theta}{\sin\theta}\right)\frac{1}{2}c\,\sin\theta\,d\theta=\pi\alpha c\rho v_\infty^2, \tag{9.5.47}$$

where $\rho$ is the constant air density.

The general solution of (9.5.42) can be represented by a sum of two terms. The first term has the form $f(\theta)$ for the symmetric airfoil given by (9.5.46) and the second term can be represented by a Fourier sine series. Thus, we have

$$f(\theta)=2\,v_\infty\left[a_0\left(\frac{1+\cos\theta}{\sin\theta}\right)+\sum_{n=1}^{\infty}a_n\,\sin n\theta\right]. \tag{9.5.48}$$

Substituting (9.5.48) into (9.5.42) gives

$$\frac{1}{\pi}\left[a_0\int_0^{\pi}\frac{(1+\cos\theta)}{(\cos\theta-\cos\theta_0)}\,d\theta+\sum_{n=1}^{\infty}a_n\int_0^{\pi}\frac{\sin\,n\theta\,\sin\theta d\theta}{(\cos\theta-\cos\theta_0)}\right]=\left(\alpha-\frac{dz}{dx}\right)_{x=\xi}, \tag{9.5.49}$$

where constants $a_n$ $(n=0,1,2,\ldots)$ are to be determined.

Using the relation (9.5.45) and the identity

$$2\sin\,n\theta\sin\theta=[\cos(n-1)\theta-\cos(n+1)\theta]\,,$$

we obtain

$$\left(\frac{dz}{dx}\right)_{x=\xi}=(\alpha-a_0)+\sum_{n=1}^{\infty}a_n\cos n\theta_0.$$

Since $\frac{dz}{dx}$ is known, we calculate the coefficients as

$$a_0 = \alpha - \frac{1}{\pi}\int_0^{\pi}\left(\frac{dz}{dx}\right)d\theta, \qquad a_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{dz}{dx}\right)\cos n\theta\, d\theta.$$

Thus, the problem is completely solved. This example of application is taken from Zayed (1996). □

The *finite Hilbert transform* was defined by Tricomi (1951) as

$$\boldsymbol{H}\{f(t), a, b\} = \hat{f}_{\boldsymbol{H}}(x, a, b) = \frac{1}{\pi}\int_a^b \frac{f(t)}{t-x}dt. \tag{9.5.50}$$

Such transforms arise naturally in aerodynamics. Tricomi (1951) studied the finite Hankel transform and its applications to airfoil theory. Subsequently, considerable attention has been given to the methods of solution of the singular integral equation for the unknown function $f(t)$ and known $\hat{f}_{\boldsymbol{H}}(x)$ as

$$\frac{1}{\pi}\int_{-1}^{1} \frac{f(t)}{t-x}dt = \hat{f}_{\boldsymbol{H}}(x), \quad -1 < x < 1, \tag{9.5.51}$$

where $f(x)$ and $\hat{f}_{\boldsymbol{H}}(x)$ satisfy the Hölder conditions on $(-1,1)$. This equation arises in boundary value problems in elasticity and in other areas. Several authors including Muskhelishvili (1963), Gakhov (1966), Peters (1972), Chakraborty (1980, 1988), Chakraborty and Williams (1980), Williams (1978), Comninou (1977), Gautesen and Dunders (1987ab), and Pennline (1976) have studied the methods of the solution of (9.5.41) and its various generalizations. The readers are referred to these papers for details.

## 9.6 Asymptotic Expansions of One-Sided Hilbert Transforms

A two-sided Hilbert transform can be written as the sum of two one-sided transforms

$$\oint_{-\infty}^{\infty} \frac{f(t)}{t-x}dt = \oint_0^{\infty} \frac{f(t)}{t-x}dt - \int_0^{\infty} \frac{f(-t)}{t+x}dt, \tag{9.6.1}$$

when $x > 0$ (with a similar expression for $x < 0$) where the second integral is actually a Stieltjes transform of $[-f(-t)]$, which has been defined by (9.7.4) in Section 9.7.

We examine the *one-sided Hilbert transform*, which is defined by

$$\boldsymbol{H}^+\{f(t)\} = \hat{f}_{\boldsymbol{H}}^+(x) = \int_0^\infty \frac{f(t)}{t-x}dt. \tag{9.6.2}$$

The Mellin transform of $\boldsymbol{H}^+\{f(t)\}$ is

$$\mathscr{M}\left[\boldsymbol{H}^+\{f(t)\}\right] = \int_0^\infty x^{p-1}\left[\int_0^\infty \frac{f(t)}{t-x}dt\right]dx = \int_0^\infty f(t)\left[\oint_0^\infty \frac{x^{p-1}}{t-x}dx\right]dt$$

$$\mathscr{M}\left\{\boldsymbol{H}^+\{f(t)\}\right\} = \pi\cot(\pi p)\int_0^\infty t^{p-1}f(t)dt$$

$$= \pi\cot(\pi p)\mathscr{M}\{f(t)\} = \pi\cot(\pi p)\tilde{f}(p).$$

Taking the inverse Mellin transform, we obtain

$$\boldsymbol{H}^+\{f(t)\} = \hat{f}_{\boldsymbol{H}}^+(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}\pi\cot(\pi p)\,\tilde{f}(p)dp. \tag{9.6.3}$$

***Example 9.6.1***
(*Asymptotic Expansion of One-Sided Hilbert Transforms*).

$$\oint_0^\infty \frac{\cos\omega t}{t-x}dt \sim -\pi\sin\omega x - \sum_{n=0}^\infty \frac{n!}{(\omega x)^{n+1}}\cos\left\{(n+1)\frac{\pi}{2}\right\}, \quad \text{as } x\to\infty, \tag{9.6.4}$$

$$\oint_0^\infty \frac{\sin\omega t}{t-x}dt \sim \pi\cos\omega x - \sum_{n=0}^\infty \frac{n!}{(\omega x)^{n+1}}\sin\left\{(n+1)\frac{\pi}{2}\right\}, \quad \text{as } x\to\infty. \tag{9.6.5}$$

We have

$$\oint_0^\infty \frac{\exp(i\omega t)}{t-x}dt = \pi i\exp(i\omega x) + \int_0^\infty \frac{\exp(i\omega t)}{t-x}dt,$$

where in the integral on the right the contour of integration passes above the pole $t=x$. The contour can be deformed into the positive imaginary axis on which $t=iu$ with $u>0$. Thus,

$$\int_0^\infty \frac{\exp(i\omega t)}{t-x}dt = -i\int_0^\infty \frac{\exp(-\omega u)}{x-iu}du$$

$$\sim -\sum_{n=0}^\infty \frac{i^{n+1}}{x^{n+1}}\int_0^\infty u^n\exp(-\omega u)du, \qquad \text{by Watson's lemma}$$

$$= -\sum_{n=0}^\infty \frac{n!}{(\omega x)^{n+1}}\exp\left\{i(n+1)\frac{\pi}{2}\right\}. \tag{9.6.6}$$

Separating the real and imaginary parts, we obtain the desired results. These results are due to Ursell (1983). □

### THEOREM 9.6.1

(*Ursell, 1983*). If $f(t)$ is analytic for real $t$, $0 \le t < \infty$, and if it has the asymptotic expansion in the form

$$f(t) \sim \sum_{n=1}^{\infty} \frac{a_n}{t^n} + \cos \omega t \sum_{n=1}^{\infty} \frac{A_n}{t^n} + \sin \omega t \sum_{n=1}^{\infty} \frac{B_n}{t^n} \quad \text{as } t \to \infty, \tag{9.6.7}$$

where the coefficients $a_n, A_n$ and $B_n$ are known and $\omega > 0$, then the one-sided Hilbert transform $\hat{f}^+(x)$ has the following asymptotic expansion

$$\hat{f}^+_{\boldsymbol{H}}(x) = \boldsymbol{H}^+\{f(t)\} = \oint_0^\infty \frac{f(t)}{t-x}dt \sim \sum_1^\infty \frac{c_n}{x^n} - \log x \sum_1^\infty \frac{a_n}{x^n}$$
$$+ \left(\sum_1^\infty \frac{A_n}{x^n}\right) \oint_0^\infty \frac{\cos \omega t}{t-x}dt + \left(\sum_1^\infty \frac{B_n}{x^n}\right) \oint_0^\infty \frac{\sin \omega t}{t-x}dt \quad \text{as } x \to \infty, \tag{9.6.8}$$

where $c_n$ is given by

$$c_n = d_n - \sum_{r=1}^{n-1} \frac{\Gamma(n-r)}{\omega^{n-r}} \left[A_r \cos\left\{\frac{\pi}{2}(n-r)\right\} + B_r \sin\left\{\frac{\pi}{2}(n-r)\right\}\right] \tag{9.6.9}$$

and

$$d_n = \lim_{p \to n} \left[\mathscr{M}\{f(t), p\} + \frac{a_n}{p-n}\right]. \tag{9.6.10}$$

Note that when $a_n = 0$, (9.6.10) becomes

$$d_n = \mathscr{M}\{f(t), n\} = \int_0^\infty t^{n-1} f(t)dt. \tag{9.6.11}$$

Substituting (9.6.9) into (9.6.8) and using (9.6.4) and (9.6.5), we obtain the following theorem:

### THEOREM 9.6.2

(*Ursell, 1983*). Under the same conditions of Theorem 9.6.1, the one-sided Hilbert transform $\hat{f}^+(x)$ has the asymptotic expansion

$$\hat{f}^+_{\boldsymbol{H}}(x) = \int_0^\infty \frac{f(t)}{t-x}dt \sim -\sum_1^\infty \frac{d_n}{x^n} - \log x \sum_1^\infty \frac{a_n}{x^n} - (\pi \sin \omega x) \sum_1^\infty \frac{A_n}{x^n}$$
$$+ (\pi \cos \omega x) \sum_1^\infty \frac{B_n}{x^n} \qquad \text{as } x \to \infty, \tag{9.6.12}$$

where $d_n$ is given by (9.6.10).

The reader is referred to Ursell (1983) for a detailed discussion of proof of Theorems 9.6.1 and 9.6.2.

## 9.7 Definition of the Stieltjes Transform and Examples

We use the Laplace transform of $\bar{f}(s) = \mathscr{L}\{f(t)\}$ with respect to $s$ to define the Stieltjes transform of $f(t)$. Clearly,

$$\mathscr{L}\{\bar{f}(s)\} = \tilde{f}(z) = \int_0^\infty e^{-sz}\bar{f}(s)ds$$

$$= \int_0^\infty e^{-sz}ds \int_0^\infty e^{-st} f(t)dt. \tag{9.7.1}$$

Interchanging the order of integration and evaluating the inner integral, we obtain

$$\tilde{f}(z) = \int_0^\infty \frac{f(t)}{t+z}dt. \tag{9.7.2}$$

The *Stieltjes transform* of a locally integrable function $f(t)$ on $0 \le t < \infty$ is denoted by $\tilde{f}(z)$ and defined by

$$\mathscr{S}\{f(t)\} = \tilde{f}(z) = \int_0^\infty \frac{f(t)}{t+z}dt, \tag{9.7.3}$$

where $z$ is a complex variable in the cut plane $|\arg z| < \pi$.

If $z = x$ is real and positive, then

$$\mathscr{S}\{f(t)\} = \tilde{f}(x) = \int_0^\infty \frac{f(t)}{t+x}dt. \tag{9.7.4}$$

Differentiating (9.7.4) with respect to $x$, we obtain

$$\frac{d^n}{dx^n}\tilde{f}(x) = (-1)^n\, n! \int_0^\infty \frac{f(t)}{(t+x)^{n+1}}dt, \qquad n = 1, 2, 3, \ldots. \tag{9.7.5}$$

We now state the *inversion theorem* for the Stieltjes transform without proof.

***THEOREM 9.7.1***

If $f(t)$ is absolutely integrable in $0 \leq t \leq T$ for every positive $T$ and is such that the integral (9.7.4) converges for $x > 0$, then $\tilde{f}(z)$ exists for complex $z (z \neq 0)$ not lying on the negative real axis and

$$\lim_{\varepsilon \to 0+} \frac{1}{2\pi i}[\tilde{f}(-x - i\varepsilon) - \tilde{f}(-x + i\varepsilon)] = \frac{1}{2}[f(x+0) + f(x-0)] \tag{9.7.6}$$

for any positive $x$ at which $f(x+0)$ and $f(x-0)$ exist.

For a rigorous proof of this theorem the reader is referred to Widder (1941, pp. 340–341).

***Example 9.7.1***

Find the Stieltjes transform of each of the following functions:

$$\text{(a) } f(t) = (t+a)^{-1}, \quad \text{(b) } f(t) = t^{\alpha-1}.$$

(a) We have, by definition,

$$\begin{aligned}\tilde{f}(z) &= \int_0^\infty \frac{dt}{(t+a)(t+z)} = \frac{1}{(a-z)}\left[\int_0^\infty \left(\frac{1}{t+z} - \frac{1}{t+a}\right)dt\right] \\ &= \frac{1}{(a-z)} \log\left|\frac{a}{z}\right|. \end{aligned} \tag{9.7.7}$$

$$\text{(b) } \tilde{f}(z) = \mathscr{S}\{t^{\alpha-1}\} = \int_0^\infty \frac{t^{\alpha-1}}{t+z}dt = z^{-1}\int_0^\infty \left(1 + \frac{t}{z}\right)^{-1} t^{\alpha-1}dt, \ \left(\frac{t}{z} = x\right),$$

$$= z^{\alpha-1}\int_0^\infty \frac{x^{\alpha-1}dx}{1+x} = z^{\alpha-1}\mathscr{M}\left\{\frac{1}{1+x}\right\}$$

which is, by Example 8.2.1(b),

$$= z^{\alpha-1}\pi \operatorname{cosec}(\pi\alpha). \tag{9.7.8}$$

▯

***Example 9.7.2***

Obtain the Stieltjes transform of $J_\nu^2(t)$.

We have

$$\tilde{f}(x) = \mathscr{S}\{J_\nu^2(t)\} = \int_0^\infty \frac{J_\nu^2(t)dt}{t+x} \tag{9.7.9}$$

satisfies the *Parseval relation*

$$\tilde{f}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p}\ \tilde{f}(p)\ \tilde{g}(1-p)\,dp. \tag{9.7.10}$$

We write $t = xu$ so that (9.7.9) becomes

$$\tilde{f}(x) = \int_0^\infty f(xu)g(u)du, \tag{9.7.11}$$

where $f(u) = J_\nu^2(u)$ and $g(u) = (1+u)^{-1}$.

Taking the Mellin transform of (9.7.11) with respect to $x$, we obtain

$$\mathscr{M}\{\tilde{f}(x), p\} = \tilde{f}(p)\ \tilde{g}(1-p)$$

where, from Oberhettinger (1974, p. 98),

$$\begin{aligned} \tilde{f}(p) &= \frac{2^{p-1}\Gamma\left(\nu + \frac{p}{2}\right)\pi \operatorname{cosec}(\pi p)}{\left\{\Gamma\left(1 - \frac{p}{2}\right)\right\}^2 \Gamma\left(1 + \nu - \frac{p}{2}\right)\Gamma(p)}, \\ \tilde{g}(1-p) &= \pi \operatorname{cosec}(\pi p). \end{aligned}$$

Thus, the inverse Mellin transform gives the desired result.

□

***Example 9.7.3***

Show that

$$\mathscr{S}\left\{\sin(k\sqrt{t})\right\} = \pi \exp(-k\sqrt{z}), \quad k > 0. \tag{9.7.12}$$

We have, by definition,

$$\mathscr{S}\left\{\sin(k\sqrt{t})\right\} = \int_0^\infty \frac{\sin(k\sqrt{t})}{t+z}\,dt, \qquad (\sqrt{t} = u),$$

$$= 2\int_0^\infty \frac{u \sin ku}{(u^2+z)}\,du = \pi \exp(-k\sqrt{z}) \quad \text{by (2.13.6).}$$

□

## 9.8 Basic Operational Properties of Stieltjes Transforms

The following properties hold for the Stieltjes transform:

(a) $$\mathscr{S}\{f(t+a)\} = \tilde{f}(z-a), \tag{9.8.1}$$

(b) $$\mathscr{S}\{f(at)\} = \tilde{f}(az), \qquad a>0, \tag{9.8.2}$$

(c) $$\mathscr{S}\{t\ f(t)\} = -z\ \tilde{f}(z) + \int_0^\infty f(t)\,dt, \tag{9.8.3}$$

provided the integral on the right hand side exists.

(d) $$\mathscr{S}\left\{\frac{f(t)}{t+a}\right\} = \frac{1}{a-z}\left[\tilde{f}(z)-\tilde{f}(a)\right], \tag{9.8.4}$$

(e) $$\mathscr{S}\left\{\frac{1}{t}\,f\left(\frac{a}{t}\right)\right\} = \frac{1}{z}\,\tilde{f}\left(\frac{a}{z}\right), \qquad a>0. \tag{9.8.5}$$

**PROOF**

(a) We have, by definition,

$$\mathscr{S}\{f(t+a)\} = \int_0^\infty \frac{f(t+a)}{t+z}\,dt$$

which is, by the change of variable $t+a=\tau$,

$$\mathscr{S}\{f(t+a)\} = \int_0^\infty \frac{f(\tau)}{\tau+(z-a)}\,d\tau = \tilde{f}(z-a).$$

(b) We have, by definition,

$$\mathscr{S}\{f(at)\} = \int_0^\infty \frac{f(at)}{t+z}\,dt, \qquad at=\tau,$$

$$= \int_0^\infty \frac{f(\tau)}{\tau+az}\,d\tau = \tilde{f}(az).$$

(c) We have from the definition

$$\mathscr{S}\{t\ f(t)\} = \int_0^\infty \frac{t\,f(t)}{t+z}\,dt = \int_0^\infty \frac{(t+z-z)f(t)}{t+z}\,dt$$

$$= \int_0^\infty f(t)\,dt - z\int_0^\infty \frac{f(t)}{t+z}\,dt.$$

This gives the desired result.

(d) We have, by definition,

$$\begin{aligned}\mathscr{S}\left\{\frac{f(t)}{t+a}\right\} &= \int_0^\infty \frac{f(t)}{(t+a)(t+z)}\,dt \\ &= \frac{1}{a-z}\left[\int_0^\infty \left\{\frac{1}{t+z}-\frac{1}{t+a}\right\} f(t)\,dt\right] \\ &= \frac{1}{a-z}\left[\tilde{f}(z)-\tilde{f}(a)\right].\end{aligned}$$

(e) We have, by definition,

$$\mathscr{S}\left\{\frac{1}{t}\,f\left(\frac{a}{t}\right)\right\} = \int_0^\infty \frac{1}{t(t+z)}\,f\left(\frac{a}{t}\right)\,dt, \qquad \left(\frac{a}{t}=\tau\right),$$

$$= \frac{1}{z}\int_0^\infty \frac{f(\tau)}{\left(\tau+\frac{a}{z}\right)}\,d\tau = \frac{1}{z}\,\tilde{f}\left(\frac{a}{z}\right).$$

∎

***THEOREM 9.8.1***

(*Stieltjes Transforms of Derivatives*). If $\mathscr{S}\{f(t)\} = \tilde{f}(z)$, then

$$\mathscr{S}\{f'(t)\} = -\frac{1}{z}\,f(0) - \frac{d}{dz}\,\tilde{f}(z), \tag{9.8.6}$$

$$\mathscr{S}\{f''(t)\} = -\left[\frac{1}{z}\,f'(0) + \frac{1}{z^2}f(0)\right] - \frac{d^2}{dz^2}\,\tilde{f}(z). \tag{9.8.7}$$

More generally,

$$\mathscr{S}\{f^{(n)}(t)\} = -\left[\frac{1}{z}f^{(n-1)}(0) + \frac{1}{z^2}f^{(n-2)}(0) + \cdots + \frac{1}{z^n}f(0)\right] - \frac{d^n}{dz^n}\tilde{f}(z). \tag{9.8.8}$$

**PROOF** Using the definition and integrating by parts, we obtain

$$\begin{aligned}\mathscr{S}\{f'(t)\} &= \int_0^\infty \frac{f'(t)}{t+z}dt \\ &= \left[\frac{f(t)}{t+z}\right]_0^\infty + \int_0^\infty \frac{f(t)\,dt}{(t+z)^2} = -\frac{1}{z}\,f(0) - \frac{d}{dz}\tilde{f}(z).\end{aligned}$$

This proves result (9.8.6).

Similarly, other results can readily be proved. ▮

## 9.9 Inversion Theorems for Stieltjes Transforms

We first introduce the following differential operator that can be used to establish inversion theorems for the Stieltjes transform.

A differential operator is defined for any real positive number $t$ by the following equations:

$$L_{k,t}[f(x)] = (-1)^{k-1}\ c_k\ t^{k-1}\ D_t^{(2k-1)}[t^k\ f(t)], \tag{9.9.1}$$

$$L_{0,t}[f(x)] = f(t), \tag{9.9.2}$$

$$L_{1,t}[f(x)] = D_t[t\ f(t)], \tag{9.9.3}$$

where $k = 2, 3, \ldots, c_k = [k!(k-2)!]^{-1},\ D_t \equiv \dfrac{d}{dt}$, and $f(x)$ has derivatives of all orders.

We state a basic theorem due to Widder (1941) without proof.

***THEOREM 9.9.1***

If $\mathscr{S}\{f(t)\} = \tilde{f}(x)$ exists and is defined by

$$\tilde{f}(x) = \int_0^\infty \frac{f(t)}{t+x}\,dt \tag{9.9.4}$$

then, for all positive $t$,

$$\text{(i)}\qquad L_{k,t}[\tilde{f}(x)] = (2k-1)!\ c_k\ t^{k-1}\int_0^\infty \frac{u^k\ f(u)}{(t+u)^{2k}}\,du, \tag{9.9.5}$$

$$\text{(ii)}\qquad \lim_{k\to\infty} L_{k,t}[\tilde{f}(x)] = f(t). \tag{9.9.6}$$

Obviously,

$$\frac{t^k}{t+u} = \frac{t^k - (-u)^k}{t+u} + \frac{(-u)^k}{t+u} = t^{k-1} - u\ t^{k-2} + \cdots \pm u^{k-1} + \frac{(-u)^k}{t+u}.$$

In view of this result, we can find

$$\begin{aligned} L_{k,t}[\tilde{f}(x)] &= (-1)^{k-1}\ c_k\ t^{k-1}\ D_t^{(2k-1)}[t^k\ \tilde{f}(t)] \\ &= c_k\ t^{k-1}(2k-1)!\int_0^\infty \frac{u^k}{(t+u)^{2k}}\,f(u)\,du. \end{aligned}$$

***THEOREM 9.9.2***
If $\tilde{f}(x)$ is the Laplace transform of $\bar{f}(s) = \mathscr{L}\{f(t)\}$ so that

$$\tilde{f}(x) = \mathscr{L}\{\tilde{f}(s)\} = \int_0^\infty \frac{f(t)}{t+x}\,dt \quad \text{for } s > 0, \text{ then}$$

$$\text{(i)} \qquad L_{k,x}[\tilde{f}(x)] = (-1)^k\; c_k\; x^{k-1} \int_0^\infty e^{-xt}\; t^{2k-1}\; f^{(k)}(t)\,dt, \tag{9.9.7}$$

$$\text{(ii)} \qquad \lim_{k\to\infty} L_{k,x}[\tilde{f}(x)] = f(x), \qquad \text{for all positive } x. \tag{9.9.8}$$

**PROOF** We have, by definition of the operator defined by (9.9.1),

$$L_{k,x}[\tilde{f}(x)] = (-1)^{k-1} x^{k-1} c_k\; D_x^{(2k-1)}[x^k \tilde{f}(x)]. \tag{9.9.9}$$

We use the result (Widder, 1941, p. 350)

$$x^{k-1}\; D_x^{(2k-1)}[x^k\; f(x)] = D_x^k[x^{2k-1}\; f^{(k-1)}(x)], \tag{9.9.10}$$

where $f(x)$ is any function that has derivatives of all orders. This can easily be verified by computing both sides of (9.9.10). Each of both sides is equal to

$$\sum_{n=0}^{k} \frac{(2k-1)!}{(2k-n-1)!} \binom{k}{n} x^{2k-n-1} f^{(2k-n-1)}(x).$$

In view of (9.9.10), result (9.9.9) becomes

$$L_{k,x}[\tilde{f}(x)] = (-1)^{k-1}\; c_k\; D_x^k[x^{2k-1} \tilde{f}^{(k-1)}(x)]. \tag{9.9.11}$$

We next show that

$$\begin{aligned}
(-1)^{k-1} \tilde{f}^{(k-1)}(x) &= (-1)^{k-1} \frac{d^{k-1}}{dx^{k-1}} \int_0^\infty e^{-xt} f(t)dt \\
&= (-1)^{2(k-1)} \int_0^\infty e^{-xt}\; t^{k-1}\; f(t)dt \\
&= \frac{1}{x} \int_0^\infty e^{-u} \left(\frac{u}{x}\right)^{k-1} f\left(\frac{u}{x}\right) du.
\end{aligned} \tag{9.9.12}$$

Using (9.9.12) in (9.9.11), we obtain

$$L_{k,x}[\tilde{f}(x)] = c_k \int_0^\infty e^{-u} u^{k-1} D_x^k \left\{x^{k-1} f\left(\frac{u}{x}\right)\right\} du,$$

which is, due to Lemma 25 (Widder 1941, p. 385),

$$= c_k \; x^{-(k+1)} \int_0^\infty e^{-u} u^{k-1} (-u)^k \; f^{(k)} \left(\frac{u}{x}\right) du. \tag{9.9.13}$$

We again set $u = xt$ in (9.9.13) to obtain the desired result

$$L_{k,x}[\tilde{f}(x)] = c_k (-1)^k x^{k-1} \int_0^\infty e^{-xt} t^{2k-1} f^{(k)}(t) dt.$$

We next take the limit as $k \to \infty$ and use Widder's result (9.9.6) to derive (9.9.8). Thus, the proof is complete. ∎

It is important to note that result (9.9.7) depends on the values of all derivatives of $f(x)$ in the domain $(0, \infty)$. This seems to be a very severe restriction on the formula (9.9.7). This restriction can be eliminated by applying the operator $L_{k,x}$ to the Laplace integral directly. Then we prove the following theorem.

***THEOREM 9.9.3***
(*Widder, 1941*). Under the same conditions of Theorem 9.9.2, the following results hold

$$\text{(i)} \qquad L_{k,x}[\tilde{f}(x)] = \int_0^\infty e^{-xs} \, P_{2k-1}(xs) \, \bar{f}(s) ds, \tag{9.9.14}$$

$$\text{(ii)} \qquad \lim_{k \to \infty} L_{k,x}[\tilde{f}(x)] = f(x), \tag{9.9.15}$$

where

$$P_{2k-1}(t) = (-1)^{k-1} \; c_k \; (2k-1)! \sum_{n=0}^{k} \binom{k}{n} \frac{(-t)^{2k-n-1}}{(2k-n-1)!}. \tag{9.9.16}$$

**PROOF** We apply the operator $L_{k,x}$ to the Laplace integral directly to obtain

$$L_{k,x}[\tilde{f}(x)] = L_{k,x}[\mathscr{L}\{\bar{f}(s)\}] = L_{k,x}\left[\int_0^\infty e^{-sx} \, \bar{f}(s) ds\right]$$

$$L_{k,x}[\tilde{f}(x)] = \int_0^\infty L_{k,x}[e^{-sx}] \, \bar{f}(s) ds$$

which is, after direct computation of $L_{k,x}[\exp(-sx)]$,

$$= \int_0^\infty e^{-xs} P_{2k-1}\ (xs)\, \bar{f}(s) ds.$$

Taking the limit as $k \to \infty$ and using result (9.9.8), we obtain

$$\lim_{k\to\infty} L_{k,x}[\tilde{f}(x)] = \lim_{k\to\infty} \int_0^\infty e^{-xs}\, P_{2k-1}(xs)\, \bar{f}(s) ds = f(x) \quad \text{for all } x > 0.$$

The significance of this result lies in the fact that the integral representation for $f(x)$ depends *only* on the values of $\bar{f}(s)$ in $(0, \infty)$ and *not* on any of its derivatives. ∎

## 9.10 Applications of Stieltjes Transforms

***Example 9.10.1***

(*Moment Problem*). If $f(t)$ has an exponential rate of decay as $t \to \infty$, then all of the *moments* exist and are given by

$$m_r = \int_0^\infty t^r\, f(t) dt, \qquad r = 0, 1, 2, \ldots. \tag{9.10.1}$$

Then it can easily be shown from (9.7.4) that

$$\tilde{f}(x) = \sum_{r=0}^{n-1} (-1)^r\, m_r\, x^{-(r+1)} + \varepsilon_n(x), \tag{9.10.2}$$

where

$$|\varepsilon_n(x)| \le x^{-(n+1)} \sup_{0<t<\infty} \left| \int_0^t \tau^n\, f(\tau)\, d\tau \right|. \tag{9.10.3}$$

□

The Stieltjes transform is found to arise in the problems of moments for the semi-infinite interval. The reader is referred to Tamarkin and Shohat (1943).

**Example 9.10.2**

(*Solution of Integral Equations*). Find the solution of the integral equation

$$\lambda \int_0^\infty \frac{f(t)}{t+x} dt = f(x), \tag{9.10.4}$$

where $\lambda$ is a real parameter.

***Case (i):*** Suppose $\lambda \neq \dfrac{1}{\pi}$.

In this case, we show that the solution of (9.10.4) is

$$f(t) = A\, t^{-\alpha} + B\, t^{\alpha-1}, \tag{9.10.5}$$

where $A$ and $B$ are arbitrary constants and $\alpha$ is a root of the equation $\sin \alpha\pi = \lambda\pi$ between zero and unity if $\lambda < \dfrac{1}{\pi}$, and with real part $\dfrac{1}{2}$ if $\lambda > \dfrac{1}{\pi}$.

If $0 < \text{Re } \alpha < 1$, then

$$\begin{aligned} \lambda \int_0^\infty \frac{t^{-\alpha}}{t+x}\, dt &= \lambda \int_0^\infty \frac{t^{p-1}}{t+x}\, dt, \qquad (p = 1-\alpha) \\ &= \lambda\pi\, x^{p-1} \text{ cosec } (\pi p) \\ &= x^{-\alpha} \left(\frac{\lambda\pi}{\sin \pi\alpha}\right) = x^{-\alpha}, \end{aligned} \tag{9.10.6}$$

so that $x^{-\alpha}$ is a solution of (9.10.4). Obviously, equation (9.10.6) holds if $\alpha$ is replaced by $1-\alpha$, and hence, $t^{\alpha-1}$ is also a solution. Thus, (9.10.5) is a solution of equation (9.10.4).

***Case (ii):*** $\lambda = \dfrac{1}{\pi}$.

In this case, we show that $\dfrac{1}{\sqrt{t}}$, and $\dfrac{1}{\sqrt{t}} \log t$ are solutions of (9.10.4).

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_0^\infty \frac{dt}{\sqrt{t}\,(t+x)} = \frac{1}{\pi} \int_0^\infty \frac{1}{(t+x)}\, t^{\frac{1}{2}-1} dt \\ &= \frac{1}{\pi}\, x^{\frac{1}{2}-1}\, \pi \,\text{cosec}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{x}}, \qquad \text{by Example 9.7.1(b).} \end{aligned}$$

Thus, $\dfrac{1}{\sqrt{t}}$ is a solution of the integral equation (9.10.4).

To show that $f(t) = \dfrac{1}{\sqrt{t}} \log t$ is a solution, we write

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\log t}{\sqrt{t}\,(t+x)}\, dt \qquad (\log t = u)$$
$$= \frac{1}{\pi} \int_{-\infty}^\infty \frac{u}{(x+e^u)} \exp\left(\frac{u}{2}\right) du.$$

Replacing $x$ by $e^x$ and multiplying both sides by $\exp\left(\frac{x}{2}\right)$, we find

$$\exp\left(\frac{x}{2}\right) f(e^x) = \frac{1}{\pi} \int_{-\infty}^\infty \left(\frac{u}{e^x+e^u}\right) \exp\left(\frac{x+u}{2}\right) du$$
$$= \frac{1}{2\pi} \int_{-\infty}^\infty u \operatorname{sech}\left(\frac{x-u}{2}\right) du, \qquad (x-u=t).$$
$$\exp\left(\frac{x}{2}\right) f(e^x) = \frac{1}{2\pi} \int_{-\infty}^\infty (x-t) \operatorname{sech}\left(\frac{t}{2}\right) dt = \frac{x}{2\pi} \int_{-\infty}^\infty \operatorname{sech}\left(\frac{t}{2}\right) dt = x,$$

or,

$$f(e^x) = x \exp\left(-\frac{x}{2}\right).$$

Thus,

$$f(t) = \frac{1}{\sqrt{t}} \log t$$

is a solution, and hence, $\dfrac{1}{\sqrt{t}}(A + B \log t)$ is also a solution of (9.10.4). ▯

## 9.11 The Generalized Stieltjes Transform

The *generalized Stieltjes transform* of a function $f(t)$ is defined by

$$\mathscr{S}_g\{f(t)\} = \tilde{f}(z,\rho) = \int_0^\infty \frac{f(t)}{(t+z)^\rho} dt, \tag{9.11.1}$$

provided the integral exists and $|\arg z| < \pi$.

***Example 9.11.1***
If Re $a > 0$, find the generalized Stieltjes transform of

(a) $t^{a-1}$, (b) $\exp(-at)$.

(a) We have, by definition,

$$\mathscr{S}_g\{t^{a-1}\} = \int_0^\infty \frac{t^{a-1}}{(t+z)^\rho} dt$$

$$= z^{-\rho} \int_0^\infty t^{a-1} \left(1 + \frac{t}{z}\right)^{-\rho} dt, \qquad (t = zu),$$

$$= z^{a-\rho} \int_0^\infty u^{a-1}(1+u)^{-\rho}\, du. \tag{9.11.2}$$

Substituting $x = \dfrac{u}{1+u}$ or $u = \dfrac{x}{1-x}$ into integral (9.11.2), we obtain

$$\mathscr{S}_g\{t^{a-1}\} = z^{a-\rho} \int_0^1 x^{a-1}(1-x)^{\rho-a-1} dx$$

$$= z^{a-\rho}\, B(a, \rho - a) = \frac{\Gamma(a)\Gamma(\rho - a)}{\Gamma(\rho)} z^{a-\rho}. \tag{9.11.3}$$

(b) We have, by definition,

$$\mathscr{S}_g\{\exp(-at)\} = \int_0^\infty \frac{\exp(-at)}{(t+z)^\rho}\, dt, \qquad (t+z = u),$$

$$= \exp(az) \int_0^\infty e^{-au} u^{-\rho}\, du.$$

Substituting $au = x$ into this integral, we obtain

$$\mathscr{S}_g\{\exp(-at)\} = a^{\rho-1} \exp(az) \int_0^\infty e^{-x} x^{1-\rho-1} dx$$

$$= a^{\rho-1} \exp(az)\, \Gamma(1-\rho). \tag{9.11.4}$$

□

The reader is referred to Erdélyi et al. (1954, pp. 234–235) where there is a table for generalized Stieltjes transforms.

## 9.12 Basic Properties of the Generalized Stieltjes Transform

The generalized Stieltjes transform satisfies the following properties:

$$\text{(a)} \quad \mathscr{S}_g\{f(at)\} = a^{\rho-1}\,\tilde{f}(az), \quad a > 0 \tag{9.12.1}$$

$$\text{(b)} \quad \mathscr{S}_g\{t\,f(t)\} = \tilde{f}(z,\rho-1) - z\,\tilde{f}(z,\rho), \tag{9.12.2}$$

$$\text{(c)} \quad \mathscr{S}_g\{f'(t)\} = \rho\,\tilde{f}(z,\rho+1) - z^{-\rho}\,f(0),\ f(t)\to 0 \,\mathrm{as}\,\mathrm{t}\to\infty. \tag{9.12.3}$$

$$\text{(d)} \quad \mathscr{S}_g\left\{\int_0^t f(x)dx\right\} = (\rho-1)^{-1}\,\tilde{f}(z,\rho-1), \quad \mathrm{Re}\ \rho > 1. \tag{9.12.4}$$

**PROOF**

(a) We have, by definition,

$$\mathscr{S}_g\{f(at)\} = \int_0^\infty \frac{f(at)\,dt}{(t+z)^\rho}, \quad (at = x),$$

$$= a^{\rho-1}\int_0^\infty \frac{f(x)\,dx}{(x+az)^\rho} = a^{\rho-1}\,\tilde{f}(az).$$

(b) It follows from the definition that

$$\begin{aligned}\mathscr{S}_g\{t\,f(t)\} &= \int_0^\infty \frac{t\,f(t)}{(t+z)^\rho}dt = \int_0^\infty \frac{(t+z-z)\,f(t)}{(t+z)^\rho}\,dt \\ &= \int_0^\infty \frac{f(t)}{(t+z)^{\rho-1}}\,dt - z\int_0^\infty \frac{f(t)\,dt}{(t+z)^\rho} \\ &= \tilde{f}(z,\rho-1) - z\,\tilde{f}(z,\rho).\end{aligned}$$

(c) We have, by definition,

$$\mathscr{S}_g\{f'(t)\} = \int_0^\infty \frac{f'(t)}{(t+z)^\rho}\,dt$$

which is, by integrating by parts,

$$\begin{aligned}\mathscr{S}_g\{f'(t)\} &= \left[\frac{f(t)}{(t+z)^\rho}\right]_0^\infty + \rho\int_0^\infty \frac{f(t)}{(t+z)^{\rho+1}}\,dt \\ &= \rho\,\tilde{f}(z,\rho+1) - z^{-\rho}\,f(0).\end{aligned}$$

(d) We write

$$g(t) = \int_0^t f(x)\, dx$$

so that $g'(t) = f(t)$ and $g(0) = 0$.

Thus,

$$\mathscr{S}_g\{f(t), \rho\} = \mathscr{S}_g\{g'(t), \rho\}$$

which is, by (9.12.3),

$$= \rho\, \tilde{g}(z, \rho + 1) - z^{-\rho}\, g(0) = \rho\, \mathscr{S}_g\{g(t), \rho + 1\}.$$

Replacing $\rho$ by $\rho - 1$, we obtain (9.12.4). ▮

## 9.13 Exercises

1. Find the Hilbert transform of each of the following functions:

   (a) $f(t) = \dfrac{1}{(a^2 + t^2)}$, $\quad \text{Re}\, a > 0$, $\qquad$ (b) $f(t) = \dfrac{t^\alpha}{(t + a)}$, $\quad |\text{Re}\, \alpha| < 1$,

   (c) $f(t) = \exp(-at)$, $\qquad$ (d) $f(t) = \dfrac{\sin t}{t}$,

   (e) $f(t) = \dfrac{1}{t} \sin(a\sqrt{t})$, $\quad a > 0$,

   (f) $f(t) = t^{-\alpha} \exp(-at)$, $\quad \text{Re}\, a > 0$, $\quad \text{Re}\, \alpha < 1$.

2. Show that

   (a) $\mathscr{S}\left\{\dfrac{1}{\sqrt{t}} \cos(a\sqrt{t})\right\} = \dfrac{\pi}{\sqrt{z}} \exp(-a\sqrt{z})$, $\quad z > 0$.

   (b) $\mathscr{S}\,\{\sin(a\sqrt{t}) J_0(b\sqrt{t})\} = \pi \exp(-a\sqrt{z}) I_0(b\sqrt{z})$, $\quad 0 < b < a$.

3. If $\tilde{f}(z) = \mathscr{S}\,\{f(t)\}$ and $f(t) = \mathscr{S}\,\{g(u)\}$, then show that

   $$\tilde{f}(z) = \int_0^\infty K(z, u)\, g(u)\, du,$$

   where $K(z, u) = (z - u)^{-1} \log\left(\dfrac{z}{u}\right)$.

4. Show that

(a) $\mathscr{S}_g\left\{t^{\rho-2} f\left(\frac{a}{t}\right)\right\} = a^{\rho-1} z^{-\rho} \tilde{f}\left(\frac{a}{z}\right), \quad a > 0.$

(b) $\mathscr{S}_g\left\{\frac{1}{\Gamma(\alpha)} \int_0^t f(x)(t-x)^{\alpha-1} dx\right\} = \frac{\Gamma(\rho-\alpha)}{\Gamma(\rho)} \tilde{f}(z, \rho-\alpha),$

where $0 < \text{Re}\ \alpha < \text{Re}\ \rho$.

5. Show that the dispersion relation associated with the linearized Benjamin–Ono equation

$$u_t + \boldsymbol{H}\{u_{xx}\} = 0 \ \text{ is } \ \omega = -k|k|.$$

6. Find the Stieltjes transforms of each of the following functions:

(a) $f(t) = \dfrac{t^{\alpha-1}}{t+a}$, (b) $f(t) = \dfrac{1}{t^2+a^2}$, (c) $f(t) = \dfrac{t}{t^2+a^2}$.

7. Show that

(a) $\mathscr{S}\left[f(te^{i\pi}) - f(te^{-i\pi})\right] = 2\pi i\, \tilde{f}(z)$,

(b) $\mathscr{S}\left[f(\sqrt{t})\right] = \tilde{f}(i\sqrt{z}) + \tilde{f}(-i\sqrt{z})$.

8. Suppose $f(t)$ is ia locally integrable function on $(0, \infty)$ and has the asymptotic representation (Wong, 1989)

$$f(t) \sim \sum_{r=0}^{\infty} a_r t^{\alpha_r} \qquad \text{as } t \to 0+$$

where Re $\alpha_r \uparrow +\infty$ as $r \to \infty$, Re $\alpha_0 > -1$, and $f(t) = O(t^{-a})$, $a > 1$.

Show that the generalized Stieltjes transform

$$\tilde{f}(x) = \int_0^{\infty} \frac{f(t)\, dt}{(t+x)^{\rho}}, \qquad \rho > 0$$

has the asymptotic representation, as $x \to 0+$,

$$\tilde{f}(x) \sim \sum_{r=0}^{\infty} a_r \, \frac{\Gamma(1+\alpha_r)\Gamma(\rho-1-\alpha_r)}{\Gamma(\rho)}\, x^{1+\alpha_r-\rho} + \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(r+\rho)\mathscr{M}\left[f; 1-\rho-r\right]}{r!\,\Gamma(\rho)}\, x^r$$

provided $1 + \alpha_r \neq \rho + n$ for all non-negative integers $r$ and $n$.

9. Show that the one-sided Hilbert transform involved in research on water waves by Hulme (1981)

$$\hat{f}_{\boldsymbol{H}}(x) = \int_0^{\infty} \frac{J_0^2(t)\, dt}{t-x}$$

satisfies the Parseval relation

$$\hat{f}_{\boldsymbol{H}}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \mathscr{M}\left[J_0^2(x); p\right] \pi \cot \pi p \, dp.$$

10. Prove the following asymptotic expansions (Ursell, 1983):

$$\oint_0^\infty \frac{J_0^2(t)}{t-x}\, dt \sim -\frac{1}{\pi x}(\log x + \gamma + 3\log 2) + \frac{1}{x}\cos 2x + \frac{1}{4x^2}\sin 2x$$
$$+\frac{1}{8\pi x^3}\left(\log x + \gamma + 3\log 2 - \frac{5}{2}\right) - \frac{5}{32x^3}\cos 2x, \quad \text{as } x \to \infty,$$

and

$$\oint_0^\infty \frac{J_0^2(t)}{t-x} dt \sim -\frac{\pi}{2} J_0(x)\, Y_0(x) - \sqrt{\pi} \sum_{r=0}^{\infty} \cos(\pi r) \frac{\Gamma(r+1)\, x^{2r+1}}{\left\{\Gamma\left(r+\frac{3}{2}\right)\right\}^3}, \quad \text{as } x \to 0.$$

11. If $\lambda = \frac{1}{\sqrt{\pi}}$, show that $f(t) = \frac{A}{\sqrt{t}}$, where $A$ is a constant, is the only solution of the integral equation

$$f(s) = \lambda \int_0^\infty e^{-st} f(t)\, dt.$$

12. If $\lambda = -\frac{1}{\sqrt{\pi}}$, show that

$$f(t) = A\left[\frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi t}} - \frac{2\log t}{\sqrt{t}}\right],$$

where $A$ is a constant, is the only solution of the integral equation as stated in Exercise 11.

13. Show that

$$L_{k,t}[(x+a)^{-1}] = c_k (2k-1)!\, t^{k-1} a^k (t+a)^{-2k}, (a > 0, t > 0, k = 2, 3, \ldots).$$

14. Prove that

$$\lim_{\varepsilon \to 0+} \frac{1}{\pi} \int_\varepsilon^\infty \left[\frac{f(x+u) - f(x-u)}{u}\right] du = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(t)}{(t-x)}\, dt.$$

15. Prove Parseval's formulas (9.3.9).

# 10

## *Finite Fourier Sine and Cosine Transforms*

> "Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them."
>
> Joseph Fourier

> "In the mathematical investigation I have usually employed such methods as present themselves naturally to a physicist. The pure mathematician will complain, and (it must be confessed) sometimes with justice, of deficient rigor. But to this question there are two sides. For, however important it may be to maintain a uniformly high standard in pure mathematics, the physicist may occasionally do well to rest content with arguments which are fairly satisfactory and conclusive from his point of view. To his mind, exercised in a different order of ideas, the more severe procedure of the pure mathematician may appear not more but less demonstrative. And further, in many cases of difficulty to insist upon highest standard would mean the exclusion of the subject altogether in view of the space that would be required."
>
> Lord Rayleigh

### 10.1 Introduction

This chapter deals with the theory and applications of finite Fourier sine and cosine transforms. The basic operational properties including convolution theorem of these transforms are discussed in some detail. Special attention is given to the use of these transforms to the solutions of boundary value and initial-boundary value problems.

The finite Fourier sine transform was first introduced by Doetsch (1935). Subsequently, the method has been developed and generalized by several authors including Kneitz (1938), Koschmieder (1941), Roettinger (1947), and Brown (1944).

## 10.2 Definitions of the Finite Fourier Sine and Cosine Transforms and Examples

Both finite Fourier sine and cosine transforms are defined from the corresponding Fourier sine and Fourier cosine series.

**DEFINITION 10.2.1** *(The Finite Fourier Sine Transform). If $f(x)$ is a continuous or piecewise continuous function on a finite interval $0 < x < a$, the finite Fourier sine transform of $f(x)$ is defined by*

$$\mathscr{F}_s\{f(x)\} = \tilde{f}_s(n) = \int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx, \tag{10.2.1}$$

*where $n = 1, 2, 3, \ldots$.*

It is a well-known result of the theory of Fourier series that the Fourier sine series for $f(x)$ in $0 < x < a$

$$\frac{2}{a}\sum_{n=1}^{\infty}\tilde{f}_s(n)\sin\left(\frac{n\pi x}{a}\right) \tag{10.2.2}$$

converges to the value $f(x)$ at each point of continuity in the interval $0 < x < a$ and to the value $\frac{1}{2}[f(x+0)+f(x-0)]$ at each point $x$ of the finite discontinuity in $0 < x < a$. In view of the definition (10.2.1), the *inverse Fourier sine transform* is given by

$$\mathscr{F}_s^{-1}\left\{\tilde{f}_s(n)\right\} = f(x) = \frac{2}{a}\sum_{n=1}^{\infty}\tilde{f}_s(n)\sin\left(\frac{n\pi x}{a}\right). \tag{10.2.3}$$

Clearly, both $\mathscr{F}_s$ and $\mathscr{F}_s^{-1}$ are linear transformations.

**DEFINITION 10.2.2** *(The Finite Fourier Cosine Transform). If $f(x)$ is a continuous or piecewise continuous function on a finite interval $0 < x < a$, the finite Fourier cosine transform of $f(x)$ is defined by*

$$\mathscr{F}_c\{f(x)\} = \tilde{f}_c(n) = \int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx, \tag{10.2.4}$$

*where $n = 0, 1, 2, \ldots$.*

It is also a well-known result of the theory of Fourier series that the Fourier cosine series for $f(x)$ in $0 < x < a$

$$\frac{1}{a}\,\tilde{f}_c(0) + \frac{2}{a}\sum_{n=1}^{\infty} \tilde{f}_c(n)\cos\left(\frac{n\pi x}{a}\right) \tag{10.2.5}$$

converges to $f(x)$ at each point of continuity in $0 < x < a$, and to $\frac{1}{2}[f(x+0) + f(x-0)]$ at each point $x$ of finite discontinuity in $0 < x < a$. By virtue of the definition (10.2.4), the *inverse Fourier cosine transform* is given by

$$\mathscr{F}_c^{-1}\left\{\tilde{f}_c(n)\right\} = f(x) = \frac{1}{a}\tilde{f}_c(0) + \frac{2}{a}\sum_{n=1}^{\infty} \tilde{f}_c(n)\cos\left(\frac{n\pi x}{a}\right). \tag{10.2.6}$$

Clearly, both $\mathscr{F}_c$ and $\mathscr{F}_c^{-1}$ are linear transformations.

When $a = \pi$, the finite Fourier sine and cosine transforms are defined, respectively, by (10.2.1) and (10.2.4) on the interval $0 < x < \pi$. The corresponding inverse transforms are given by the same results (10.2.3) and (10.2.6) with $a = \pi$. The transform of a function defined over an interval $0 < x < a$ can be written easily in terms of a transform on the standard interval $0 < x < \pi$. We substitute $\xi = \dfrac{\pi x}{a}$ to write (10.2.1) and (10.2.4) as follows:

$$\tilde{f}_s(n) = \int_0^a \sin\left(\frac{n\pi x}{a}\right) f(x)dx = \frac{a}{\pi}\int_0^{\pi} \sin(n\xi) f\left(\frac{a\xi}{\pi}\right) d\xi = \frac{a}{\pi}\mathscr{F}_s\left\{f\left(\frac{ax}{\pi}\right)\right\}$$

$$\tilde{f}_c(n) = \int_0^a \cos\left(\frac{n\pi x}{a}\right) f(x)dx = \frac{a}{\pi}\int_0^{\pi} \cos(n\xi) f\left(\frac{a\xi}{\pi}\right) d\xi = \frac{a}{\pi}\mathscr{F}_c\left\{f\left(\frac{ax}{\pi}\right)\right\}.$$

***Example 10.2.1***

Find the finite Fourier sine and cosine transforms of

$$\text{(a)} \quad f(x) = 1 \quad \text{and} \quad (b) \quad f(x) = x.$$

(a) We have

$$\mathscr{F}_s(1) = \tilde{f}_s(n) = \int_0^a \sin\left(\frac{n\pi x}{a}\right) dx = \frac{a}{n\pi}[1 - (-1)^n], \tag{10.2.7}$$

$$\mathscr{F}_c\{1\} = \tilde{f}_c(n) = \int_0^a \cos\left(\frac{n\pi x}{a}\right) dx = \left\{\begin{matrix} a, & n = 0 \\ 0, & n \neq 0 \end{matrix}\right\}. \tag{10.2.8}$$

(b) $$\mathscr{F}_s\{x\} = \int_a^a x \sin\left(\frac{n\pi x}{a}\right) dx = \frac{(-1)^{n+1}a^2}{n\pi}. \tag{10.2.9}$$

$$\mathscr{F}_c\{x\} = \int_0^a x \cos\left(\frac{n\pi x}{a}\right) dx = \left\{\begin{array}{ll} \dfrac{a^2}{2}, & n = 0 \\ \left(\dfrac{a}{n\pi}\right)^2 [(-1)^n - 1], & n \neq 0 \end{array}\right\}. \tag{10.2.10}$$

□

## 10.3 Basic Properties of Finite Fourier Sine and Cosine Transforms

As a preliminary to the solution of differential equations by the finite Fourier sine and cosine transforms, we now establish the transforms of derivatives of $f(x)$.

$$\mathscr{F}_s\{f'(x)\} = -\left(\frac{n\pi}{a}\right)\tilde{f}_c(n), \tag{10.3.1}$$

$$\mathscr{F}_s\{f''(x)\} = -\left(\frac{n\pi}{a}\right)^2 \tilde{f}_s(n) + \left(\frac{n\pi}{a}\right)[f(0) + (-1)^{n+1}f(a)], \tag{10.3.2}$$

$$\mathscr{F}_c\{f'(x)\} = \left(\frac{n\pi}{a}\right)\tilde{f}_s(n) + (-1)^n f(a) - f(0), \tag{10.3.3}$$

$$\mathscr{F}_c\{f''(x)\} = -\left(\frac{n\pi}{a}\right)^2 \tilde{f}_c(n) + (-1)^n f'(a) - f'(0). \tag{10.3.4}$$

Similar results can be obtained for the finite Fourier sine and cosine transforms of higher derivatives of $f(x)$.

Results (10.3.1)–(10.3.4) can be proved by integrating by parts. For example, we have

$$\mathscr{F}_s\{f'(x)\} = \int_0^a f'(x) \sin\left(\frac{n\pi x}{a}\right) dx,$$

which is, integrating by parts,

$$= \left[f(x)\sin\left(\frac{n\pi x}{a}\right)\right]_0^a - \frac{n\pi}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right) dx$$

$$= -\left(\frac{n\pi}{a}\right)\tilde{f}_c(n).$$

Similarly, we find that

$$\begin{aligned}
\mathscr{F}_c\{f'(x)\} &= \int_0^a f'(x)\cos\left(\frac{n\pi x}{a}\right)dx \\
&= \left[f(x)\cos\left(\frac{n\pi x}{a}\right)\right]_0^a + \frac{n\pi}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx \\
&= (-1)^n f(a) - f(0) + \frac{n\pi}{a}\tilde{f}_s(n).
\end{aligned}$$

This proves the result (10.3.3).

Results (10.3.2) and (10.3.4) and the results for higher derivatives can be obtained by the repeated application of the fundamental results (10.3.1) and (10.3.3).

***DEFINITION 10.3.1*** *(Odd Periodic Extension). A function $f_1(x)$ is said to be the odd periodic extension of the function $f(x)$, with period $2\pi$ if*

$$f_1(x) = \left\{\begin{array}{lll} f(x) & \textit{for} & 0 < x < \pi \\ -f(-x) & \textit{for} & -\pi < x < 0 \end{array}\right\}. \tag{10.3.5}$$

*Or, equivalently,*

$$\begin{aligned}
f_1(x) &= f(x) \qquad \textit{when} \quad 0 < x < \pi, \\
f_1(-x) &= -f_1(x), \qquad f_1(x+2\pi) = f_1(x) \quad \textit{for} -\infty < x < \infty.
\end{aligned} \tag{10.3.6}$$

*Similarly, the even periodic extension $f_2(x)$ of $f(x)$, with period $2\pi$ is defined in $-\pi < x < \pi$ by the equations*

$$f_2(x) = \left\{\begin{array}{lll} f(x) & \textit{for} & 0 < x < \pi \\ f(-x) & \textit{for} & -\pi < x < 0 \end{array}\right\}. \tag{10.3.7}$$

*Or, equivalently,*

$$\begin{aligned}
f_2(x) &= f(x) \qquad \textit{when} \quad 0 < x < \pi, \\
f_2(-x) &= f_2(x), \qquad f_2(x+2\pi) = f_2(x) \quad \textit{for} -\infty < x < \infty.
\end{aligned} \tag{10.3.8}$$

***THEOREM 10.3.1***

If $f_1(x)$ is the odd periodic extension of $f(x)$ with period $2\pi$, then, for any constant $\alpha$,

$$\mathscr{F}_s\{f_1(x-\alpha) + f_1(x+\alpha)\} = 2\cos n\alpha\,\mathscr{F}_s\{f(x)\}. \tag{10.3.9}$$

In particular, when $\alpha = \pi$, and $n = 1, 2, 3, \ldots,$

$$\mathscr{F}_s\{f(x-\pi)\} = (-1)^n\ \mathscr{F}_s\{f(x)\}. \tag{10.3.10}$$

Similarly, we obtain

$$\mathscr{F}_c\{f_1(x+\alpha)-f_1(x-\alpha)\}=2\sin(n\alpha)\,\mathscr{F}_s\{f(x)\}. \tag{10.3.11}$$

**PROOF** To prove (10.3.9), we follow Churchill (1972) and write the right hand side of (10.3.9) as

$$\begin{aligned}2\cos n\alpha \tilde{f}_s(n) &= 2\cos(n\alpha)\int_0^\pi \sin(nx)f(x)dx\\ &= 2\int_0^\pi \cos n\alpha \sin nx f_1(x)dx\\ &= \int_0^\pi [\sin n(x+\alpha)+\sin n(x-\alpha)]f_1(x)dx,\end{aligned}$$

which is, since the integrand is even function of $x$,

$$=\frac{1}{2}\int_{-\pi}^{\pi}[\sin n(x+\alpha)+\sin n(x-\alpha)]f_1(x)dx,$$

which is, by putting $x+\alpha=t$ and $x-\alpha=t$,

$$=\frac{1}{2}\int_{-\pi+\alpha}^{\pi+\alpha}\sin nt\, f_1(t-\alpha)\,dt+\frac{1}{2}\int_{-(\pi+\alpha)}^{\pi-\alpha}\sin nt\, f_1(t+\alpha)\,dt,$$

which is, since the integrands are periodic function of $t$ with period $2\pi$, and hence, the limits of integration can be replaced with limits $-\pi$ to $\pi$,

$$\begin{aligned}&=\frac{1}{2}\int_{-\pi}^{\pi}\sin nt\, f_1(t-\alpha)\,dt+\frac{1}{2}\int_{-\pi}^{\pi}\sin nt\, f_1(t+\alpha)\,dt\\ &=\frac{1}{2}\left[\int_{-\pi}^{0}+\int_{0}^{\pi}\right]\{\sin nt\, f_1(t-\alpha)\}dt\\ &\qquad\qquad+\frac{1}{2}\left[\int_{-\pi}^{0}+\int_{0}^{\pi}\right]\{\sin\, nt\, f_1(t+\alpha)\}dt. \end{aligned} \tag{10.3.12}$$

Furthermore,

$$\int_{-\pi}^{0}\sin nt\, f_1(t-\alpha)\,dt=\int_0^\pi \sin nx\, f_1(x+\alpha)\,dx$$

in which $f_1(-x-\alpha) = -f_1(x+\alpha)$ is used.

Making a similar change of variables in the third integral of (10.3.12), we obtain the formula

$$2\cos n\alpha\,\tilde{f}_s(n) = \int_0^\pi \sin\, nt\, f_1(t-\alpha)dt + \int_0^\pi \sin\, nx\, f_1(x+\alpha)dx,$$

which gives the desired result (10.3.9).

Finally, $f_1(x+\pi) = f_1(2\pi + x - \pi) = f_1(x-\pi) = -f_1(\pi - x)$, and when $0 < x < \pi$, $f_1(\pi - x) = f(\pi - x)$. Thus, when $\alpha = \pi$, result (10.3.9) becomes

$$\tilde{f}_s(n)\,\cos\, n\pi = \int_0^\pi \sin nx\, f(x-\pi)dx = \mathscr{F}_s\{f(x-\pi)\},$$

which reduces to (10.3.10).

The proof of (10.3.11) is similar to that of (10.3.9), and hence, is left to the reader. ∎

***THEOREM 10.3.2***

If $f_2(x)$ is the even periodic extension of $f(x)$ with period $2\pi$, then, for any constant $\alpha$,

$$\mathscr{F}_c\{f_2(x-\alpha) + f_2(x+\alpha)\} = 2\cos\, n\alpha\, \mathscr{F}_c\{f(x)\}, \tag{10.3.13}$$

$$\mathscr{F}_c\{f_2(x-\alpha) - f_2(x+\alpha)\} = 2\sin\, n\alpha\, \mathscr{F}_c\{f(x)\}. \tag{10.3.14}$$

This theorem is very much similar to that of Theorem 10.3.1, and hence, the proof is left to the reader.

In the notation of Churchill (1972), we introduce the *convolution* of two sectionally continuous periodic functions $f(x)$ and $g(x)$ defined in $-\pi < x < \pi$ by

$$f(x) * g(x) = \int_{-\pi}^{\pi} f(x-u)\, g(u)\, du. \tag{10.3.15}$$

Clearly, $f(x) * g(x)$ is continuous and periodic with period $2\pi$. The convolution is symmetric, that is, $f * g = g * f$. Furthermore, the convolution is an even function if $f(x)$ and $g(x)$ are both even or both odd. It is odd if either $f(x)$ or $g(x)$ is even or the other odd. We next prove the convolution theorem.

***THEOREM 10.3.3***

(*Convolution*). If $f_1(x)$ and $g_1(x)$ are the odd periodic extensions of $f(x)$ and $g(x)$ respectively on $0 < x < \pi$, and if $f_2(x)$ and $g_2(x)$ are the even periodic extensions of $f(x)$ and $g(x)$ respectively on $0<x<\pi$, then

$$\mathscr{F}_c\{f_1(x)*g_1(x)\} = -2\,\tilde{f}_s(n)\,\tilde{g}_s(n), \tag{10.3.16}$$
$$\mathscr{F}_c\{f_2(x)*g_2(x)\} = 2\,\tilde{f}_c(n)\,\tilde{g}_c(n), \tag{10.3.17}$$
$$\mathscr{F}_s\{f_1(x)*g_2(x)\} = 2\,\tilde{f}_s(n)\,\tilde{g}_c(n), \tag{10.3.18}$$
$$\mathscr{F}_s\{f_2(x)*g_1(x)\} = 2\,\tilde{f}_c(n)\,\tilde{g}_s(n). \tag{10.3.19}$$

Or, equivalently,

$$\mathscr{F}_c^{-1}\left\{\tilde{f}_s(n)\ \tilde{g}_s(n)\right\} = -\frac{1}{2}\{f_1(x)*g_1(x)\}, \tag{10.3.20}$$
$$\mathscr{F}_c^{-1}\left\{\tilde{f}_c(n)\ \tilde{g}_c(n)\right\} = \frac{1}{2}\{f_2(x)*g_2(x)\}, \tag{10.3.21}$$
$$\mathscr{F}_s^{-1}\left\{\tilde{f}_s(n)\ \tilde{g}_c(n)\right\} = \frac{1}{2}\{f_1(x)*g_2(x)\}, \tag{10.3.22}$$
$$\mathscr{F}_s^{-1}\left\{\tilde{f}_c(n)\ \tilde{g}_s(n)\right\} = \frac{1}{2}\{f_2(x)*g_1(x)\}. \tag{10.3.23}$$

**PROOF** To prove (10.3.16), we consider the product

$$2\,\tilde{f}_s(n)\,\tilde{g}_s(n) = 2\int_0^\pi \tilde{f}_s(n)\ \sin nu\, g(u)du,$$

which is, by using (10.3.11),

$$= \int_0^\pi g(u)\,[\mathscr{F}_c\{f_1(x+u)-f_1(x-u)\}]\,du$$
$$= \int_0^\pi g(u)\left[\int_0^\pi \{f_1(x+u)-f_1(x-u)\}\cos nx\right]du,$$

which is, by interchanging the order of integration,

$$= \int_0^\pi \cos(nx)\left[\int_0^\pi \{f_1(x+u)-f_1(x-u)\}\,g(u)\,du\right]dx. \tag{10.3.24}$$

Using the definition of convolution (10.3.15), introducing new variables of integration, and invoking the odd extension properties of $f_1(x)$ and $g_1(x)$, we obtain

$$f_1(x)*g_1(x) = \int_0^\pi [f_1(x-u)-f_1(x+u)]\,g(u)\,du \tag{10.3.25}$$
$$= I_1 - I_2 - I_3 + I_4, \tag{10.3.26}$$

where

$$I_1 = \int_0^x f(u)\, g(x+u)\, du, \qquad I_2 = \int_x^\pi f(u)\, g(u-x) du, \tag{10.3.27ab}$$

$$I_3 = \int_0^{\pi-x} f(u)\, g(x+u)\, du, \quad I_4 = \int_x^\pi f(u)\, g(2\pi - x - u) du. \tag{10.3.28ab}$$

In view of (10.3.25), we thus obtain the desired result (10.3.16) from (10.3.24). This completes the proof.

The other results included in Theorem 10.3.3 can be proved by the above method of proof.

As an example of convolution theorem, we evaluate the inverse cosine Fourier transform of $(n^2 - a^2)^{-1}$. We write, for $n \neq 0$,

$$\frac{1}{(n^2-a^2)} = \frac{n(-1)^{n+1}}{(n^2-a^2)} \cdot \frac{(-1)^{n+1}}{n} = \tilde{f}_s(n)\, \tilde{g}_s(n),$$

where $\tilde{f}_s(n) = n(-1)^{n+1}(n^2-a^2)^{-1}$ and $\tilde{g}_s(n) = \dfrac{(-1)^{n+1}}{n}$ so that $f(x) = \left(\dfrac{\sin ax}{\sin a\pi}\right)$ and $g(x) = \dfrac{x}{\pi}$.

Evidently,

$$\frac{1}{(n^2-a^2)} = \tilde{f}_s(n)\, \tilde{g}_s(n) = \mathscr{F}_s\left\{\frac{\sin ax}{\sin a\pi}\right\} \mathscr{F}_s\left\{\frac{x}{\pi}\right\}.$$

According to result (10.3.20),

$$\mathscr{F}_c^{-1}\left\{\frac{1}{(n^2-a^2)}\right\} = \mathscr{F}_c^{-1}\left\{\tilde{f}_s(n)\, \tilde{g}_s(n)\right\} = -\frac{1}{2}\, f_1(x) * g_1(x), \tag{10.3.29}$$

where $f_1(x)$ is the periodic extension of the odd function $f(x)$ with period $2\pi$ and $g_1(x) = \dfrac{x}{\pi}$. Thus, it turns out that

$$\mathscr{F}_c^{-1}\left\{\frac{1}{(n^2-a^2)}\right\} = -\frac{1}{2}\int_{-\pi}^{\pi} f_1(x-u)\, g_1(u)\, du = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(x-u)\, u\, du.$$

This integral can easily be evaluated by splitting up the interval of integration or, by using (10.3.26), and hence,

$$\mathscr{F}_c^{-1}\left\{\frac{1}{(n^2-a^2)}\right\} = -\frac{\cos\{a(\pi-x)\}}{a \sin a\pi}. \tag{10.3.30}$$

■

## 10.4 Applications of Finite Fourier Sine and Cosine Transforms

In this section we illustrate the use of finite Fourier sine and cosine transforms to the solutions of boundary value and initial-boundary value problems.

***Example 10.4.1***
(*Heat Conduction Problem in a Finite Domain with the Dirichlet Data at the Boundary*). We began by considering the solution of the temperature distribution $u(x,t)$ of the diffusion equation

$$u_t = \kappa\, u_{xx}, \qquad 0 \le x \le a,\ t > 0, \tag{10.4.1}$$

with the boundary and initial conditions

$$u(0,\,t) = 0 = u(a,t), \tag{10.4.2ab}$$

$$u(x,\,0) = f(x) \quad \text{for } 0 \le x \le a. \tag{10.4.3}$$

Application of the finite Fourier sine transform (10.2.1) to this diffusion problem gives the initial value problem

$$\frac{d\,\tilde{u}_s}{dt} + \kappa\left(\frac{n\pi}{a}\right)^2 \tilde{u}_s = 0, \tag{10.4.4}$$

$$\tilde{u}_s(n,\,0) = \tilde{f}_s(n). \tag{10.4.5}$$

The solution of (10.4.4)–(10.4.5) is

$$\tilde{u}_s(n,\,t) = \tilde{f}_s(n)\ \exp\left\{-\kappa\left(\frac{n\pi}{a}\right)^2 t\right\}. \tag{10.4.6}$$

The inverse finite Fourier sine transform (10.2.3) leads to the solution

$$\begin{aligned}
u(x,\,t) &= \frac{2}{a}\sum_{n=1}^{\infty} \tilde{f}_s(n)\ \exp\left\{-\kappa\left(\frac{n\pi}{a}\right)^2 t\right\}\sin\left(\frac{n\pi x}{a}\right)\\
u(x,t) &= \frac{2}{a}\sum_{n=1}^{\infty} \exp\left\{-\kappa\left(\frac{n\pi}{a}\right)^2 t\right\}\sin\left(\frac{n\pi x}{a}\right)\\
&\qquad\qquad \times \int_0^a f(\xi)\sin\left(\frac{n\pi\xi}{a}\right)d\xi.
\end{aligned} \tag{10.4.7}$$

If, in particular, $f(x) = T_0 =$ constant, then (10.4.7) becomes

$$u(x,t) = \left(\frac{2\,T_0}{\pi}\right)\sum_{n=1}^{\infty}\frac{1}{n}[1-(-1)^n]\exp\left\{-\kappa\left(\frac{n\pi}{a}\right)^2 t\right\}\sin\left(\frac{n\pi x}{a}\right). \tag{10.4.8}$$

This series solution can be evaluated numerically using the *Fast Fourier transform* which is an algorithm for the efficient calculation of the finite Fourier transform. ▯

**Example 10.4.2**
(*Heat Conduction Problem in a Finite Domain with the Neumann Data at the Boundary*). We consider the solution of the diffusion equation (10.4.1) with the prescribed heat flux at $x=0$ and $x=a$, and the associated boundary and initial data are

$$u_x(0, t)=0=u_x(a, t) \quad \text{for} \quad t>0, \tag{10.4.9}$$
$$u(x, 0)=f(x) \qquad \text{for} \quad 0 \le x \le a. \tag{10.4.10}$$

In this case, it is appropriate to use the finite Fourier cosine transform (10.2.4). So, the application of this transform gives the initial value problem

$$\frac{d\tilde{u}_c}{dt}+\kappa\left(\frac{n\pi}{a}\right)^2\tilde{u}_c=0, \tag{10.4.11}$$
$$\tilde{u}_c(n, 0)=\tilde{f}_c(n). \tag{10.4.12}$$

The solution of this problem is

$$\tilde{u}_c(n, t)=\tilde{f}_c(n)\exp\left\{-\kappa\left(\frac{n\pi}{a}\right)^2 t\right\}. \tag{10.4.13}$$

The inverse finite cosine transform (10.2.5) gives the formal solution

$$\begin{aligned} u(x, t) &= \frac{1}{a}\tilde{f}_c(0)+\frac{2}{a}\sum_{n=1}^{\infty}\tilde{f}_c(n)\exp\left\{-\kappa\left(\frac{n\pi}{a}\right)^2 t\right\}\cos\left(\frac{n\pi x}{a}\right) \\ &= \frac{1}{a}\int_0^a f(\xi)d\xi + \frac{2}{a}\sum_{n=1}^{\infty}\left[\int_0^a f(\xi)\cos\left(\frac{n\pi\xi}{a}\right)d\xi\right] \\ &\qquad \times\exp\left\{-\kappa\left(\frac{n\pi}{a}\right)^2 t\right\}\cos\left(\frac{n\pi x}{a}\right). \end{aligned} \tag{10.4.14}$$

▯

**Example 10.4.3**
(*The Static Deflection of a Uniform Elastic Beam*). We consider the static deflection $y(x)$ of a uniform elastic beam of finite length $\ell$ which satisfies the equilibrium equation

$$\frac{d^4y}{dx^4}=\frac{W(x)}{EI}=w(x), \;\; 0 \le x \le \ell, \tag{10.4.15}$$

where $W(x)$ is the applied load per unit length of the beam, $E$ is the Young's modulus of the beam, and $I$ is the moment of inertia of the cross section of the beam. If the beam is freely hinged at its ends, then

$$y(x) = y''(x) = 0 \quad \text{at } x = 0 \text{ and } x = \ell. \tag{10.4.16}$$

Application of the finite Fourier sine transform of $y(x)$ to (10.4.15) and (10.4.16) gives

$$\tilde{y}_s(n) = \left(\frac{\ell}{n\pi}\right)^4 \tilde{w}_s(n). \tag{10.4.17}$$

Inverting this result, we find

$$\begin{aligned} y(x) &= \frac{2\ell^3}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin\left(\frac{n\pi x}{\ell}\right) \tilde{w}_s(n) \\ &= \frac{2\ell^3}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin\left(\frac{n\pi x}{\ell}\right) \int_0^{\ell} w(\xi) \sin\left(\frac{n\pi \xi}{\ell}\right) d\xi. \end{aligned} \tag{10.4.18}$$

In particular, if the applied load of magnitude $W_0$ is confined to the point $x = \alpha$, where $0 < \alpha < \ell$, then $w(x) = W_0\ \delta(x - \alpha)$ where $W_0$ is a constant. Consequently, the static deflection is

$$y(x) = \frac{2\ell^3 W_0}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{n\pi \alpha}{\ell}\right). \tag{10.4.19}$$

□

***Example 10.4.4***

(*Transverse Displacement of an Elastic Beam of Finite Length*). We consider the transverse displacement of an elastic beam at a point $x$ in the downward direction where the equilibrium position of the beam is along the $x$-axis. With the applied load $W(x, t)$ per unit length of the beam, the displacement function $y(x, t)$ satisfies the equation of motion

$$\frac{\partial^4 y}{\partial x^4} + \frac{1}{a^2} \frac{\partial^2 y}{\partial t^2} = \frac{W(x, t)}{EI}, \quad 0 \le x \le \ell,\ t > 0, \tag{10.4.20}$$

where $a^2 = EI/(\rho\alpha)$, $\alpha$ is the cross-sectional area and $\rho$ is the line density of the beam.

If the beam is freely hinged at its ends, then

$$y(x, t) = \frac{\partial^2 y}{\partial x^2} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \ell. \tag{10.4.21}$$

The initial conditions are

$$y(x, t) = f(x),\ \frac{\partial y}{\partial t} = g(x) \quad \text{at} \quad t = 0 \quad \text{for} \quad 0 < x < \ell. \tag{10.4.22}$$

We use the joint Laplace transform with respect to $t$ and the finite Fourier sine transform with respect to $x$ defined by

$$\bar{\tilde{u}}_s(n,\, s) = \int_0^\infty e^{-st} dt \int_0^\ell u(x,\, t)\, \sin\left(\frac{n\pi x}{\ell}\right) dx. \tag{10.4.23}$$

Application of the double transform to (10.4.20)–(10.4.22) gives the solution for $\bar{\tilde{y}}_s(n,\, s)$ as

$$\bar{\tilde{y}}_s(n,\, s) = \frac{s\tilde{f}_s(n) + \tilde{g}_s(n)}{(s^2 + c^2)} + \left(\frac{a^2}{EI}\right) \frac{\overline{\tilde{W}}_s(n,\, s)}{(s^2 + c^2)}, \tag{10.4.24}$$

where $c = a\left(\frac{n\pi}{\ell}\right)^2$.

The inverse Laplace transform gives

$$\begin{aligned}\tilde{y}_s(n,\, t) &= \tilde{f}_s(n)\, \cos(ct) + \frac{\tilde{g}_s(n)}{c}\, \sin(ct)\\ &\quad + \left(\frac{a^2}{EI}\right) \frac{1}{c} \int_0^t \sin c(t-\tau) \tilde{W}_s(n,\, \tau) d\tau. \end{aligned} \tag{10.4.25}$$

Thus, the inverse finite Fourier sine transform yields the formal solution as

$$\begin{aligned} y(x,\, t) &= \frac{2}{\ell} \sum_{n=1}^\infty y_s(n, t) \sin\left(\frac{\pi n x}{\ell}\right),\\ &= \frac{2}{\ell} \sum_{n=1}^\infty \sin\left(\frac{n\pi x}{\ell}\right) \left[\left\{\tilde{f}_s(n) \cos(ct) + \frac{\tilde{g}_s(n)}{c} \sin(ct)\right\}\right.\\ &\quad \left. + \left(\frac{a^2}{EI}\right) \frac{1}{c} \int_0^t \sin c(t-\tau)\, \tilde{W}_s(n,\, \tau) d\tau \right], \end{aligned} \tag{10.4.26}$$

where

$$\tilde{f}_s(n) = \int_0^\ell f(\xi) \sin\left(\frac{n\pi\xi}{\ell}\right) d\xi, \quad \tilde{g}_s(n) = \int_0^\ell g(\xi) \sin\left(\frac{n\pi\xi}{\ell}\right) d\xi. \tag{10.4.27ab}$$

The case of free vibrations is of interest. In this case, $W(x,\, t) \equiv 0$ and hence, $\tilde{W}_s(n,\, t) \equiv 0$. Consequently, solution (10.4.26) reduces to a simple form

$$y(x, t) = \frac{2}{\ell} \sum_{n=1}^\infty \left[\tilde{f}_s(n) \cos ct + \frac{\tilde{g}_s(n)}{c} \sin ct\right] \sin\left(\frac{n\pi x}{\ell}\right), \tag{10.4.28}$$

where $\tilde{f}_s(n)$ and $\tilde{g}_s(n)$ are given by (10.4.27ab). ▯

**Example 10.4.5**

(*Free Transverse Vibrations of an Elastic String of Finite Length*). We consider the free vibration of a string of length $\ell$ stretched to a constant tension $T$ between two points $(0,0)$ and $(0,\ell)$ lying on the $x$-axis. The free transverse displacement function $u(x,\, t)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le \ell,\, t > 0, \tag{10.4.29}$$

where $c^2 = \dfrac{T}{\rho}$ and $\rho$ is the line density of the string.

The initial and boundary conditions are

$$u(x,\, t) = f(x), \quad \frac{\partial u}{\partial t} = g(x) \quad \text{at} \quad t = 0 \quad \text{for} \quad 0 \le x \le \ell, \tag{10.4.30ab}$$

$$u(x,\, t) = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \ell \quad \text{for} \quad t > 0. \tag{10.4.31ab}$$

Application of the joint Laplace transform with respect to $t$ and the finite Fourier sine transform with respect to $x$ defined by a similar result (10.4.23) to (10.4.29)–(10.4.31ab) gives

$$\bar{\tilde{u}}_s(n,\, s) = \frac{s\tilde{f}_s(n)}{(s^2 + a^2)} + \frac{\tilde{g}_s(n)}{(s^2 + a^2)}, \tag{10.4.32}$$

where $a^2 = \left(\dfrac{n\pi c}{\ell}\right)^2$.

The inverse Laplace transform gives

$$\tilde{u}_s(n, t) = \tilde{f}_s(n) \cos at + \frac{\tilde{g}_s(n)}{a} \sin at. \tag{10.4.33}$$

The inverse finite Fourier sine transform leads to the solution for $u(x, t)$ as

$$u(x,\, t) = \frac{2}{\ell} \sum_{n=1}^{\infty} \left[\tilde{f}_s(n) \cos at + \frac{\tilde{g}_s(n)}{a} \sin at\right] \sin\left(\frac{n\pi x}{\ell}\right), \tag{10.4.34}$$

where $\tilde{f}_s(n)$ and $\tilde{g}_s(n)$ are given by (10.4.27ab). ▯

**Example 10.4.6**

(*Two-Dimensional Unsteady Couette Flow*). We consider two-dimensional unsteady viscous flow between the plate at $z = 0$ at rest and the plate $z = h$ in motion parallel to itself with a variable velocity $U(t)$ in the $x$ direction. The fluid velocity $u(z, t)$ satisfies the equation of motion

$$\frac{\partial u}{\partial t} = -\frac{P(t)}{\rho} + \nu \frac{\partial^2 u}{\partial z^2}, \quad 0 \le z \le h,\, t > 0, \tag{10.4.35}$$

and the boundary and initial conditions

$$u(z,\, t) = 0 \qquad \text{on} \quad z = 0,\ t > 0; \tag{10.4.36}$$

$$u(z,\, t) = U(t) \quad \text{on} \quad z = h,\ t > 0; \tag{10.4.37}$$

$$u(z,\, t) = 0 \qquad \text{at} \quad t \le 0,\ \text{for } 0 \le z \le h; \tag{10.4.38}$$

where the pressure gradient $p_x = P(t)$ and $\nu$ is the kinematic viscosity of the fluid.

Application of the double transform defined by (10.4.23) to this initial boundary value problem gives the solution for $\bar{\tilde{u}}_s(n, s)$ as

$$\left(s + \frac{\nu\, n^2\pi^2}{h^2}\right) \bar{\tilde{u}}_s(n,\, t) = -\frac{h\bar{P}(s)}{n\pi\rho}[1 + (-1)^{n+1}] + \frac{\nu\, n\pi}{h}(-1)^{n+1}\bar{U}(s). \tag{10.4.39}$$

The inverse Laplace transform yields

$$\tilde{u}_s(n,\, t) = -\frac{h}{n\pi\rho}[1 + (-1)^{n+1}] \int_0^t P(t-\tau) \exp\left(-\frac{\nu\, n^2\pi^2\tau}{h^2}\right) d\tau + \frac{\nu\, n\pi}{h}(-1)^{n+1} \int_0^t U(t-\tau) \exp\left(-\frac{\nu\, n^2\pi^2\tau}{h^2}\right) d\tau. \tag{10.4.40}$$

Finally, the inverse finite Fourier sine transform gives the formal solution

$$u(z,\, t) = \frac{2}{h}\sum_{n=1}^{\infty} \tilde{u}_s(n,\, t) \sin\left(\frac{n\pi z}{h}\right). \tag{10.4.41}$$

If, in particular, $P(t) = \text{constant } P$ and $U(t) = \text{constant} = U$, then (10.4.41) reduces to

$$u(z,\, t) = -\frac{2P}{\mu h}\sum_{n=1}^{\infty}\left(\frac{h}{n\pi}\right)^3 [1 + (-1)^{n+1}] \sin\left(\frac{n\pi z}{h}\right)\left[1 - \exp\left(-\frac{\nu\, n^2\pi^2 t}{h^2}\right)\right] + \frac{2U}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{h}{n\pi}\right) \sin\left(\frac{n\pi z}{h}\right)\left[1 - \exp\left(-\frac{\nu\, n^2\pi^2 t}{h^2}\right)\right]. \tag{10.4.42}$$

This solution for the velocity field consists of both steady-state and transient components. In the limit as $t \to \infty$, the transient component decays to zero, and the steady state is attained in the form

$$u(z,\, t) = -\frac{2P}{\mu h}\sum_{n=1}^{\infty}\left(\frac{h}{n\pi}\right)^3 [1 + (-1)^{n+1}] \sin\left(\frac{n\pi z}{\hbar}\right) + \frac{2U}{h^2}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{h^2}{n\pi}\right) \sin\left(\frac{n\pi z}{h}\right). \tag{10.4.43}$$

In view of the inverse finite Fourier sine transforms

$$\mathscr{F}_s^{-1}\left\{2\left(\frac{h}{n\pi}\right)^3\left[1+(-1)^{n+1}\right]\right\}=z(h-z), \tag{10.4.44}$$

$$\mathscr{F}_s^{-1}\left\{(-1)^{n+1}\left(\frac{h^2}{n\pi}\right)\right\}=z, \tag{10.4.45}$$

solution (10.4.43) can be rewritten in the closed form

$$u(z,\,t)=\frac{Uz}{h}-\frac{h}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left(1-\frac{z}{h}\right)z. \tag{10.4.46}$$

This is known as the *generalized Couette flow*. In the absence of the pressure gradient term, solution (10.4.46) reduces to the linear profile of *simple Couette flow*. On the other hand, if $U(t)\equiv 0$ and $P(t)\neq 0$, the solution (10.4.46) represents the parabolic profile of Poiseuille flow between two parallel stationary plates due to an imposed pressure gradient. $\square$

## 10.5 Multiple Finite Fourier Transforms and Their Applications

The above analysis for the finite Fourier sine and cosine transforms of a function of one independent variable can readily be extended to a function of several independent variables. In particular, if $f(x,\,y)$ is a function of two independent variables $x$ and $y$, defined in a region $0\le x\le a,\quad 0\le y\le b$, its *double finite Fourier sine transform* is defined by

$$\mathscr{F}_s\{f(x,y)\}=\tilde{f}_s(m,n)=\int_0^a\int_0^b \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy. \tag{10.5.1}$$

The *inverse transform* is given by the double series

$$\mathscr{F}_s^{-1}\left\{\tilde{f}_s(m,n)\right\}=f(x,\,y)=\left(\frac{4}{ab}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{f}_s(m,n)\,\sin\left(\frac{m\pi x}{a}\right)\times\sin\left(\frac{n\pi y}{b}\right). \tag{10.5.2}$$

Similarly, we can define the double finite Fourier cosine transform and its inverse.

The double Fourier sine transforms of the partial derivatives of $f(x,y)$ can easily be obtained. If $f(x,y)$ vanishes on the boundary of the rectangular region $D\{0 \le x \le a, \quad 0 \le y \le b\}$, then

$$\mathscr{F}_s\left[\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right]=-\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\tilde{f}_s(m,n). \tag{10.5.3}$$

**Example 10.5.1**

(*Free Vibrations of a Rectangular Elastic Membrane*). The initial value problem for the transverse displacement field $u(x,y,t)$ satisfies the following equation and the boundary and initial data

$$c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2 u}{\partial t^2}, \qquad \text{for all } (x,y) \text{ in } D,\ t>0, \tag{10.5.4}$$

$$u(x,y,t)=0 \qquad \text{on the boundary } \partial D \text{ for all } t>0, \tag{10.5.5}$$

$$u(x,y,t)=f(x,y),\ u_t(x,y,t)=g(x,y) \quad \text{at } t=0,\ \text{for } (x,y)\in D. \tag{10.5.6ab}$$

Application of the double finite Fourier sine transform defined by

$$\tilde{u}_s(m,n)=\int_0^a\int_0^b u(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy, \tag{10.5.7}$$

to the system (10.5.4)–(10.5.6ab) gives

$$\frac{d^2\tilde{u}_s}{dt^2}+c^2\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\tilde{u}_s=0, \qquad t>0 \tag{10.5.8}$$

$$\tilde{u}_s(m,n,0)=\tilde{f}_s(m,n),\ \left(\frac{d\tilde{u}_s}{dt}\right)_{t=0}=\tilde{g}_s(m,n). \tag{10.5.9}$$

The solution of this transformed problem is

$$\begin{aligned}\tilde{u}_s(m,n,t) &= \tilde{f}_s(m,n)\cos(c\pi\omega_{mn}t)\\ &\quad+(c\pi\omega_{mn})^{-1}\,\tilde{g}_s(m,n)\sin(c\pi\omega_{mn}t),\end{aligned} \tag{10.5.10}$$

where

$$\omega_{mn}=\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{\frac{1}{2}}. \tag{10.5.11}$$

The inverse transform gives the formal solution for $u(x,y,t)$ in the form

$$\begin{aligned}u(x,y,t) &= \left(\frac{4}{ab}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\big[\tilde{f}_s(m,n)\cos(c\pi\omega_{mn}t)\\ &\quad+(c\pi\omega_{mn})^{-1}\,\tilde{g}_s(m,n)\sin(c\pi\omega_{mn}t)\big],\end{aligned} \tag{10.5.12}$$

where

$$\tilde{f}_s(m,n) = \int_0^a \int_0^b f(\xi,\eta) \sin\left(\frac{m\pi\xi}{a}\right) \sin\left(\frac{n\pi\eta}{b}\right) d\xi d\eta, \tag{10.5.13}$$

$$\tilde{g}_s(m,n) = \int_0^a \int_0^b g(\xi,\eta) \sin\left(\frac{m\pi\xi}{a}\right) \sin\left(\frac{n\pi\eta}{b}\right) d\xi d\eta. \tag{10.5.14}$$

$\square$

***Example 10.5.2***

(*Deflection of a Simply Supported Rectangular Elastic Plate*). The deflection $u(x,y)$ of the plate satisfies the *biharmonic equation*

$$\nabla^4 u \equiv \frac{\partial^4 u}{\partial x^4} + 2\,\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = \frac{w(x,y)}{\mathscr{D}},$$
$$\text{in } D\{0 \le x \le a, \quad 0 \le y \le b\}, \tag{10.5.15}$$

where $w(x,y)$ represents the applied load at a point $(x,y)$ and $\mathscr{D} = \dfrac{2Eh^3}{3(1-\sigma^2)}$ is the constant flexural rigidity of the plate.

On the edge of the simply supported plate the deflection and bending moments are zero; hence, equation (10.5.15) has to be solved subject to the boundary conditions

$$\left.\begin{array}{lll} u(x,y) = 0 & \text{on } x=0 \text{ and } x=a \\ u(x,y) = 0 & \text{on } y=0 \text{ and } y=b \\ \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{on } x=0 \text{ and } x=a \\ \dfrac{\partial^2 u}{\partial y^2} = 0 & \text{on } y=0 \text{ and } y=b \end{array}\right\}. \tag{10.5.16}$$

We first solve the problem due to a concentrated load $W_0$ at the point $(\xi,\eta)$ inside $D$ so that $w(x,y) = P\ \delta(x-\xi)\ \delta(y-\eta)$, where $P$ is a constant.

Application of the double finite Fourier sine transform (10.5.7) to (10.5.15)–(10.5.16) gives

$$\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \tilde{u}_s(m,n) = \left(\frac{P}{\mathscr{D}}\right) \sin\left(\frac{m\pi\xi}{a}\right) \sin\left(\frac{n\pi\eta}{b}\right),$$

or,

$$\tilde{u}_s(m,n) = \left(\frac{P}{\mathscr{D}\,\pi^4 \omega_{mn}^4}\right) \sin\left(\frac{m\pi\xi}{a}\right) \sin\left(\frac{n\pi\eta}{b}\right), \tag{10.5.17}$$

where $\omega_{mn}$ is defined by (10.5.11).

The inverse transform gives the formal solution

$$u(x,y) = \left(\frac{4P}{\pi^4 ab\,\mathscr{D}}\right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\omega_{mn}^{-4} \sin\left(\frac{m\pi\xi}{a}\right) \sin\left(\frac{n\pi\eta}{b}\right)\right] \times \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right). \quad (10.5.18)$$

For an arbitrary load $w(x,y)$ over the region $\alpha \le x \le \beta$, $\gamma \le y \le \delta$ inside the region $D$, we can replace $P$ by $w(\xi,\eta)\,d\xi d\eta$ and integrate over the rectangle $\alpha \le \xi \le \beta$, $\gamma \le \eta \le \delta$.

Consequently, the formal solution is obtained from (10.5.18) and has the form

$$u(x,y) = \left(\frac{4}{\pi^4 ab\mathscr{D}}\right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \int_{\alpha}^{\beta} \int_{\gamma}^{\delta} w(\xi,\eta) \sin\left(\frac{m\pi\xi}{a}\right) \sin\left(\frac{n\pi\eta}{b}\right) d\xi d\eta \right\} \times \omega_{mn}^{-4} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right). \quad (10.5.19)$$

□

## 10.6 Exercises

1. Find the finite Fourier cosine transform of $f(x) = x^2$.

2. Use the result (10.3.2) to prove

   (a) $\mathscr{F}_s\{x^2\} = \dfrac{a^3}{n\pi}(-1)^{n+1} - 2\left(\dfrac{a}{n\pi}\right)^3 [1 + (-1)^{n+1}]$,

   (b) $\mathscr{F}_s\{x^3\} = (-1)^n \dfrac{a^4}{\pi^2}\left(\dfrac{6}{n^3\pi^3} - \dfrac{1}{n\pi}\right)$.

3. Solve the initial-boundary value problem in a finite domain

$$\begin{aligned} u_t &= \kappa\, u_{xx}, && 0 \le x \le a,\ t > 0, \\ u(x,0) &= 0 && \text{for} \quad 0 \le x \le a, \\ u(0,t) &= f(t) && \text{for} \quad t > 0, \\ u(a,t) &= 0 && \text{for} \quad t > 0. \end{aligned}$$

4. Solve Exercise 3 above by replacing the only condition at $x = a$ with the radiation condition

$$u_x + hu = 0 \qquad \text{at } x = a,$$

where $h$ is a constant.

5. Solve the heat conduction problem

$$\begin{aligned} &u_t = \kappa\, u_{xx}, && 0 \le x \le a,\ t > 0, \\ &\left.\begin{aligned} &u_x(0,\, t) = f(t) \\ &u_x(a,\, t) + h\, u = 0 \end{aligned}\right\} && \text{for } t > 0, \\ &u(x,\, 0) = 0 && \text{for } 0 \le x \le a. \end{aligned}$$

6. Solve the diffusion equation (10.4.1) with the following boundary and initial data

$$\begin{aligned} &u_x(0,\, t) = f(t), \quad u_x(a,\, t) = 0 \text{ for } t > 0 \\ &u(x, 0) = 0 \quad \text{for} \quad 0 \le x \le a. \end{aligned}$$

7. Solve the problem of free vibrations described in Example 10.4.4, when the beam is at rest in its equilibrium position at time $t = 0$, and an impulse $I$ is applied at $x = \eta$, that is,

$$f(x) \equiv 0 \quad \text{and} \quad g(x) = \left(\frac{I}{\rho\alpha}\right) \delta(x - \eta).$$

8. Find the solution of the problem in Example 10.4.4 when

   (i) $W(x,\, t) = W_0\, \phi(t)\, \delta(x - \eta), \quad 0 < \eta < \ell;$

   (ii) a concentrated applied load is moving along the beam with a constant speed $U$, that is, $W(x,\, t) = W_0\, \phi(t) \delta(x - Ut)\, H(Ut - \ell)$, where $W_0$ is a constant.

9. Find the solution of the forced vibration of an elastic string of finite length $\ell$ which satisfies the forced wave equation

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + F(x,\, t), \quad 0 \le x \le \ell,\ t > 0,$$

with the initial and boundary data

$$\begin{aligned} u(x,\, t) = f(x),\ u_t = g(x) \quad \text{at } t = 0 \quad &\text{for } 0 \le x \le \ell, \\ u(0,\, t) = 0 = u(\ell,\, t) \quad &\text{for } t > 0. \end{aligned}$$

Derive the solution for special cases when $f(x) = 0 = g(x)$ with

   (i) an arbitrary non-zero $F(x, t)$, and

   (ii) $F(x,\, t) = \dfrac{P(t)}{T}\, \delta(x - a), \quad 0 \le a \le \ell$, where $T$ is a constant.

10. For the finite Fourier sine transform defined over $(0,\pi)$, show that

(a) $\mathscr{F}_s\left\{\dfrac{x}{2}(\pi-x)\right\}=\dfrac{1}{n^3}[1+(-1)^{n+1}]$

(b) $\mathscr{F}_s\left\{\dfrac{\sinh a(\pi-x)}{\sinh a\pi}\right\}=\dfrac{n}{(n^2+a^2)},\quad a\neq 0.$

11. For the finite Fourier cosine transform defined over $(0,\pi)$, show that

(a) $\mathscr{F}_c\{(\pi-x)^2\}=\dfrac{2\pi}{n^2}$ for $n=1,2,\ldots$; $\mathscr{F}_s\{(\pi-x)^2\}=\dfrac{\pi^3}{3}$ for $n=0$.

(b) $\mathscr{F}_c\{\cosh\ a(\pi-x)\}=\dfrac{a\sinh(a\pi)}{(n^2+a^2)}\quad$ for $a\neq 0$.

12. Use the finite Fourier sine transform to solve the problem of diffusion of electricity along a cable of length $a$. The potential $V(x,t)$ at any point $x$ of the cable of resistance $R$ and capacitance $C$ per unit length satisfies the diffusion equation

$$V_t=\kappa\ V_{xx},\qquad 0\le x\le a,\ t>0,$$

where $\kappa=(RC)^{-1}$ and the boundary conditions (the ends of the cable are earthed)

$$V(0,\,t)=0=V(a,t)\quad\text{for } t>0,$$

and the initial conditions

$$V(x,0)=\left\{\begin{array}{ll}\left(\dfrac{2V_0}{a}\right)x, & 0\le x\le\dfrac{a}{2}\\ \left(\dfrac{2V_0}{a}\right)(a-x), & \dfrac{a}{2}\le x\le a\end{array}\right\},$$

where $V_0$ is a constant.

13. Establish the following results

(a) $\mathscr{F}_s\left[\dfrac{d}{dx}\{f_1(x)*g_1(x)\}\right]=2n\tilde{f}_s(n)\tilde{g}_s(n),$

(b) $\mathscr{F}_s\left[\displaystyle\int_0^x\{f_1(u)*g_1(u)\}\,du\right]=\dfrac{2}{n}\,\tilde{f}_s(n)\tilde{g}_s(n).$

14. If $p$ is not necessarily an integer, we write

$$\tilde{f}_c(p)=\int_0^\pi f(x)\cos px\,dx\quad\text{and}\quad\tilde{f}_s(p)=\int_0^\pi f(x)\sin px\,dx.$$

Show that, for any constant $\alpha$,

(a) $\mathscr{F}_c\{2\,f(x)\cos\alpha x\} = \tilde{f}_c(n-\alpha) + \tilde{f}_c(n+\alpha)$,
(b) $\mathscr{F}_c\{2\,f(x)\sin\alpha x\} = \tilde{f}_s(n+\alpha) - \tilde{f}_s(n-\alpha)$,
(c) $\mathscr{F}_s\{2\,f(x)\cos\alpha x\} = \tilde{f}_s(n+\alpha) + \tilde{f}_s(n-\alpha)$,
(d) $\mathscr{F}_s\{2\,f(x)\sin\alpha x\} = \tilde{f}_c(n-\alpha) - \tilde{f}_c(n+\alpha)$.

15. Solve the problem in Example 10.5.2 for uniform load $W_0$ over the region $\alpha \le x \le \beta$ and $\gamma \le y \le \delta$.

16. Solve the problem of free oscillations of a rectangular elastic plate of density $\rho$ bounded by the two parallel planes $z = \pm h$. The deflection $u(x, y, t)$ satisfies the equation

$$\mathscr{D}\,\nabla^4 u + \rho h\, u_{tt} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b, \quad t > 0,$$

where the deflection and the bending moments are all zero at the edges.

# 11

# *Finite Laplace Transforms*

"The genius of Laplace was a perfect sledgehammer in bursting purely mathematical obstacles, but like that useful instrument, it gave neither finish nor beauty to the results ... nevertheless, Laplace never attempted the investigation of a subject without leaving upon it the marks of difficulties conquered: sometimes clumsily, sometimes indirectly, but still his end is obtained and the difficulty is conquered."

Anonymous

"It seems to be one of the fundamental features of nature that fundamental physics laws are described in terms of great beauty and power. As time goes on, it becomes increasingly evident that the rules that the mathematicians find interesting are the same as those that nature has chosen."

Paul Dirac

## 11.1 Introduction

The Laplace transform method is normally used to find the response of a linear system at any time $t$ to the initial data at $t = 0$ and the disturbance $f(t)$ acting for $t \geq 0$. If the disturbance or input function is $f(t) = \exp(at^2),\ a > 0$, the usual Laplace transform cannot be used to find the solution of an initial value problem because the Laplace transform of $f(t)$ does not exist. From a physical point of view, there seems to be no reason at all why the function $f(t)$ cannot be used as an acceptable disturbance for a system. It is often true that the solution at times later than $t$ would not affect the state at time $t$. This leads to the idea of introducing the *finite Laplace transform* in $0 \leq t \leq T$ in order to extend the power and usefulness of the usual Laplace transform in $0 \leq t < \infty$.

This chapter deals with the definition and basic operational properties of the finite Laplace transform. In Section 11.4, the method of the finite Laplace transform is used to solve the initial value problems and the boundary value problems. This chapter is essentially based on papers by Debnath and Thomas (1976) and Dunn (1967).

## 11.2 Definition of the Finite Laplace Transform and Examples

The *finite Laplace transform* of a continuous (or an almost piecewise continuous) function $f(t)$ in $(0,T)$ is denoted by $\mathscr{S}_T\{f(t)\}=\bar{f}(s,T)$, and defined by

$$\mathscr{S}_T\{f(t)\}=\bar{f}(s,T)=\int_0^T f(t)\,e^{-st}dt, \tag{11.2.1}$$

where $s$ is a real or complex number and $T$ is a finite number that may be positive or negative so that (11.2.1) can be defined in any interval $(-T_1,T_2)$. Clearly, $\mathscr{S}_T$ is a linear integral transformation.

The *inverse finite Laplace transform* is defined by the complex integral

$$f(t)=\mathscr{S}_T^{-1}\{\bar{f}(s,T)\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\bar{f}(s,T)\,e^{st}ds, \tag{11.2.2}$$

where the integral is taken over any open contour $\Gamma$ joining any two points $c-iR$ and $c+iR$ in the finite complex $s$ plane as $R\to\infty$.

If $f(t)$ is almost piecewise continuous, that is, it has at most a finite number of simple discontinuities in $0\le t\le T$. Moreover, in the intervals where $f(t)$ is continuous, it satisfies a *Lipschitz condition* of order $\alpha>0$. Under these conditions, it can be shown that the inversion integral (11.2.2) is equal to

$$\frac{1}{2\pi i}\int_\Gamma \bar{f}(s,T)e^{st}ds=\frac{1}{2}[f(t-0)+f(t+0)], \tag{11.2.3}$$

where $\Gamma$ is an arbitrary open contour that terminates with finite constant $c$ as $R\to\infty$. This is due to the fact that $\bar{f}(s,T)$ is an entire function of $s$.

It follows from (11.2.1) that if

$$\int f(t)\,e^{-st}dt=-F(s,t)\,e^{-st}, \tag{11.2.4}$$

then

$$\bar{f}(s,T)=F(s,0)-F(s,T)\,e^{-sT} \tag{11.2.5}$$

$$=\bar{f}(s)-F(s,T)\,e^{-sT}, \tag{11.2.6}$$

where $\bar{f}(s)$ is the usual Laplace transform defined by (3.2.1) and hence,

$$\bar{f}(s) = F(s,0) = \int_0^\infty e^{-st} f(t)\,dt. \tag{11.2.7}$$

Further, using (11.2.2) and (11.2.6), the inversion formula can be written as

$$f(t) = \frac{1}{2\pi i}\int_\Gamma F(s,0)\,e^{st}ds - \frac{1}{2\pi i}\int_\Gamma F(s,T)\,e^{s(t-T)}ds. \tag{11.2.8}$$

It is noted that the first integral may be closed in the left half of the complex plane. On the other hand, for $t < T$, the contour of the second integral must be closed in the right half-plane. We select $\Gamma$ so that all poles of $F(s,0)$ lie to the left of $\Gamma$. Thus, the first integral represents the solution of the initial value problem, and for $t < T$, the second integral vanishes. When $t > T$, the second integral may be closed in the left half of the complex plane so that $f(t) = 0$ for $t > T$. Thus, for the solution of the initial value problem, there is no need to consider the second integral, and this case is identical with the usual Laplace transform.

So, unlike the usual Laplace transform of a function $f(t)$, there is no restriction needed on the transform variable $s$ for the existence of the finite Laplace transform $\mathscr{L}_T\{f(t)\} = \bar{f}(s,T)$. Further, the existence of (11.2.1) does not require the exponential order property of $f(t)$. If a function $f(t)$ has the usual Laplace transform, then it also has the finite Laplace transform. In other words, if $\bar{f}(s) = \mathscr{L}\{f(t)\}$ exists, then $\mathscr{L}_T\{f(t)\} = \bar{f}(s,T)$ exists as shown below. We have

$$\bar{f}(s) = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt. \tag{11.2.9}$$

Since $\bar{f}(s)$ exists, both the integrals on the right of (11.2.9) exist. Hence, the first integral in (11.2.9) exists and defines $\bar{f}(s,T)$.

However, the converse of this result is not necessarily true. This can be shown by an example. It is well known that the ususal Laplace transform of $f(t) = \exp(at^2),\ a > 0$, does not exist. But the finite Laplace transform of this function exists as shown below.

$$\begin{aligned}\bar{f}(s,T) &= \mathscr{L}_T\{\exp(at^2)\} = \int_0^T \exp(-st + at^2)dt \\ &= \exp\left(-\frac{s^2}{4a}\right)\int_0^T \exp\left[-\left(\sqrt{a}\,t - \frac{s}{2\sqrt{a}}\right)i\right]^2 dt \\ &= \frac{1}{2i}\left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left(-\frac{s^2}{4a}\right)\left[erf\left\{\left(\sqrt{a}\,T - \frac{s}{2\sqrt{a}}\right)i\right\}\right. \\ &\qquad \left. + erf\left(\frac{s\,i}{2\sqrt{a}}\right)\right].\end{aligned} \tag{11.2.10}$$

In the limit as $T \to \infty$, (11.2.10) does not exist as seen below.

We use the result (see Carslaw and Jeager, 1953, p. 48), to obtain

$$\text{erf}(z) = \exp(-z^2)\left[1 + \frac{2i}{\pi}\int_0^z e^{x^2}\,dx\right] \to \infty \quad \text{as } z \to \infty, \tag{11.2.11}$$

where

$$z = \left(T\sqrt{a} - \frac{s}{2\sqrt{a}}\right) i.$$

This ensures that the right-hand side of (11.2.10) tends to infinity as $T \to \infty$. Thus, the usual Laplace transform of $\exp(at^2)$ does not exist as expected.

The solution of the final value problem is denoted by $f_{fi}$ and defined by

$$f_{fi}(t) = \frac{1}{2\pi i}\int_\Gamma F(s,T)\, e^{s(t-T)} ds, \tag{11.2.12}$$

where the contour $\Gamma$ lies to the left of the singularities of $F(s,t)$ or $F(s,0)$.

***THEOREM 11.2.1***

The solution of an initial value problem is identical with that of the final value problem.

**PROOF** Suppose $f_{in}$ is the solution of the initial value problem, and it is given by

$$f_{in}(t) = \frac{1}{2\pi i}\int_{Br} F(s,0)\, e^{st} ds, \tag{11.2.13}$$

where $Br$ is the *Bromwich contour* extending from $c - iR$ to $c + iR$ as $R \to \infty$. We next reverse the direction of $\Gamma$ in (11.2.12) and then subtract (11.2.13) from (11.2.12) to obtain

$$\begin{aligned} f_{in}(t) - f_{fi}(t) &= \frac{1}{2\pi i}\int_C \{F(s,0) - F(s,T)\, e^{-sT}\}\, e^{st} ds \\ &= \frac{1}{2\pi i}\int_C \bar{f}(s,T)\, e^{st} ds, \end{aligned} \tag{11.2.14}$$

where $C$ is a closed contour which contains all the singularities of $F(s,0)$ or $F(s,T)$. Thus, the integrand of (11.2.14) is an entire function of $s$ and hence, the integral around a contour $C$ must vanish by *Cauchy's Fundamental Theorem.*

Hence,

$$f_{in}(t) = f_{fi}(t) = f(t). \tag{11.2.15}$$

This completes the proof. ▮

We next calculate the finite Laplace transform of several elementary functions:

***Example 11.2.1***
If $f(t) = 1$, then

$$\mathscr{S}_T\{1\} = \bar{f}(s,T) = \int_0^T e^{-st}dt = \frac{1}{s}(1 - e^{-sT}). \tag{11.2.16}$$

□

***Example 11.2.2***
If $f(t) = e^{at}$, then

$$\mathscr{S}_T\{e^{at}\} = \bar{f}(s,T) = \int_0^T e^{-(s-a)t}dt = \frac{1 - e^{-(s-a)T}}{(s-a)}. \tag{11.2.17}$$

□

***Example 11.2.3***
If $f(t) = \sin\, at$ or $\cos\, at$, then

$$\begin{aligned}\mathscr{S}_T\{\sin\, at\} &= \int_0^T \sin\, at\, e^{-st}dt \\ &= \frac{a}{s^2+a^2} - \frac{e^{-sT}}{s^2+a^2}(s\,\sin\, aT + a\cos\, aT). \end{aligned} \tag{11.2.18}$$

$$\mathscr{S}_T\{\cos\, at\} = \frac{s}{s^2+a^2} + \frac{e^{-st}}{s^2+a^2}(a\,\sin\, aT - s\cos\, aT). \tag{11.2.19}$$

□

***Example 11.2.4***
If $f(t) = t$, then

$$\mathscr{S}_T\{t\} = \int_0^T t\, e^{-st}dt = \frac{1}{s^2} - \frac{e^{-sT}}{s}\left(\frac{1}{s} + T\right). \tag{11.2.20}$$

□

***Example 11.2.5***
If $f(t) = t^2$, then

$$\mathscr{S}_T\{t^2\} = \int_0^T t^2 e^{-st} dt = \frac{2}{s^3} - \frac{e^{-sT}}{s}\left(T^2 + \frac{2T}{s} + \frac{2}{s^2}\right). \tag{11.2.21}$$

More generally, if $f(t) = t^n$, then

$$\mathscr{S}_T\{t^n\} = \int_0^T t^n e^{-st} dt = \frac{n!}{s^{n+1}} - \frac{e^{-sT}}{s} \times \left\{T^n + \frac{n}{s} T^{n-1} + \frac{n(n-1)}{s^2} T^{n-2} + \cdots + \frac{n!\,T}{s^{n-1}} + \frac{n!}{s^n}\right\}. \tag{11.2.22}$$

□

***Example 11.2.6***
If $f(t) = t^a$, $a(> -1)$ is a real number, then

$$\mathscr{S}_T\{t^a\} = s^{-(a+1)}\,\gamma(a+1, s\mathrm{T}), \tag{11.2.23}$$

where $\gamma(\alpha, x)$ is called the *incomplete gamma function* and is defined by

$$\gamma(\alpha, x) = \int_0^x e^{-u} u^{\alpha-1} du. \tag{11.2.24}$$

We have

$$\begin{aligned} \mathscr{S}_T\{t^a\} &= \int_0^T t^a e^{-st} dt, \quad (u = st) \\ &= s^{-(a+1)} \int_0^{sT} u^a e^{-u} du = s^{-(a+1)}\,\gamma(a+1, sT). \end{aligned}$$

□

***Example 11.2.7***
If $0 < a < T$ and $f(t) = 0$ in $-a < t < 0$, then

$$\mathscr{S}_T\{f(t-a)\} = e^{-as} \int_0^{T-a} e^{-\tau s} f(\tau) d\tau = e^{-as} \bar{f}(s, T-a). \tag{11.2.25}$$

In particular,

$$\mathscr{S}_T\{H(t-a)\} = \int_a^T e^{-st}dt = \frac{1}{s}\,(e^{-as} - e^{-sT}). \tag{11.2.26}$$

□

**Example 11.2.8**

$$\mathscr{S}_T\{erf\ at\} = s^{-1}\exp\left(\frac{s^2}{4a^2}\right)\left[erf\left(aT + \frac{s}{2a}\right) - erf\left(\frac{s}{2a}\right)\right] - \frac{e^{-sT}}{s}\,erf(aT). \tag{11.2.27}$$

We have, by definition,

$$\mathscr{S}_T\{erf(at)\} = \int_0^T erf(at)\,e^{-st}dt$$

which is, integrating by parts and using the definition of $erf(at)$,

$$\begin{aligned}
&= -[s^{-1}e^{-st}erf(at)]_0^T + \frac{1}{s}\frac{2}{\sqrt{\pi}}\int_0^T \exp[-(st + a^2t^2)]\,dt \\
&= -\frac{e^{-sT}}{s}\,erf(aT) + \frac{1}{s}\exp\left(\frac{s^2}{4a^2}\right)\frac{2}{\sqrt{\pi}}\int_{\frac{s}{2a}}^{aT+\frac{s}{2a}} e^{-u^2}du \\
&= -\frac{e^{-sT}}{s}erf(aT) + \frac{1}{s}\exp\left(\frac{s^2}{4a^2}\right)\left[erf\left(aT + \frac{s}{2a}\right) - erf\left(\frac{s}{2a}\right)\right].
\end{aligned}$$

□

**Example 11.2.9**

If $f(t)$ is a periodic function with period $\omega$, then

$$\bar{f}(s,T) = \mathscr{S}_T\{f(t)\} = \frac{(1 - e^{-sT})}{(1 - e^{-s\omega})}\,\bar{f}(s,\omega), \tag{11.2.28}$$

where $T = n\omega$, and $n$ is a finite positive integer.

By definition, we have

$$\begin{aligned}\mathscr{S}_T\{f(t)\} &= \int_0^T f(t)\,e^{-st}dt \\ &= \int_0^{\omega} f(t)\,e^{-st}dt + \int_{\omega}^{2\omega} f(t)\,e^{-st}dt + \cdots + \int_{(n-1)\omega}^{n\omega} f(t)\,e^{-st}dt,\end{aligned}$$

which is, substituting $t = u + \omega,\ t = u + 2\omega, \cdots, t = u + (n-1)\omega$ in the second, third, and the last integral, respectively,

$$\begin{aligned}&= \int_0^{\omega} f(u)\,e^{-su}du + e^{-s\omega}\int_0^{\omega} f(u)\,e^{-su}du + \cdots + e^{-s(n-1)\omega}\int_0^{\omega} f(u)\,e^{-su}du \\ &= \left[1 + e^{-s\omega} + \cdots + e^{-s\omega(n-1)}\right]\int_0^{\omega} e^{-su} f(u)du \\ &= \frac{(1 - e^{-ns\omega})}{(1 - e^{-s\omega})}\bar{f}(s,\,\omega).\end{aligned}$$

In the limit as $n \to \infty$ $(T \to \infty)$, (11.2.28) reduces to the known result

$$\bar{f}(s) = (1 - e^{-s\omega})^{-1}\int_0^{\omega} e^{-su} f(u)du. \tag{11.2.29}$$

□

## 11.3 Basic Operational Properties of the Finite Laplace Transform

***THEOREM 11.3.1***
If $\mathscr{S}_T\{f(t)\} = \bar{f}(s, T)$, then

(a) (Shifting) $$\mathscr{S}_T\{e^{-at}f(t) = \bar{f}(s+a, T), \tag{11.3.1}$$

(b) (Scaling) $$\mathscr{S}_T\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a},\, aT\right). \tag{11.3.2}$$

The proofs are easy exercises for the reader.

***THEOREM 11.3.2***
(*Finite Laplace Transforms of Derivatives*). If $\mathscr{S}_T\{f(t)\} = \bar{f}(s,T)$, then

$$\mathscr{S}_T\{f'(t)\} = s\bar{f}(s,T) - f(0) + e^{-sT} f(T), \tag{11.3.3}$$
$$\mathscr{S}_T\{f''(t)\} = s^2\bar{f}(s,T) - sf(0) - f'(0) + sf(T)\,e^{-sT} + f'(T)\,e^{sT}. \tag{11.3.4}$$

More generally,

$$\mathscr{S}_T\{f^{(n)}(t)\} = s^n\bar{f}(s,T) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0) + e^{-sT}\sum_{k=1}^{n} s^{n-k} f^{(k-1)}(T). \tag{11.3.5}$$

**PROOF** We have, integrating by parts,

$$\mathscr{S}_T\{f'(t)\} = \int_0^T f'(t)\,e^{-st}dt = [f(t)\,e^{-st}]_0^T + s\int_0^T f(t)\,e^{-st}dt$$
$$= s\bar{f}(s,T) - f(0) + f(T)\,e^{-sT}.$$

Repeating this process gives (11.3.4). By induction, we can prove (11.3.5).

▮

***THEOREM 11.3.3***
(*Finite Laplace Transform of Integrals*). If

$$F(t) = \int_0^t f(u)du \tag{11.3.6}$$

so that $F'(t) = f(t)$ for all $t$, then

$$\mathscr{S}_T\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\,\{\bar{f}(s,T) - e^{-sT}F(T)\}. \tag{11.3.7}$$

**PROOF** We have from (11.3.3)

$$\mathscr{S}_T\{F'(t)\} = s\mathscr{S}_T\{F(t)\} - F(0) + e^{-sT}F(T).$$

Or,

$$\bar{f}(s,T) = s\,\mathscr{S}_T\left\{\int_0^t f(u)du\right\} + e^{-sT}F(T).$$

Hence,

$$\mathscr{S}_T\left\{\int_0^t f(u)du\right\} = \frac{1}{s}[\bar{f}(s,T) - e^{-sT}F(T)].$$

▮

***THEOREM 11.3.4***
If $\mathscr{S}_T\{f(t)\} = \bar{f}(s,T)$, then

$$\frac{d}{ds}\bar{f}(s,T) = \mathscr{S}_T\{(-t)f(t)\}, \tag{11.3.8}$$

$$\frac{d^2}{ds^2}\bar{f}(s,T) = \mathscr{S}_T\{(-t)^2 f(t)\}. \tag{11.3.9}$$

More generally,

$$\frac{d^n}{ds^n}\bar{f}(s,T) = \mathscr{S}_T\{(-t)^n f(t)\}. \tag{11.3.10}$$

The proofs of these results are easy exercises.

These results can be used to find the finite Laplace transform of the product $t^n$ and any derivatives of $f(t)$ in terms of $\bar{f}(s,T)$. In other words,

$$\begin{aligned}\mathscr{S}_T\{t^n f'(t)\} &= (-1)^n \frac{d^n}{ds^n}[\mathscr{S}_T\{f'(t)\}]\\ &= (-1)^n \frac{d^n}{ds^n}[s\,\bar{f}(s,T) - f(0) + f(T)e^{-sT}].\end{aligned}$$

Similarly, we obtain a more general result

$$\mathscr{S}_T\{t^n f^{(m)}(t)\} = (-1)^n \frac{d^n}{ds^n}[\mathscr{S}_T\{f^{(m)}(t)\}],$$

which is, by (11.3.5),

$$\begin{aligned}= (-1)^n \frac{d^n}{ds^n}\Bigg[s^m \bar{f}(s,T) - \sum_{k=1}^{m} s^{m-k} f^{(k-1)}(0)\\ + e^{-sT}\sum_{k=1}^{m} s^{m-k} f^{(k-1)}(T)\Bigg].\end{aligned}$$

Finally, we can find

$$\begin{aligned}\int_s^T \bar{f}(s,T)\,ds &= \int_s^T ds \int_0^T f(t)\,e^{-st}dt = \int_0^T f(t)\,dt \int_s^T e^{-st}ds\\ &= \int_0^T \frac{f(t)}{t}\,e^{-st}dt - \int_0^T \frac{f(t)}{t}\,e^{-Tt}dt = \bar{g}(s,T) - \bar{g}(T,T),\end{aligned}$$

where $g(t) = \dfrac{f(t)}{t}$ and the existence of $\bar{g}(s,T)$ is assumed.

## 11.4 Applications of Finite Laplace Transforms

***Example 11.4.1***
Use the finite Laplace transform to solve the initial value problem

$$\frac{dx}{dt} + \alpha x = At, \qquad 0 \le t \le T, \tag{11.4.1}$$

$$x(t=0) = a. \tag{11.4.2}$$

Application of the finite Laplace transform gives

$$s\,\bar{x}(s,T) - x(0) + e^{-sT}x(T) + \alpha\,\bar{x}(s,T) = A\left[\frac{1}{s^2} - \frac{1}{s}\,e^{-sT}\left(\frac{1}{s} + T\right)\right].$$

Or,

$$\bar{x}(s,T) = \frac{a}{s+\alpha} - \frac{e^{-sT}x(T)}{s+\alpha} + \frac{A}{s+\alpha}\left[\frac{1}{s^2} - \frac{1}{s}\,e^{-sT}\left(\frac{1}{s} + T\right)\right]. \tag{11.4.3}$$

This is not an entire function, but it becomes an entire function by setting

$$\bar{x}(T) = \frac{AT}{\alpha} - \frac{A}{\alpha^2} + \frac{A}{\alpha^2}e^{-\alpha T} + a\,e^{-\alpha T} \tag{11.4.4}$$

so that

$$\bar{x}(s,T) = \left(a + \frac{A}{\alpha^2}\right)\left[\frac{1 - e^{-(s+\alpha)T}}{s+\alpha}\right] + \frac{A}{\alpha}\left[\frac{1}{s^2}(1 - e^{-sT}) - \frac{T}{s}e^{-sT}\right] - \frac{A}{\alpha^2}\left[\frac{1}{s}(1 - e^{-sT})\right]. \tag{11.4.5}$$

Using Table B-11 of finite Laplace transforms gives the final solution

$$x(t) = \left(a + \frac{A}{\alpha^2}\right)e^{-\alpha t} + \frac{At}{\alpha} - \frac{A}{\alpha^2}. \tag{11.4.6}$$

□

***Example 11.4.2***
Solve the simple harmonic oscillator governed by

$$\frac{d^2x}{dt^2} + \omega^2 x = F, \tag{11.4.7}$$

$$x(t=0)=a, \quad \dot{x}(t=0)=u, \tag{11.4.8ab}$$

where $F, a$, and $u$ are constants.

Application of the finite Laplace transform gives the solution

$$\bar{x}(s,T)=\frac{as}{s^2+\omega^2}+\frac{u}{s^2+\omega^2}-\frac{s\,e^{-sT}x(T)}{s^2+\omega^2} - \frac{e^{-sT}\dot{x}(T)}{s^2+\omega^2}+\frac{F}{s(s^2+\omega^2)}(1-e^{-sT}), \tag{11.4.9}$$

Since $\bar{x}(s,T)$ is not an entire function, we choose $x(T)$ such that

$$x(T)=\left(a-\frac{F}{\omega^2}\right)\cos\,\omega T+\frac{u}{\omega}\sin\,\omega T+\frac{F}{\omega^2} \tag{11.4.10}$$

$\bar{x}(s,T)$ becomes an entire function. Consequently, (11.4.9) becomes

$$\bar{x}(s,T)=\left(a-\frac{F}{\omega^2}\right)\left[\frac{s}{s^2+\omega^2}+\frac{e^{-sT}}{s^2+\omega^2}\{\omega\sin\omega T-s\,\cos\omega T\}\right] +\frac{u}{\omega}\left[\frac{\omega}{s^2+\omega^2}-\frac{e^{-sT}}{s^2+\omega^2}\{s\,\sin\omega T+\omega\cos\omega T\}\right] +\frac{F}{\omega^2}\left\{\frac{1-e^{-sT}}{s}\right\}. \tag{11.4.11}$$

Using Table B-11 of finite Laplace transforms, we invert (11.4.11) so that the solution becomes

$$x(t)=\left(a-\frac{F}{\omega^2}\right)\cos\omega t+\frac{u}{\omega}\,\sin\omega t+\frac{F}{\omega^2}. \tag{11.4.12}$$

□

***Example 11.4.3***

(*Boundary Value Problem*). The equation for the upward displacement of a taut string caused by a concentrated or distributed load $W(x)$ normalized with respect to the tension of the string of length $L$ is

$$\frac{d^2y}{dx^2}=W(x), \qquad 0\le x\le L \tag{11.4.13}$$

and the associated boundary conditions are

$$y(0)=y(L)=0. \tag{11.4.14}$$

We solve this boundary value problem due to a concentrated load of unit magnitude at a point $a$ where

$$W(x)=\delta(x-a), \qquad 0<a<L.$$

The use of the finite Laplace transform defined by

$$\bar{y}(s,L) = \int_0^L y(x)\, e^{-sx}\, dx \tag{11.4.15}$$

leads to the solution of (11.4.13)–(11.4.14) in the form

$$\bar{y}(s,L) = \frac{1}{s^2}[e^{-sa} + y'(0) - e^{-sL} y'(L)], \tag{11.4.16}$$

where $y'(x)$ denotes the derivative of $y(x)$ with respect to $x$. The function $\bar{y}(s,L)$ is not an entire function of $s$ unless the condition $y'(0) = y'(L) - 1$ is satisfied. Using this condition, solution (11.4.16) can be put in the form

$$\begin{aligned}\bar{y}(s,L) &= \frac{y'(0)}{s^2}\left[1 - e^{sL} - s\,L\,e^{-sL}\right] \\ &\quad + \frac{e^{-sa}}{s^2}\left[1 - e^{-s(L-a)} + y'(0) s\,L\,e^{-s(L-a)}\right]. \end{aligned} \tag{11.4.17}$$

In order to complete the inversion of (11.4.17), we set $Ly'(0) = a - L$ so that the inversion gives the solution

$$y(x) = x\,y'(0) + (x-a)\,H(x-a). \tag{11.4.18}$$

□

***Example 11.4.4***
(*Transient Current in a Simple Circuit*). The current $I(t)$ in a simple circuit (see Figure 4.4) containing a resistance $R$, and an inductance $L$ with an oscillating voltage $E(t) = E_0 \cos \omega t$ is given by

$$L\frac{dI}{dt} + RI = E_0 \cos \omega t, \qquad 0 \le t \le T, \tag{11.4.19}$$

$$I(t) = 0 \quad \text{at} \quad t = 0. \tag{11.4.20}$$

Application of the finite Laplace transform to (11.4.19)–(11.4.20) gives

$$\begin{aligned} s\bar{I}(s,T) &+ e^{-sT} I(T) + \frac{R}{L}\,\bar{I}(s,T) \\ &= \frac{E_0}{L}\left[\frac{s}{s^2+\omega^2} + \frac{e^{-sT}}{s^2+\omega^2}(\omega \sin \omega T - s\,\cos \omega T)\right]. \end{aligned}$$

Or,

$$\begin{aligned}\bar{I}(s,T) &= -\frac{e^{-sT} I(T)}{\left(s + \frac{R}{L}\right)} + \frac{E_0}{L\left(s+\frac{R}{L}\right)} \\ &\quad \times \left[\frac{s}{s^2+\omega^2} + \frac{e^{-sT}}{s^2+\omega^2}(\omega \sin \omega T - s\,\cos \omega T)\right]. \end{aligned} \tag{11.4.21}$$

Since $\bar{I}(s,T)$ is not an entire function, we make it entire by setting

$$I(T)=\frac{E_0}{L}\frac{\omega}{\left(\omega^2+\frac{R^2}{L^2}\right)}\left(\frac{R}{\omega L}\cos\omega T+\sin\omega T-\frac{R}{\omega L}e^{-\frac{RT}{L}}\right). \tag{11.4.22}$$

Putting this into (11.4.21) gives

$$\begin{aligned}\bar{I}(s,T) = &\frac{\omega E_0}{L\left(\omega^2+\frac{R^2}{L^2}\right)}\left[\frac{R}{\omega L}\left\{\frac{s}{s^2+\omega^2}+\frac{e^{-sT}}{s^2+\omega^2}(\omega \sin \omega T - s \cos \omega T)\right\}\right] \\ &+\frac{\omega E_0}{L\left(\omega^2+\frac{R^2}{L^2}\right)}\left[\left\{\frac{\omega}{s^2+\omega^2}-\frac{e^{-sT}}{s^2+\omega^2}(s \sin \omega T + \omega \cos \omega T)\right\}\right] \\ &-\frac{\omega E_0}{L\left(\omega^2+\frac{R^2}{L^2}\right)}\left[\frac{R}{\omega L\left(s+\frac{R}{L}\right)}\left\{1-e^{-\left(s+\frac{R}{L}\right)T}\right\}\right]. \end{aligned} \tag{11.4.23}$$

Using the table of the finite Laplace transform, we can invert (11.4.23) to obtain the solution

$$I(t)=\frac{\omega E_0}{L\left(\omega^2+\frac{R^2}{L^2}\right)}\left\{\frac{R}{\omega L}\cos \omega t+\sin \omega t-\frac{R}{\omega L}\exp\left(-\frac{Rt}{L}\right)\right\}. \tag{11.4.24}$$

Obviously, the first two terms in the curly brackets represent the steady-state current field, and the last term represents the transient current. In the limit $t\to\infty$, the transient term decays and the steady state is attained. $\square$

**Example 11.4.5**

(*Moments of a Random Variable*). Find the $n$th order moments of a random variable $X$ with the density function $f(x)$ in $0\le x\le T$.

It follows from definition (11.2.1) that the finite Laplace transform $\bar{f}(s,T)$ of the density function $f(x)$ can be interpreted as the mathematical expectation of $\exp(sX)$. In other words.

$$\bar{f}(s,T)=\mathscr{L}_T\{f(x)\}=E\{\exp(sX)\}=\int_0^T e^{sx}f(x)dx, \tag{11.4.25}$$

where $s$ is a real parameter.

Consequently,

$$\frac{d}{ds}\bar{f}(s,T)=\int_0^T x\,e^{sx}f(x)dx.$$

This result gives the definition of the expectation of $X$ as

$$m_1=\int_0^T x\,f(x)\,dx=\left[\frac{d}{ds}\,\bar{f}(s,T)\right]_{s=0}. \tag{11.4.26}$$

This implies that the mean of $X$ is expressed in terms of the derivative of the finite Laplace transform of the density function $f(x)$.

Similarly, differentiating (11.4.25) $n$ times with respect to $s$, we obtain

$$m_n = \int_0^T x^n f(x)\,dx = \left[\frac{d^n}{ds^n}\bar{f}(s,T)\right]_{s=0}. \tag{11.4.27}$$

In view of the result, the standard deviation and the variance of $X$ can be obtained in terms of the derivatives of the finite Laplace transform of the density function. ▯

## 11.5 Tauberian Theorems

***THEOREM 11.5.1***

If $\mathscr{S}_T\{f(t)\} = \bar{f}(s,T)$ exists, then

$$\lim_{s\to\infty} \bar{f}(s,T) = 0. \tag{11.5.1}$$

If, in addition, $\mathscr{S}_T\{f'(t)\}$ exists, then

$$\lim_{s\to\infty} [s\,\bar{f}(s,T)] = \lim_{t\to 0} f(t). \tag{11.5.2}$$

***THEOREM 11.5.2***

If $\mathscr{S}_T\{f(t)\} = \bar{f}(s,T)$ exists, then

$$\lim_{s\to 0} \bar{f}(s,T) = \int_0^T f(t)dt. \tag{11.5.3}$$

If, in addition, $\mathscr{S}_T\{f'(t)\}$ exists, then

$$\lim_{s\to 0} s\,\bar{f}(s,T) = 0. \tag{11.5.4}$$

The proofs of these theorems are similar to those for the usual Laplace transforms discussed in Section 3.8.

## 11.6 Exercises

1. Find the finite Laplace transform of each of the following functions:

(a) $\cosh at$, (b) $\sinh at$,

(c) $\exp(-at^2), \quad a > 0$, (d) $H(t)$,

(e) $t^n e^{-at}, \quad a > 0$, (f) $e^{-at} \sin bt$.

2. If $f(t)$ has a finite discontinuity at $t = a$, where $0 < a < T$, show that

$$\mathscr{S}_T \{f'(t)\} = s\,\bar{f}(s, T) + f(T)e^{-sT} - f(0) - e^{-sa}\,[f]_a,$$

where $[f]_a = f(a+0) - f(a-0)$.
Generalize this result if $f(t)$ has a finite number of finite discontinuities at $t = a_1,\ a_2, \ldots, a_n$ in [0,T].

3. Verify the result (11.3.7) when $f(t) = \sin\, at$.

4. Verify the Tauberian theorems for the function $f(t) = \exp(-at)$, $a > 0$.

5. Solve the initial value problem

$$\frac{d^2x}{dt^2} + \omega^2 x = A \exp(\alpha t^2), \quad (\alpha > 0), \quad 0 \le t \le T$$
$$x(0) = a, \quad \dot{x}(0) = u,$$

where $\omega, A, \alpha, a$ and $u$ are constants.

# 12

## *Z Transforms*

> "Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?"
>
> Paul R. Halmos

> "The shortest path between two truths in the real domain passes through the complex domain."
>
> Jacques Hadamard

## 12.1 Introduction

We begin this chapter with a brief introduction to the input-output characteristics of a linear dynamic system. Some special features of linear dynamic systems are briefly discussed. Analogous to the Fourier and Laplace transforms applied to the continuous linear systems, the $Z$ transform applicable to linear time-invariant discrete-time systems is studied in this chapter. The basic operational properties including the convolution theorem, initial and final value theorems, the $Z$ transform of partial derivatives, and the inverse $Z$ transform are presented in some detail. Applications of the $Z$ transform to difference equations and to the summation of infinite series are discussed with examples.

## 12.2 Dynamic Linear Systems and Impulse Response

In physical and engineering problems, a *system* is referred to as a physical device that can transform a *forcing* or *input function* (*input signal* or simply *signal*) $f(t)$ into an *output function* (*output signal* or *response*) $g(t)$ where $t$ is an independent time variable. In other words, the output is simply the response of the input due to the action of the physical device. Both input and output signals are functions of the continuous time variable. These may include steps or impulses. However, the input or the output or both may be sequences in the sense that they can assume values defined only for discrete values of time $t$. One of the essential features of a system is that the output $g(t)$ is completely determined by the given input function $f(t)$, and the characteristics of the system, and in some instances by the initial data. Usually, the action of the system is mathematically represented by

$$g(t) = L\, f(t), \tag{12.2.1}$$

where $L$ is a transformation (or operator) that transforms the input signal $f(t)$ to the output signal $g(t)$. The system is called *linear* if its operator $L$ is linear, that is, $L$ satisfies the *principle of superposition*. Obvious examples of linear operators are integral transformations.

Another fundamental characteristic of linear systems is that the response to an arbitrary input can be found by analyzing the input components of standard type and adding the responses to the individual components. The very nature of the delta function, $\delta(t)$, suggests that it can be used to represent the *unit impulse function*. In Section 2.4 it was shown that $\delta(t)$ satisfies the following fundamental property

$$f(t)\delta(t-t_n) = f(t_n)\delta(t-t_n), \tag{12.2.2}$$

where $t_n$ ($n$ is an integer) is any particular value of $t$ and $f(t)$ is a continuous function in any interval containing the point $t = t_n$. Result (12.2.2) is very important in the theory of sampling systems. Sampling of signals is very common in communication and digital systems. It is also used in pulse modulation systems and in all kinds of feedback systems where a digital computer is one of the common elements.

Summing (12.2.2) over all integral $n$ gives the *sampled function* $f^*(t)$ as

$$f^*(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t-t_n) = \sum_{n=-\infty}^{\infty} f(t_n)\,\delta(t-t_n). \tag{12.2.3}$$

Thus, the sampled function is approximately represented by a train of impulse functions, each having an area equal to the function at the sampling instant.

With $t_n = nT$, the series $\sum\limits_{n=-\infty}^{\infty} \delta(t-nT)$ is called the *impulse train* as shown in Figure 12.1.

As the sampling period $T$ assumes a small value $d\tau$, the function $f(t)$ in-

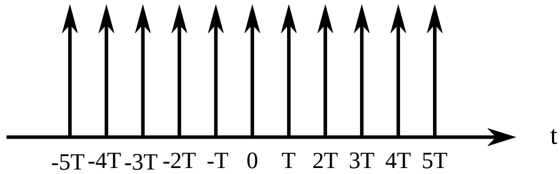


**Figure 12.1** The impulse train $\sum_{n=-\infty}^{\infty} \delta(t - nT)$.

volved in (12.2.3) can be written in the form

$$f(t) = \frac{\sum_{n=-\infty}^{\infty} f(n\,d\tau)\,\delta(t - n\,d\tau)}{\sum_{n=-\infty}^{\infty} \delta(t - n\,d\tau)}. \tag{12.2.4}$$

Multiplying the numerator and the denominator of (12.2.4) by $nd\tau$ and replacing $n\,d\tau$ by $d\tau$, we obtain

$$f(t) = \frac{\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau}{\int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau} = \int_{-\infty}^{\infty} f(t)\,\delta(t-\tau)\,d\tau. \tag{12.2.5}$$

Denoting the impulse response of the system to the input $\delta(t)$ by $h(t)$, the output is mathematically represented by the Fourier convolution as

$$g(t) = \int_{-\infty}^{\infty} f(\tau)\,h(t-\tau)\,d\tau = f(t) * h(t). \tag{12.2.6}$$

The convolution of $f(t)$ and the impulse train is $g(t)$ so that

$$\begin{aligned} g(t) &= f(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)\,\delta(t - nT - \tau)\,d\tau \\ &= \sum_{n=-\infty}^{\infty} f(t - nT). \end{aligned} \tag{12.2.7}$$

This represents the superposition of all translations of $f(t)$ by $nT$.

If the input function $f(\tau)$ is the impulse $\delta(\tau-\tau_n)$ located at $\tau_n$, then $h(t-\tau_n)$ is the response (output) of the system to the above impulse. This follows from (12.2.6) as

$$g(t)=\int_{-\infty}^{\infty} h(t-\tau)\,\delta(\tau-\tau_n)\,d\tau = h(t-\tau_n).$$

If the impulse is located at $\tau_0=0$, then $g(t)=h(t)$. This explains why the term impulse response of $h(t)$ was coined in the system's analysis, for the system's response to this particular case of an impulse located at $\tau_0=0$. The most important result in this section is (12.2.6), which gives the output $g(t)$ as the Fourier convolution product of the input signal $f(t)$ and the impulse response of the system $h(t)$. This shows an application of Fourier integral analysis to the analysis of linear dynamic systems.

Usually, the input is applied only for $t\ge 0$, and $h(t)=0$ for $t<0$. Hence, the output represented by (12.2.6) reduces to the Laplace convolution as

$$g(t)=\int_0^t f(\tau)\,h(t-\tau)\,d\tau = f(t)*h(t). \tag{12.2.8}$$

Physically, this represents the response for any input when the impulse response of any linear time invariant system is known.

We consider a certain waveform $f(t)$ shown in Figure 12.2 which is sampled periodically by a switch.

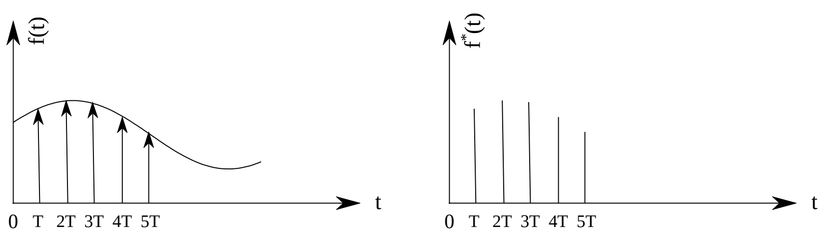


**Figure 12.2** Input and sampled functions.

We have seen earlier that the delta function takes on the value of the function at the instant at which it is applied, the sampled function $f^*(t)$ can be expressed as

$$f^*(t)=f(t)\sum_{n=0}^{\infty}\delta(t-nT)=\sum_{n=0}^{\infty} f(nT)\,\delta(t-nT). \tag{12.2.9}$$

Result (12.2.9) can be considered as the amplitude modulation of unit impulses by the waveform $f(t)$. Evidently, this result is very useful for analyzing the systems where signals are sampled at a time interval $T$. Thus, the above discussion enables us to introduce the $Z$ transform in the next section.

## 12.3 Definition of the $Z$ Transform and Examples

We take the Laplace transform of the sampled function given by (12.2.9) so that

$$\mathscr{L}\{f^*(t)\} = \bar{f}^*(s) = \sum_{n=0}^{\infty} f(nT)\exp(-nsT). \tag{12.3.1}$$

It is convenient to make a change of variable $z = \exp(sT)$ so that (12.3.1) becomes

$$\mathscr{L}\{f^*(t)\} = F(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}. \tag{12.3.2}$$

Thus, $F(z)$ is called the $Z$ *transform* of $f(nT)$. Since the interval $T$ between the samples has no effect on the properties and the use of the $Z$ transform, it is convenient to set $T = 1$. We now define the $Z$ *transform* of a sequence $\{f(n)\}$ as the function $F(z)$ of a complex variable $z$ defined by

$$Z\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}. \tag{12.3.3}$$

Thus, $Z$ is a linear transformation and can be considered as an operator mapping sequences of scalars into functions of the complex variable $z$. It is assumed in this chapter that there exists an $R$ such that (12.3.3) converges for $|z| > R$. Since $|z| = |\exp(sT)| = |\exp(\sigma + i\mu)T| = |\exp(\sigma T)|$, it follows that, when $\sigma < 0$ (that is, in the left half of the complex $s$ plane), $|z| < 1$, and thus, the left half of the $s$ plane corresponds to the interior of the unit circle in the complex $z$ plane. Similarly, the right half of the $s$ plane corresponds to the exterior ($|z| > 1$) of the unit circle in the $z$ plane. And $\sigma = 0$ in the $s$ plane corresponds to the unit circle in the $z$ plane.

The inverse $Z$ transform is given by the complex integral

$$Z^{-1}\{F(z)\} = f(n) = \frac{1}{2\pi i}\oint_C F(z)\, z^{n-1}\, dz, \tag{12.3.4}$$

where $C$ is a simple closed contour enclosing the origin and lying outside the circle $|z| = R$. The existence of the inverse imposes restrictions on $f(n)$ for uniqueness. We require that $f(n) = 0$ for $n < 0$.

To obtain the inversion integral, we consider

$$\begin{aligned}F(z) &= \sum_{n=0}^{\infty} f(n)\, z^{-n} \\ &= f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + \cdots + f(n)\, z^{-n} + f(n+1)\, z^{-(n+1)} + \cdots .\end{aligned}$$

Multiplying both sides by $(2\pi i)^{-1} z^{n-1}$ and integrating along the closed contour $C$, which usually encloses all singularities of $F(z)$, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\oint_C F(z) z^{n-1}\, dz = \frac{1}{2\pi i}\bigg[\oint_C f(0) z^{n-1}\, dz + \oint_C f(1) z^{n-2}\, dz \\ + \cdots + \oint_C f(n) z^{-1}\, dz + \oint_C f(n+1) z^{-2}\, dz + \cdots\bigg].\end{aligned}$$

By Cauchy's Fundamental Theorem all integrals on the right vanish except

$$\frac{1}{2\pi i}\oint_C f(n)\,\frac{dz}{z} = f(n).$$

This leads to the inversion integral for the $Z$ transform in the form

$$Z^{-1}\{F(z)\} = f(n) = \frac{1}{2\pi i}\oint_C F(z)\, z^{n-1}\, dz.$$

Similarly, we can define the so called *bilateral $Z$ transform* by

$$Z\{f(n)\} = F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n}, \tag{12.3.5}$$

for all complex numbers $z$ for which the series converges. This reduces to the unilateral $Z$ transform (12.3.3) if $f(n) = 0$ for $n < 0$. The inverse $Z$ transform is given by a complex integral similar to (12.3.4). Substituting $z = re^{i\theta}$ in (12.3.5), we obtain the $Z$ transform evaluated at $r = 1$

$$\mathscr{F}\{f(n)\} = F(\theta) = \sum_{n=-\infty}^{\infty} f(n) e^{-in\theta}.$$

This is known as the *Fourier transform of the sequence* $\{f(n)\}_{-\infty}^{\infty}$.

***Example 12.3.1***

If $f(n) = a^n$, $n \geq 0$, then

$$Z\{a^n\} = \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}, \quad |z| > a. \tag{12.3.6}$$

When $a = 1$, we obtain

$$Z\{1\} = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1}, \quad |z| > 1. \tag{12.3.7}$$

If $f(n) = n\,a^n$ for $n \geq 0$, then

$$Z\{n\,a^n\} = \sum_{n=0}^{\infty} n\,a^n\,z^{-n} = \frac{az}{(z-a)^2}, \quad |z| > |a|. \tag{12.3.8}$$

□

***Example 12.3.2***
If $f(n) = \exp(inx)$, then

$$Z\{\exp(inx)\} = \frac{z}{z - \exp(ix)}. \tag{12.3.9}$$

This follows immediately from (12.3.6).
Furthermore,

$$Z\{\cos nx\} = \frac{z(z - \cos x)}{z^2 - 2z\cos x + 1}, \quad Z\{\sin nx\} = \frac{z \sin x}{z^2 - 2z\cos x + 1}. \tag{12.3.10}$$

These follow readily from (12.3.9) by writing $\exp(inx) = \cos nx + i \sin nx$. □

***Example 12.3.3***
If $f(n) = n$, then

$$\begin{aligned} Z\{n\} &= \sum_{n=0}^{\infty} n\,z^{-n} = z \sum_{n=0}^{\infty} n\,z^{-(n+1)} \\ &= -z\frac{d}{dz}\left(\sum_{n=0}^{\infty} z^{-n}\right) = \frac{z}{(z-1)^2}, \quad |z| > 1. \end{aligned} \tag{12.3.11}$$

□

***Example 12.3.4***
If $f(n) = \frac{1}{n!}$, then

$$Z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!}\,z^{-n} = \exp\left(\frac{1}{z}\right) \quad \text{for all } z. \tag{12.3.12}$$

□

***Example 12.3.5***

If $f(n) = \cosh nx$, then

$$Z\{\cosh nx\} = \frac{z(z-\cosh x)}{z^2 - 2z\cosh x + 1}. \tag{12.3.13}$$

We have

$$\begin{aligned} Z\{\cosh nx\} &= \frac{1}{2} Z\{e^{nx} + e^{-nx}\} \\ &= \frac{1}{2}\left[\frac{z}{z-e^x} + \frac{z}{z-e^{-x}}\right] \\ &= \frac{z(z-\cosh x)}{z^2 - 2z\cosh x + 1}. \end{aligned}$$

□

***Example 12.3.6***

Show that

$$Z\{n^2\} = \frac{z(z+1)}{(z-1)^3}. \tag{12.3.14}$$

We have, from (12.4.13) in section 12.4,

$$Z\{n \cdot n\} = -z\frac{d}{dz} Z\{n\} = -z\frac{d}{dz}\frac{z}{(z-1)^2} = \frac{z(z+1)}{(z-1)^3}.$$

□

***Example 12.3.7***

If $f(n)$ is a periodic sequence of integral period $N$, then

$$F(z) = Z\{f(n)\} = \frac{z^N}{z^N - 1} F_1(z),$$

where

$$F_1(z) = \sum_{k=0}^{N-1} f(k)\, z^{-k}. \tag{12.3.15}$$

We have, by definition,

$$\begin{aligned} F(z) = Z\{f(n)\} &= \sum_{n=0}^{\infty} f(n)\, z^{-n} = z^N \sum_{n=0}^{\infty} f(n+N)\, z^{-(n+N)} \\ &= z^N \sum_{k=N}^{\infty} f(k)\, z^{-k}, \quad (n+N=k) \\ &= z^N \left[\sum_{k=0}^{\infty} f(k)\, z^{-k} - \sum_{k=0}^{N-1} f(k)\, z^{-k}\right] \\ &= \left\{z^N F(z) - z^N F_1(z)\right\}. \end{aligned}$$

Thus,

$$F(z) = \frac{z^N}{(z^N - 1)}\, F_1(z).$$

□

## 12.4 Basic Operational Properties of *Z* Transforms

***THEOREM 12.4.1***
(*Translation*). If $Z\{f(n)\} = F(z)$ and $m \geq 0$, then

$$Z\{f(n-m)\} = z^{-m}\left[F(z) + \sum_{r=-m}^{-1} f(r)\, z^{-r}\right], \tag{12.4.1}$$

$$Z\{f(n+m)\} = z^{m}\left[F(z) - \sum_{r=0}^{m-1} f(r)\, z^{-r}\right]. \tag{12.4.2}$$

In particular, if $m = 1, 2, 3, \ldots$, then

$$Z\{f(n-1)\} = z^{-1}F(z) - f(-1)\, z. \tag{12.4.3}$$

$$Z\{f(n-2)\} = z^{-2}\left[F(z) + \sum_{r=-2}^{-1} f(r) z^{-r}\right]. \tag{12.4.4}$$

and so on.

Similarly, it follows from (12.4.2) that

$$Z\{f(n+1)\} = z\{F(z) - f(0)\}, \tag{12.4.5}$$

$$Z\{f(n+2)\} = z^2\{F(z) - f(0)\} - z\, f(1), \tag{12.4.6}$$

$$Z\{f(n+3)\} = z^3\{F(z) - f(0)\} - z^2\, f(1) - z\, f(2). \tag{12.4.7}$$

More generally, for $m > 0$,

$$Z\{f(n+m)\} = z^m\{F(z) - f(0)\} - z^{m-1} f(1) - \cdots - z\, f(m-1). \tag{12.4.8}$$

All these results are widely used for the solution of initial value problems involving difference equations. Result (12.4.8) is somewhat similar to (3.4.12) for this Laplace transform, and has been used to solve initial value problems involving differential equations.

**PROOF** We have, by definition,

$$Z\{f(n-m)\} = \sum_{n=0}^{\infty} f(n-m)z^{-n}, \qquad (n-m=r),$$
$$= z^{-m} \sum_{r=-m}^{\infty} f(r)\, z^{-r} = z^{-m} \sum_{r=0}^{\infty} f(r) z^{-r} + z^{-m} \sum_{r=-m}^{-1} f(r) z^{-r}.$$

When $m=1$, we get (12.4.3).

If $f(r)=0$ for all $r<0$, then

$$Z\{f(n-m)\} = z^{-m} \sum_{r=0}^{\infty} f(r)\, z^{-r}. \tag{12.4.9}$$

When $m=1$, this result gives

$$Z\{f(n-1)\} = z^{-1} F(z). \tag{12.4.10}$$

Similarly, we prove (12.4.2) by writing

$$Z\{f(n+m)\} = \sum_{n=0}^{\infty} f(n+m) z^{-n}, \qquad (n+m=r),$$
$$= z^{m} \sum_{r=m}^{\infty} f(r)\, z^{-r} = z^{m} \sum_{r=0}^{\infty} f(r)\, z^{-r} - z^{m} \sum_{r=0}^{m-1} f(r)\, z^{-r}$$
$$= z^{m} \left[ F(z) - \sum_{r=0}^{m-1} f(r) z^{-r} \right].$$

When $m=1,2,3,\ldots$, results (12.4.5)–(12.4.7) follow immediately. ∎

***THEOREM 12.4.2***

(*Multiplication*). If $Z\{f(n)\} = F(z)$, then

$$Z\{a^n f(n)\} = F\left(\frac{z}{a}\right), \qquad |z| > |a|. \tag{12.4.11}$$

$$Z\{e^{-nb} f(n)\} = F(ze^{b}), \qquad |z| > |e^{-b}|. \tag{12.4.12}$$

$$Z\{n\, f(n)\} = -z \frac{d}{dz} F(z). \tag{12.4.13}$$

More generally,

$$Z\left[n^k\, f(n)\right] = (-1)^k \left( z \frac{d}{dz} \right)^k F(z), \quad k = 0, 1, 2, \ldots, \tag{12.4.14}$$

where

$$\left(z\frac{d}{dz}\right)^k F(z) = \left(z\frac{d}{dz}\right)^{(k-1)} \left(z\,\frac{d}{dz}\right) F.$$

**PROOF** Result (12.4.11) follows immediately from the definition (12.3.3), and (12.4.12) follows from (12.4.11) by writing $a = e^{-b}$.

If $f(n) = 1$ so that $Z\{f(n)\} = \frac{z}{z-1}$, and if $a = e^b$, then (12.4.1) gives

$$Z\{(e^b)^n\} = \frac{ze^{-b}}{ze^{-b}-1} = \frac{z}{z-e^b}, \quad |z| > |e^b|. \tag{12.4.15}$$

Putting $b = ix$ also gives (12.3.9)

To prove (12.4.13), we use the definition (12.3.3) to obtain

$$\begin{aligned} Z\{n\,f(n)\} &= \sum_{n=0}^{\infty} n\,f(n)\,z^{-n} = z\sum_{n=0}^{\infty} n\,f(n)\,z^{-(n+1)} \\ &= z\sum_{n=0}^{\infty} f(n)\left\{-\frac{d}{dz}z^{-n}\right\} = -z\frac{d}{dz}\left\{\sum_{n=0}^{\infty} f(n)\,z^{-n}\right\} = -z\frac{d}{dz}F(z). \end{aligned}$$

▮

***THEOREM 12.4.3***
(*Division*).

$$Z\left\{\frac{f(n)}{n+m}\right\} = -z^m \int_0^z \frac{F(\xi)d\xi}{\xi^{m+1}}. \tag{12.4.16}$$

**PROOF** We have

$$\begin{aligned} Z\left\{\frac{f(n)}{n+m}\right\} &= \sum_{n=0}^{\infty} \frac{f(n)}{n+m}\,z^{-n}, \quad (m \geq 0), \\ &= -z^m \sum_{n=0}^{\infty} f(n)\left[-\int_0^z \xi^{-(n+m+1)}\,d\xi\right] \\ &= -z^m \int_0^z \xi^{-(m+1)}\left[\sum_{n=0}^{\infty} f(n)\,\xi^{-n}\right] d\xi \\ &= -z^m \int_0^z \xi^{-(m+1)} F(\xi)\,d\xi. \end{aligned}$$

When $m = 0, 1, 2, \ldots$, several particular results follow from (12.4.16). ▮

***THEOREM 12.4.4***

(*Convolution*). If $Z\{f(n)\}=F(z)$ and $Z\{g(n)\}=G(z)$, then the $Z$ transform of the convolution $f(n)*g(n)$ is given by

$$Z\{f(n)*g(n)\}=Z\{f(n)\}Z\{g(n)\}, \tag{12.4.17}$$

where the *convolution* is defined by

$$f(n)*g(n)=\sum_{m=0}^{\infty} f(n-m)g(m). \tag{12.4.18}$$

Or, equivalently,

$$Z^{-1}\{F(z)G(z)\}=\sum_{m=0}^{\infty} f(n-m)g(m). \tag{12.4.19}$$

**PROOF** We proceed formally to obtain

$$Z\{f(n)*g(n)\}=\sum_{n=0}^{\infty} z^{-n}\sum_{m=0}^{\infty} f(n-m)g(m),$$

which is, interchanging the order of summation,

$$=\sum_{m=0}^{\infty} g(m)\sum_{n=0}^{\infty} f(n-m)z^{-n}.$$

Substituting $n-m=r$, we obtain

$$Z\{f(n)*g(n)\}=\sum_{m=0}^{\infty} g(m)z^{-m}\sum_{r=-m}^{\infty} f(r)z^{-r},$$

which is, in view of $f(r)=0$ for $r<0$,

$$\begin{aligned}&=\sum_{m=0}^{\infty} g(m)\,z^{-m}\sum_{r=0}^{\infty} f(r)\,z^{-r}\\ &=Z\{f(n)\}Z\{g(n)\}.\end{aligned}$$

This proves the theorem.

More generally, the convolution $f(n)*g(n)$ is defined by

$$f(n)*g(n)=\sum_{m=-\infty}^{\infty} f(n-m)\,g(m). \tag{12.4.20}$$

If we assume $f(n)=0=g(n)$ for $n<0$, then (12.4.20) becomes (12.4.18).

However, the $Z$ transform of (12.4.20) gives

$$Z\{f(n) * g(n)\} = \sum_{n=-\infty}^{\infty} z^{-n} \sum_{m=-\infty}^{\infty} f(n-m)\, g(m),$$

which is, interchanging the order of summation,

$$\begin{aligned}
&= \sum_{m=-\infty}^{\infty} g(m) \sum_{n=-\infty}^{\infty} f(n-m)\, z^{-n} \\
&= \sum_{m=-\infty}^{\infty} z^{-m}\, g(m) \sum_{n=-\infty}^{\infty} f(n-m)\, z^{-(n-m)} \\
&= \sum_{m=-\infty}^{\infty} z^{-m} g(m) \sum_{r=-\infty}^{\infty} f(r) z^{-r}, \quad (r = n-m) \\
&= Z\{f(n)\} Z\{g(n)\}. \qquad (12.4.21)
\end{aligned}$$

This is the convolution theorem for the bilateral $Z$ transform. ∎

The $Z$ transform of the product $f(n)\, g(n)$ is given by

$$Z\{f(n)\, g(n)\} = \frac{1}{2\pi i} \oint_C F(w)\, G\left(\frac{z}{w}\right) \frac{dw}{w}, \qquad (12.4.22)$$

where $C$ is a closed contour enclosing the origin in the domain of convergence of $F(w)$ and $G\left(\frac{z}{w}\right)$.

***THEOREM 12.4.5***
(*Parseval's Formula*). If $F(z) = Z\{f(n)\}$ and $G(z) = Z\{g(n)\}$, then

$$\sum_{n=-\infty}^{\infty} f(n)\, \overline{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta})\; \overline{G}(e^{i\theta})\, d\theta. \qquad (12.4.23)$$

In particular,

$$\sum_{n=-\infty}^{\infty} |f(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{i\theta})|^2\, d\theta. \qquad (12.4.24)$$

***THEOREM 12.4.6***
(*Initial Value Theorem*). If $Z\{f(n)\} = F(z)$, then

$$f(0) = \lim_{z\to\infty} F(z). \qquad (12.4.25)$$

Also, if $f(0) = 0$, then

$$f(1) = \lim_{z\to\infty} z\, F(z). \qquad (12.4.26)$$

**PROOF** We have, by definition,

$$F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n} = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots . \tag{12.4.27}$$

The initial value of $f(n)$ at $n=0$ is obtained from (12.4.27) by letting $z \to \infty$, and hence

$$f(0) = \lim_{z\to\infty} F(z).$$

If $f(0)=0$, then (12.4.27) gives

$$f(1) = \lim_{z\to\infty} zF(z).$$

This proves the theorem. ∎

***THEOREM 12.4.7***
(*Final Value Theorem*). If $Z\{f(n)\} = F(z)$, then

$$\lim_{n\to\infty} f(n) = \lim_{z\to 1}\{(z-1)F(z)\}. \tag{12.4.28}$$

provided the limits exist.

**PROOF** We have, from (12.3.3) and (12.4.5),

$$Z\{f(n+1) - f(n)\} = z\{F(z) - f(0)\} - F(z).$$

Or, equivalently,

$$\sum_{n=0}^{\infty}[f(n+1) - f(n)]\, z^{-n} = (z-1)\, F(z) - zf(0).$$

In the limit as $z \to 1$, we obtain

$$\lim_{z\to 1}\sum_{n=0}^{\infty}[f(n+1) - f(n)]\, z^{-n} = \lim_{z\to 1}(z-1)\, F(z) - f(0).$$

Or,

$$\lim_{n\to\infty}\, [f(n+1) - f(0)] = f(\infty) - f(0) = \lim_{z\to 1}(z-1)F(z) - f(0)$$

Thus,

$$\lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)F(z),$$

provided the limits exist.

This proves the theorem. The reader is referred to Zadeh and Desoer (1963) for a rigorous proof. ∎

***Example 12.4.1***
Verify the initial value theorem for the function

$$F(z) = \frac{z}{(z-a)(z-b)}.$$

We have

$$f(0) = \lim_{z\to\infty} \frac{z}{(z-a)(z-b)} = 0, \qquad f(1) = \lim_{z\to\infty} z\,F(z) = 1.$$

□

***THEOREM 12.4.8***
(*The Z Transform of Partial Derivatives*).

$$Z\left\{\frac{\partial}{\partial a} f(n,a)\right\} = \frac{\partial}{\partial a}[Z\{f(n,a)\}]. \tag{12.4.29}$$

**PROOF**

$$\begin{aligned} Z\left\{\frac{\partial}{\partial a} f(n,a)\right\} &= \sum_{n=0}^{\infty} \left[\frac{\partial}{\partial a} f(n,a)\right] z^{-n} \\ &= \frac{\partial}{\partial a}\left[\sum_{n=0}^{\infty} f(n,a) z^{-n}\right] = \frac{\partial}{\partial a}[Z\{f(n,a)\}]. \end{aligned}$$

As an example of this result, we show

$$Z\{n\,e^{an}\} = Z\left\{\frac{\partial}{\partial a}\, e^{na}\right\} = \frac{\partial}{\partial a} Z\{e^{na}\} = \frac{\partial}{\partial a}\left(\frac{z}{z-e^a}\right) = \frac{z\,e^a}{(z-e^a)^2}.$$

■

## 12.5 The Inverse *Z* Transform and Examples

The inverse $Z$ transform is given by the complex integral (12.3.4), which can be evaluated by using the Cauchy residue theorem. However, we discuss other simple ways of finding the inverse transform of a given $F(z)$. These include a method from the definition (12.3.3), which leads to the expansion of $F(z)$ as a series of inverse powers of $z$ in the form

$$F(z) = f(0) + f(1)\,z^{-1} + f(2)\,z^{-2} + \cdots + f(n) z^{-n} + \cdots\,. \tag{12.5.1}$$

The coefficient of $z^{-n}$ in this expansion is

$$f(n) = Z^{-1}\{F(z)\}. \tag{12.5.2}$$

If $F(z)$ is given by a series

$$F(z) = \sum_{n=-\infty}^{\infty} a_n\, z^{-n}, \qquad r_1 < z < r_2,$$

then its inverse $Z$ transform is unique and is equal to $\{f(n) = a_n\}$ for all $n$. If the domain of analyticity of $F(z)$ contains the unit circle $|z| = 1$, and if $F$ is single valued therein, then $F(e^{i\theta})$ is a periodic function with period $2\pi$ and hence, it can be expanded in a Fourier series. The coefficients of this series represent the inverse $Z$ transform of $F(z)$ and are given by

$$Z^{-1}\{F(z)\} = f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{i\theta})\, e^{in\theta} d\theta.$$

***Example 12.5.1***
Find the inverse $Z$ transform of $F(z) = z(z-a)^{-1}$.
We have

$$\begin{aligned} F(z) &= \frac{z}{z-a} = \left(1 - \frac{a}{z}\right)^{-1} \\ &= 1 + az^{-1} + a^2 z^{-2} + \cdots + a^n z^{-n} + \cdots . \end{aligned}$$

so that $f(0) = 1,\ f(1) = a,\ f(2) = a^2, \ldots, f(n) = a^n, \ldots$.
Obviously,

$$f(n) = Z^{-1}\left\{\frac{z}{(z-a)}\right\} = a^n.$$

□

***Example 12.5.2***
Find $Z^{-1}\left\{\exp\left(\frac{1}{z}\right)\right\}$.
Obviously,

$$\exp\left(\frac{1}{z}\right) = 1 + 1\cdot z^{-1} + \frac{1}{2!}z^{-2} + \cdots + \frac{1}{n!}z^{-n} + \cdots .$$

This gives

$$f(n) = \frac{1}{n!} = Z^{-1}\left\{\exp\left(\frac{1}{z}\right)\right\}.$$

Other methods for inversion use partial fractions and the Convolution Theorem 12.4.4. We illustrate these methods by the following examples. ∎

***Example 12.5.3***
Find the inverse $Z$ transform of

$$F(z) = \frac{z}{z^2 - 6z + 8}.$$

We write

$$F(z) = \frac{z}{(z-2)(z-4)} = \frac{1}{2}\left(\frac{z}{z-4} - \frac{z}{z-2}\right).$$

It follows from the table of $Z$ transforms that

$$f(n) = Z^{-1}\{F(z)\} = \frac{1}{2}\left[Z^{-1}\left\{\frac{z}{z-4}\right\} - Z^{-1}\left\{\frac{z}{z-2}\right\}\right] = \frac{1}{2}(4^n - 2^n).$$

∎

***Example 12.5.4***
Use the Convolution Theorem 12.4.4 to find the inverse of $\dfrac{z^2}{(z-a)(z-b)}$.

We set

$$F(z) = \frac{z}{z-a}, \qquad G(z) = \frac{z}{z-b}$$

so that

$$f(n) = Z^{-1}\{F(z)\} = a^n, \quad g(n) = Z^{-1}\{G(z)\} = b^n.$$

Thus, the convolution theorem gives

$$\begin{aligned} Z^{-1}\{F(z)\,G(z)\} &= \sum_{m=0}^{n} a^{n-m}\,b^m = a^n \sum_{m=0}^{n}\left(\frac{b}{a}\right)^m \\ &= a^n \cdot \left\{\frac{1-\left(\frac{b}{a}\right)^{n+1}}{1-\frac{b}{a}}\right\} = \frac{a^{n+1}}{(a-b)}\left\{1 - \left(\frac{b}{a}\right)^{n+1}\right\}. \end{aligned}$$

∎

***Example 12.5.5***
Find the inverse $Z$ transform of

$$F(z) = \frac{3z^2 - z}{(z-1)(z-2)^2}.$$

We write $F(z)$ as partial fractions

$$F(z) = \frac{3z^2 - z}{(z-1)(z-2)^2} = 2\cdot\frac{z}{(z-1)} - 2\cdot\frac{z}{(z-2)} + \frac{5}{2}\cdot\frac{2z}{(z-2)^2}$$

so that its inverse is

$$f(n) = Z^{-1}\left\{\frac{2z}{(z-1)}\right\} - Z^{-1}\left\{2\cdot\frac{z}{(z-2)}\right\} + \frac{5}{2}\,Z^{-1}\left\{\frac{2z}{(z-2)^2}\right\}$$

which is, by (12.3.6), and (12.4.13) with $f(n) = 2^n$,

$$= 2 - 2^{n+1} + \frac{5}{2}\cdot n\,2^n = 2 - 2^{n+1} + 5\cdot n\,2^{n-1}.$$

□

***Example 12.5.6***
Use the Convolution Theorem to show that

$$Z^{-1}\left\{\frac{z(z+1)}{(z-1)^3}\right\} = n^2.$$

We write

$$\frac{z(z+1)}{(z-1)^3} = \frac{z}{(z-1)^2}\left(\frac{z+1}{z-1}\right) = \frac{z}{(z-1)^2}\left[\frac{z}{z-1} + \frac{1}{z-1}\right].$$

Letting

$$F(z) = \frac{z}{(z-1)^2} \quad \text{and} \quad G(z) = \frac{z}{z-1} + \frac{1}{z-1},$$

we obtain

$$f(n) = n \quad \text{and} \quad g(n) = H(n) + H(n-1).$$

Thus,

$$Z^{-1}\left\{\frac{z(z+1)}{(z-1)^3}\right\} = f(n) * g(n) = \sum_{m=0}^{n} m[H(n-m) + H(n-m-1)] = n^2.$$

□

***Example 12.5.7***
(*Reconstruction of a Sequence from its Z Transform*).
Suppose

$$F(z) = \frac{z}{z-1}, \quad |z| > 1 \qquad \text{and} \qquad G(z) = \frac{z}{z-1}, \quad |z| < 1, \tag{12.5.3}$$

$$Z^{-1}\{F(z)\} = f(n) = \left\{\begin{matrix} 1, & n \ge 0, \\ 0, & n < 0, \end{matrix}\right\} \text{ and } Z^{-1}\{G(z)\} = g(n) = \left\{\begin{matrix} 1, & n \le 0, \\ 0, & n \ge 0, \end{matrix}\right\}. \tag{12.5.4}$$

This shows that the inverse $Z$ transform of $z(z-1)^{-1}$ is not unique. In general, the inverse $Z$ transform is not unique, unless its region of convergence is specified. □

## 12.6 Applications of Z Transforms to Finite Difference Equations

***Example 12.6.1***
(*First Order Difference Equation*). Solve the initial value problem for the difference equation

$$f(n+1) - f(n) = 1, \qquad f(0) = 0. \tag{12.6.1}$$

Application of the $Z$ transform to (12.6.1) combined with (12.4.5) gives

$$z[F(z) - f(0)] - F(z) = \frac{z}{z-1}.$$

Or,
$$F(z) = \frac{z}{(z-1)^2}.$$

The inverse $Z$ transform (see result (12.3.11)) gives the solution

$$f(n) = Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = n. \tag{12.6.2}$$

▯

***Example 12.6.2***
(*First Order Difference Equation*). Solve the equation

$$f(n+1) + 2f(n) = n, \quad f(0) = 1. \tag{12.6.3}$$

The use of the $Z$ transform to this problem gives

$$z\{F(z) - f(0)\} + 2F(z) = \frac{z}{(z-1)^2}.$$

Or,
$$\begin{aligned} F(z) &= \frac{z}{z+2} + \frac{z}{(z+2)(z-1)^2} \\ &= \frac{z}{z+2} + \frac{1}{9}\cdot\frac{z}{z+2} + \frac{3}{9}\cdot\frac{z}{(z-1)^2} - \frac{1}{9}\cdot\frac{z}{z-1} \\ &= \left(\frac{10}{9}\right)\frac{z}{(z+2)} + \frac{3}{9}\cdot\frac{z}{(z-1)^2} - \frac{1}{9}\frac{z}{(z-1)}. \end{aligned}$$

The inverse $Z$ transform yields the solution

$$f(n) = \frac{1}{9}[10(-2)^n + 3n - 1]. \tag{12.6.4}$$

▯

**Example 12.6.3**

(*The Fibonacci Sequence*). The Fibonacci sequence is defined as a sequence in which every term is the sum of the two proceeding terms. So it satisfies the difference equation

$$u_{n+1} = u_n + u_{n-1}, \qquad u_1 = u(0) = 1. \tag{12.6.5}$$

Application of the $Z$ transform gives

$$U(z) = \frac{z^2}{z^2 - z - 1}, \quad \text{where} \quad U(z) = Z\{u_n\}.$$

Thus, the inverse transform leads to the solution

$$u_n = Z^{-1}\left\{\frac{z^2}{z^2 - z - 1}\right\} = Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\},$$

where $a = \frac{1}{2}(1 + \sqrt{5})$ and $b = \frac{1}{2}(1 - \sqrt{5})$.

Using Example 12.5.4, the Fibonacci sequence is

$$u_n = \frac{a^{n+1} - b^{n+1}}{(a-b)}, \quad n = 0, 1, 2, \ldots \tag{12.6.6}$$

More explicitly, the Fibonacci sequence is given by $1, 1, 2, 3, 5, \ldots$ . ▯

**Example 12.6.4**

(*Second Order Difference Equation*). Solve the initial value problem

$$f(n+2) - 3f(n+1) + 2f(n) = 0, \;\; f(0) = 1, \;\; f(1) = 2. \tag{12.6.7}$$

Application of the $Z$ transform gives

$$z^2\{F(z) - f(0)\} - zf(1) - 3[z\{F(z) - f(0)\}] + 2F(z) = 0.$$

Or,

$$(z^2 - 3z + 2)F(z) = (z^2 - z).$$

Hence,

$$F(z) = \frac{z}{(z-2)}.$$

Thus, the inversion gives the solution

$$f(n) = Z^{-1}\left\{\frac{z}{(z-2)}\right\} = 2^n. \tag{12.6.8}$$

▯

***Example 12.6.5***
(*Periodic Solution*). Find the solution of the initial value problem

$$u(n+2) - u(n+1) + u(n) = 0, \tag{12.6.9}$$
$$u(0) = 1 \quad \text{and} \quad u(1) = 2. \tag{12.6.10}$$

The $Z$ transform of (12.6.9)–(12.6.10) gives

$$\{z^2 U(z) - z^2 - 2z\} - \{zU(z) - z\} + U(z) = 0.$$

Or,

$$U(z) = \frac{z^2 + z}{(z^2 - z + 1)} = \frac{\left(z^2 - \frac{1}{2}z\right)}{z^2 - z + 1} + \frac{\sqrt{3}\left(\frac{\sqrt{3}}{2}z\right)}{z^2 - z + 1}. \tag{12.6.11}$$

Writing $x = \frac{\pi}{3}$ in (12.3.10), the inverse $Z$ transform of (12.6.11) gives the periodic solution

$$u(n) = \cos\left(\frac{n\pi}{3}\right) + \sqrt{3}\,\sin\left(\frac{n\pi}{3}\right). \tag{12.6.12}$$

□

***Example 12.6.6***
(*Second Order Nonhomogeneous Difference Equation*). Solve the initial value problem

$$u(n+2) - 5\,u(n+1) + 6\,u(n) = 2^n, \quad u(0) = 1, u(1) = 0. \tag{12.6.13}$$

The $Z$ transform of (12.6.13) yields

$$(z^2 - 5z + 6)U(z) = z^2 - 5z + \frac{z}{z-2}. \tag{12.6.14}$$

Or,

$$\begin{aligned} U(z) &= z\left[\frac{z-5}{(z-2)(z-3)} + \frac{1}{(z-2)^2(z-3)}\right] \\ &= z\left[\left(\frac{3}{z-2} - \frac{2}{z-3}\right) + \left(\frac{1}{z-3} - \frac{1}{z-2} - \frac{1}{(z-2)^2}\right)\right] \\ &= z\left[\frac{2}{z-2} - \frac{1}{z-3} - \frac{1}{(z-2)^2}\right]. \end{aligned} \tag{12.6.15}$$

The inverse $Z$ transform of (12.6.15) gives the solution

$$u(n) = 2^{n+1} - 3^n - n\,2^{n-1}. \tag{12.6.16}$$

□

***Example 12.6.7***
(*Chebyshev Polynomials*). Solve the second order difference equation

$$u_{n+2} - 2xu_{n+1} + u_n = 0, \quad |x| \le 1, \tag{12.6.17}$$

$$u(0) = u_0 \quad \text{and} \quad u(1) = u_1, \tag{12.6.18}$$

where $u_0$ and $u_1$ are constants.

The $Z$ transform of equation (12.6.17) with (12.6.18) gives

$$z^2 U(z) - z^2 u_0 - zu_1 - 2x[z\,U(z) - zu_0] + U(z) = 0.$$

Or,

$$U(z) = u_0 \left[\frac{z^2 - zx}{z^2 - 2xz + 1}\right] + (u_1 - xu_0)\left[\frac{z}{z^2 - 2xz + 1}\right] \tag{12.6.19}$$

$$= u_0 \left[\frac{z^2 - zx}{z^2 - 2xz + 1}\right] + \frac{(u_1 - xu_0)}{\sqrt{1 - x^2}}\left[\frac{z\sqrt{1 - x^2}}{z^2 - 2xz + 1}\right]$$

$$= u_0 \left[\frac{z^2 - zx}{z^2 - 2xz + 1}\right] + v_0\left[\frac{z\sqrt{1 - x^2}}{z^2 - 2xz + 1}\right], \tag{12.6.20}$$

where $v_0 = (u_1 - xu_0)(1 - x^2)^{-\frac{1}{2}}$ is independent of $z$.

Since $|x| \le 1$, we may write $x = \cos t$ and then take the inverse $Z$ transform with the aid of (12.3.10) to obtain the solution

$$u_n = u_0 \cos nt + v_0 \sin nt \tag{12.6.21}$$

$$= u_0 \cos(n \cos^{-1} x) + v_0 \sin(n \cos^{-1} x). \tag{12.6.22}$$

Usually, the function

$$T_n(x) = \cos(n \cos^{-1} x) \tag{12.6.23}$$

is called the *Chebyshev polynomial of the first kind of degree n.*

The properties of this polynomial are presented in Appendix A-4. This polynomial plays an important role in the theory of special functions, and is found to be extremely useful in approximation theory and modern numerical analysis. □

## 12.7 Summation of Infinite Series

***THEOREM 12.7.1***
If $Z\{f(n)\} = F(z)$, then

(i) $$\sum_{k=1}^{n} f(k) = Z^{-1}\left\{\frac{z}{z-1}\,F(z)\right\}, \tag{12.7.1}$$

and

$$\text{(ii)} \quad \sum_{k=1}^{\infty} f(k) = \lim_{z\to 1} F(z) = F(1). \tag{12.7.2}$$

**PROOF** We write

$$g(n) = \sum_{k=0}^{n} f(k) \qquad \text{so that} \qquad g(n) = f(n) + g(n-1).$$

Application of the $Z$ transform gives

$$G(z) = F(z) + z^{-1}G(z)$$

so that

$$G(z) = \frac{z}{(z-1)}F(z).$$

Or,

$$Z\{g(n)\} = Z\left\{\sum_{k=0}^{n} f(k)\right\} = \frac{z}{(z-1)}F(z).$$

In the limit as $z \to 1$ together with the Final Value Theorem 12.4.7 gives

$$\lim_{n\to\infty} \sum_{k=0}^{n} f(k) = \lim_{z\to 1}(z-1)\cdot\frac{z}{z-1}\,F(z) = F(1).$$

This proves the theorem. ∎

***Example 12.7.1***
Use the $Z$ transform to show that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x. \tag{12.7.3}$$

We have, from (12.4.11),

$$Z\{x^n f(n)\} = F\left(\frac{z}{x}\right).$$

Setting $f(n) = \dfrac{1}{n!}$ so that $F(z) = \exp\left(\dfrac{1}{z}\right)$, we find

$$Z\left\{\frac{x^n}{n!}\right\} = \exp\left(\frac{x}{z}\right).$$

The use of Theorem 12.7.1(ii) gives

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{z \to 1} \exp\left(\frac{x}{z}\right) = e^x.$$

▯

***Example 12.7.2***
Show that

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \log(1+x). \tag{12.7.4}$$

Using (12.3.6), we find

$$Z\{x^{n+1}\} = \frac{zx}{z-x}$$

whence, in view of 4(b) in 12.8 Exercises,

$$\begin{aligned} Z\left\{\frac{x^{n+1}}{n+1}\right\} &= z\int_z^{\infty} \frac{zx}{(z-x)} \cdot \frac{dz}{z^2} \\ &= xz\int_z^{\infty} \frac{dz}{z(z-x)} \\ &= xz\left[\frac{1}{x}\log\left(\frac{z-x}{z}\right)\right]_z^{\infty} \\ &= -z\log\left(\frac{z-x}{z}\right). \end{aligned}$$

Replacing $x$ by $(-x)$ in this result, we obtain

$$Z\left\{(-1)^n \frac{x^{n+1}}{n+1}\right\} = z\log\left(\frac{z+x}{z}\right).$$

Application of Theorem 12.7.1(ii) gives

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1} = \lim_{z \to 1} z\log\left(\frac{z+x}{z}\right) = \log(1+x).$$

▯

***Example 12.7.3***
Find the sum of the series

$$\sum_{n=0}^{\infty} a^n \sin nx.$$

We know from (12.3.10) and (12.4.11) that

$$Z\{f(n)\} = Z\{\sin nx\} = \frac{z \sin x}{z^2 - 2z \cos x + 1},$$
$$Z\{a^n \sin nx\} = F\left(\frac{z}{a}\right) = \frac{az \sin x}{a^2 - 2az \cos x + z^2}.$$

Hence, Theorem 12.7.1(ii) gives

$$\sum_{n=0}^{\infty} a^n \sin nx = \lim_{z \to 1} F\left(\frac{z}{a}\right) = \frac{a \sin x}{a^2 - 2a \cos x + 1}. \tag{12.7.5}$$

$\square$

## 12.8 Exercises

1. Find the $Z$ transform of the following functions:

   (a) $n^3$, (b) $\frac{a^n}{n!}$, (c) $n \exp\{(n-1)\alpha\}$,

   (d) $H(n) - H(n-2)$, (e) $n^2 a^n$, (f) $\delta(n) = \left\{\begin{array}{ll} 1, & n = 0, \\ 0, & \text{otherwise} \end{array}\right\}$.

2. Show that

   (a) $Z\{\sinh na\} = \dfrac{z\,(\sinh a)}{z^2 - 2z \cosh a + 1}$,

   (b) $Z\{\exp(-an) \cos bn\} = \dfrac{z\,(z - e^{-a} \cos b)}{z^2 - 2\,z\,e^{-a} \cos b + e^{-2a}}$.

   (c) $Z\left\{e^{-an} \sin bn\right\} = \dfrac{e^a\, z\, \sin b}{e^{2a}\, z^2 - 2\, e^a\, z \cos b + 1}, \qquad |z| > e^{-a}$.

3. Show that

$$Z\{n\, a^n f(n)\} = -z \frac{d}{dz}\left\{F\left(\frac{z}{a}\right)\right\}.$$

4. Prove that

   (a) $Z\left\{\dfrac{f(n)}{n}\right\} = \displaystyle\int_z^{\infty} \frac{F(z)}{z}\, dz$,

   (b) $Z\left\{\dfrac{f(n)}{n+m}\right\} = z^m \displaystyle\int_z^{\infty} \frac{F(z)\, dz}{z^{m+1}}$.

Hence, deduce that

$$Z\left\{\frac{1}{n+1}\right\} = z\log\left(\frac{z}{z-1}\right).$$

5. Show that

   (a) $Z\{na^{n-1}\} = \dfrac{z}{(z-a)^2}$,

   (b) $Z\left\{\dfrac{n(n-1)\cdots(n-m+1)}{m!}\,a^{n-m}\right\} = \dfrac{z}{(z-a)^{m+1}}$.

6. Find the inverse $Z$ transform of the following functions:

   (a) $\dfrac{z^2}{(z-2)(z-3)}$, (b) $\dfrac{z^2-1}{z^2+1}$, (c) $\dfrac{z}{(z-1)^2}$,

   (d) $\dfrac{z}{(z-a)^2}$, (e) $\dfrac{1}{(z-a)^2}$, (f) $\dfrac{1}{(z-1)^2(z-2)}$,

   (g) $\dfrac{z+3}{(z+1)(z+2)}$, (h) $\dfrac{z^3}{(z^2-1)(z-2)}$, (i) $\dfrac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}$.

   (j) $F(z) = \frac{z^2}{(z-e^{-a})(z-e^{-b})}$, $a$, $b$ are constants.

   (k) $F(z) = (z-a)^{-k}$, $k = 1, 2, \ldots$, $|z| > |a| > 0$.

   (l) $F(z) = \frac{z^4+5}{(z-1)^2(z-2)}$, $|z| > 2$, (m) $F(z) = \frac{(z-1)}{(z+2)(z-\frac{1}{2})}$, $|z| > 2$.

7. Solve the following difference equations:

   (a) $f(n+1) + 3f(n) = n$, $f(0) = 1$.

   (b) $f(n+1) - 5f(n) = \sin n$, $f(0) = 0$.

   (c) $f(n+1) - af(n) = a^n$, $f(0) = x_0$.

   (d) $f(n+1) - f(n) = a[1 - f(n)]$, $f(0) = x_0$.

   (e) $f(n+2) - f(n+1) - 6f(n) = 0$, $f(0) = 0$, $f(1) = 3$.

   (f) $f(n+2) + 4f(n+1) + 3f(n) = 0$, $f(0) = 1$, $f(1) = 1$.

   (g) $f(n+2) - f(n+1) - 6f(n) = \sin\left(\dfrac{n\pi}{2}\right)$ $(n \geq 2)$, $f(0) = 0$, $f(1) = 3$.

   (h) $f(n+2) - 2f(n+1) + f(n) = 0$, $f(0) = 2$, $f(1) = 0$.

   (i) $f(n+2) - 2af(n+1) + a^2 f(n) = 0$, $f(0) = 0$, $f(1) = a$.

   (j) $f(n+3) - f(n+2) - f(n+1) + f(n) = 0$, $f(0) = 1, f(1) = f(2) = 0$.

   (k) $f(n) = f(n-1) + 2f(n-2)$, $f(0) = 1, f(1) = 2$.

   (l) $f(n) - a\,f(n-1) = 1$, $f(-1) = 2$.

(m) $f(n+2)+3\,f(n+1)+2\,f(n)=0,\ f(0)=1,\ f(1)=2.$

(n) $f(n+1)-2\,f(n)=0, \quad f(0)=3.$

8. Show that the solution of the resistive ladder network governed by the difference equation for the current field $i(n)$

$$i(n+2)-3\,i(n+1)+i(n)=0, \quad i(0)=1, \quad i(1)=2\,i(0)-\frac{V}{R}$$

is

$$i(n)=\cosh(xn)+\frac{2}{\sqrt{5}}\left(\frac{1}{2}-\frac{V}{R}\right)\sinh(nx),$$

where $\cosh\, x=\dfrac{3}{2}$ and $\sinh x=\dfrac{\sqrt{5}}{2}$.

9. Use the Initial Value Theorem to find $f(0)$ for $F(z)$ given by

(a) $\dfrac{z}{z-\alpha}$, (b) $\dfrac{z}{(z-\alpha)(z-\beta)}$,

(c) $\dfrac{z(z-\cos x)}{z^2-2z\cos x+1}$, (d) $\dfrac{1}{(z-a)^m}$.

10. Use the Final Value Theorem to find $\lim\limits_{n\to\infty} f(n)$ for $F(z)$:

(a) $F(z)=\dfrac{z}{z-a}$, (b) $F(z)=\dfrac{z^2-z\ \cos a}{(z^2-2z\ \cos a+1)}$.

11. Find the sum of the following series using the $Z$ transform:

(a) $\displaystyle\sum_{n=0}^{\infty} a^n e^{inx}$, (b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{e^{-n}}{n+1}$, (c) $\displaystyle\sum_{n=0}^{\infty}\exp[-x(2n+1)]$.

12. Solve the second order difference equation

$$3f(n+2)-2f(n+1)-f(n)=0, \quad f(0)=1, \quad f(1)=2$$

and then show that $f(n)\to\dfrac{7}{4}$ as $n\to\infty$.

13. Solve the simultaneous difference equations

$$\begin{aligned} u(n+1) &= 2\,v(2)+2,\\ v(n+1) &= 2\,u(n)-1, \end{aligned}$$

with the initial data $u(0)=v(0)=0$.

14. Show that the solution of the third order difference equation

$$u(n+3) - 3\,u(n+2) + 3\,u(n+1) - u(n) = 0,$$
$$u(0) = 1,\ u(1) = 0,\ u(2) = 1,$$

is

$$u(n) = (n-1)^2.$$

15. Show that the solution of the initial value problem

$$u(n+2) - 4\,u(n+1) + 3\,u(n) = 0, \quad u(0) = u_0 \text{ and } u(1) = u_1$$

is

$$u_n = \frac{1}{2}(3u_0 - u_1) + \frac{1}{2}(u_1 - u_0)3^n.$$

16. Find the solution of the following initial value problems:

(a) $u_{n+2} + 2\,u_{n+1} - 3\,u_n = 0, \quad u_0 = 1, \quad u_1 = 0,$

(b) $3\,u_{n+2} - 5\,u_{n+1} + 2\,u_n = 0, \quad u_0 = 1, \quad u_1 = 0,$

(c) $u_{n+2} - 4\,u_{n+1} + 5\,u_n = 0, \quad u_0 = \frac{1}{2}, \quad u_1 = 3.$

# 13

# *Finite Hankel Transforms*

"No human investigation can be called real science if it cannot be demonstrated mathematically."

Leonardo da Vinci

"The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics."

Godfrey H. Hardy

## 13.1 Introduction

This chapter is devoted to the study of the *finite Hankel transform* and its basic operational properties. The usefulness of this transform is shown by solving several initial-boundary problems of physical interest. The method of finite Hankel transforms was first introduced by Sneddon (1946).

## 13.2 Definition of the Finite Hankel Transform and Examples

Just as problems on finite invervals $-a < x < a$ lead to Fourier series, problems on finite intervals $0 < r < a$, where $r$ is the cylindrical polar coordinate, lead to the *Fourier-Bessel series* representation of a function $f(r)$ which can be stated in the following theorem:

***THEOREM 13.2.1***
If $f(r)$ is defined in $0 \le r \le a$ and

$$\tilde{f}_n(k_i) = \int_o^a r f(r) J_n(r k_i) dr, \tag{13.2.1}$$

then $f(r)$ can be represented by the Fourier-Bessel series as

$$f(r) = \frac{2}{a^2} \sum_{i=1}^{\infty} \tilde{f}_n(k_i) \frac{J_n(r k_i)}{J_{n+1}^2(a k_i)}, \tag{13.2.2}$$

where $k_i (0 < k_1 < k_2 < \cdots)$ are the roots of the equation $J_n(a k_i) = 0$, that means

$$J_n'(a k_i) = J_{n-1}(a k_i) = -J_{n+1}(a k_i), \tag{13.2.3}$$

due to the standard recurrence relations among $J_n'(x)$, $J_{n-1}(x)$, and $J_{n+1}(x)$.

**PROOF** We write formally the Bessel series expansion of $f(r)$ as

$$f(r) = \sum_{i=1}^{\infty} c_i \, J_n(r k_i), \tag{13.2.4}$$

where the summation is taken over all the positive zeros $k_1, k_2, \ldots$ of the Bessel function $J_n(a k_i)$. Multiplying (13.2.4) by $r J_n(r k_i)$, integrating the both sides of the result from 0 to $a$, and then using the orthogonal property of the Bessel functions, we obtain

$$\int_o^a r f(r) J_n(r k_i) \, dr = c_i \int_o^a r \, J_n^2(r k_i) \, dr.$$

Or,

$$\tilde{f}_n(k_i) = \frac{a^2 c_i}{2} J_{n+1}^2(a k_i),$$

hence, we obtain

$$c_i = \frac{2}{a^2} \, \frac{\tilde{f}_n(k_i)}{J_{n+1}^2(a k_i)}. \tag{13.2.5}$$

Substituting the value of $c_i$ into (13.2.4) gives (13.2.2). ∎

***DEFINITION 13.2.1*** *The finite Hankel transform of order $n$ of a function $f(r)$ is denoted by $\mathscr{H}_n\{f(r)\} = \tilde{f}_n(k_i)$ and is defined by*

$$\mathscr{H}_n \{f(r)\} = \tilde{f}_n(k_i) = \int_0^a r f(r) J_n(r k_i) \, dr. \tag{13.2.6}$$

*The inverse finite Hankel transform is then defined by*

$$\mathscr{H}_n^{-1}\left\{\tilde{f}_n(k_i)\right\} = f(r) = \frac{2}{a^2}\sum_{i=1}^{\infty}\tilde{f}_n(k_i)\frac{J_n(rk_i)}{J_{n+1}^2(ak_i)}, \tag{13.2.7}$$

*where the summation is taken over all positive roots of* $J_n(ak) = 0$.

*The zero-order finite Hankel transform and its inverse are defined by*

$$\mathscr{H}_0\{f(r)\} = \tilde{f}_0(k_i) = \int_0^a r f(r) J_0(rk_i)\,dr, \tag{13.2.8}$$

$$\mathscr{H}_0^{-1}\left\{\tilde{f}_0(k_i)\right\} = f(r) = \frac{2}{a^2}\sum_{i=1}^{\infty}\tilde{f}_0(k_i)\frac{J_0(rk_i)}{J_1^2(ak_i)}, \tag{13.2.9}$$

*where the summation is taken over the positive roots of* $J_0(ak) = 0$.

*Similarly, the first-order finite Hankel transform and its inverse are*

$$\mathscr{H}_1\{f(r)\} = \tilde{f}_1(k_i) = \int_0^a r f(r) J_1(rk_i)\,dr, \tag{13.2.10}$$

$$\mathscr{H}_1^{-1}\left\{\tilde{f}_1(k_i)\right\} = f(r) = \frac{2}{a^2}\sum_{i=1}^{\infty}\tilde{f}_1(k_i)\frac{J_1(rk_i)}{J_2^2(ak_i)}, \tag{13.2.11}$$

*where* $k_i$ *is chosen as a positive root of* $J_1(ak) = 0$.

We now give examples of *finite Hankel transforms* of some functions.

***Example 13.2.1***

If $f(r) = r^n$, then

$$\mathscr{H}_n\{r^n\} = \int_0^a r^{n+1} J_n(rk_i)\,dr = \frac{a^{n+1}}{k_i}J_{n+1}(ak_i). \tag{13.2.12}$$

When $n = 0$,

$$\mathscr{H}_0\{1\} = \frac{a}{k_i}J_1(ak_i). \tag{13.2.13}$$

□

***Example 13.2.2***

If $f(r) = (a^2 - r^2)$, then

$$\mathscr{H}_0\{(a^2 - r^2)\} = \int_0^a r\,(a^2 - r^2) J_0(ak_i)\,dr = \frac{4a}{k_i^3}J_1(ak_i) - \frac{2a^2}{k_i^2}J_0(ak_i).$$

Since $k_i$ are the roots of $J_0(ak)=0$, we find

$$\mathscr{H}_0\{(a^2-r^2)\}=\frac{4a}{k_i^3}\,J_1(ak_i). \tag{13.2.14}$$

□

## 13.3 Basic Operational Properties

We state the following operational properties of the *finite Hankel transforms:*

$$\mathscr{H}_n\{f'(r)\} = \frac{k_i}{2n}\left[(n-1)\mathscr{H}_{n+1}\{f(r)\} - (n+1)\mathscr{H}_{n-1}\{f(r)\}\right], \quad n\geq 1, \tag{13.3.1}$$

provided $f(r)$ is finite at $r=0$.

When $n=1$, we obtain the finite Hankel transform of derivatives

$$\mathscr{H}_1\{f'(r)\}=-k_i\mathscr{H}_0\{f(r)\}=-k_i\tilde{f}_0(k_i). \tag{13.3.2}$$

$$\mathscr{H}_n\left[\frac{1}{r}\frac{d}{dr}\{rf'(r)\}-\frac{n^2}{r^2}f(r)\right]=-k_i^2\tilde{f}_n(k_i)-ak_if(a)\,J_n'(ak_i)\,. \tag{13.3.3}$$

When $n=0$

$$\mathscr{H}_0\left[f''(r)+\frac{1}{r}\,f'(r)\right]=-k_i^2\,\tilde{f}_0(k_i)+ak_if(a)\,J_1(ak_i). \tag{13.3.4}$$

If $n=1$, (13.3.3) becomes

$$\mathscr{H}_1\left[f''(r)+\frac{1}{r}\,f'(r)-\frac{1}{r^2}\,f(r)\right]=-k_i^2\tilde{f}_1(k_i)-ak_if(a)\,J_1'(ak_i). \tag{13.3.5}$$

Results (13.3.4) and (13.3.5) are very useful for finding solutions of differential equations in cylindrical polar coordinates.

The proofs of the above results are elementary exercises for the reader.

## 13.4 Applications of Finite Hankel Transforms

***Example 13.4.1***

(*Temperature Distribution in a Long Circular Cylinder*). Find the solution of the axisymmetric heat conduction equation

$$\frac{\partial u}{\partial t} = \kappa \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right), \quad 0 \le r \le a, \; t > 0 \tag{13.4.1}$$

with the boundary and initial conditions

$$u(r,t) = f(t) \quad \text{on} \quad r = a, \; t > 0 \tag{13.4.2}$$

$$u(r,0) = 0, \qquad 0 \le r \le a. \tag{13.4.3}$$

Application of the *finite Hankel transform* defined by

$$\tilde{u}(k_i,t) = \mathscr{H}_0 \{u(r,t)\} = \int_0^a r J_o(rk_i) u(r,t)\, dr, \tag{13.4.4}$$

yields the given system with the boundary condition

$$\begin{aligned} \tilde{u}_t + \kappa k_i^2 \tilde{u} &= \kappa a k_i J_1(ak_i) f(t), \\ \tilde{u}(k_i,0) &= 0. \end{aligned} \tag{13.4.5ab}$$

The solution of the first order system is

$$\tilde{u}(k_i,t) = \kappa a k_i J_1(ak_i) \int_0^t f(\tau) \exp\left\{-\kappa k_i^2 (t-\tau)\right\} \, d\tau. \tag{13.4.6}$$

The inverse transform gives the formal solution

$$u(r,t) = \left(\frac{2\kappa}{a}\right) \sum_{i=1}^{\infty} \frac{k_i J_0(rk_i)}{J_1(ak_i)} \int_0^t f(\tau) \exp\left\{-\kappa k_i^2 (t-\tau)\right\} \, d\tau. \tag{13.4.7}$$

In particular, if $f(t) = T_0 =$ constant,

$$u(r,t) = \left(\frac{2T_0}{a}\right) \sum_{i=1}^{\infty} \frac{J_0(rk_i)}{k_i J_1(ak_i)} \left[1 - \exp\left(-\kappa k_i^2 t\right)\right]. \tag{13.4.8}$$

Using the inverse version of (13.2.7) gives the final solution

$$u(r,t) = T_0 - \left(\frac{2T_0}{a}\right) \sum_{i=1}^{\infty} \frac{J_0(rk_i)}{k_i J_1(ak_i)} \exp\left(-\kappa k_i^2 t\right). \tag{13.4.9}$$

This solution representing the temperature distribution consists of the steady-state term, and the transient term which decays to zero as $t \to \infty$. Consequently, the steady temperature is attained in the limit as $t \to \infty$. □

**Example 13.4.2**

(*Unsteady Viscous Flow in a Rotating Long Circular Cylinder*). The axisymmetric unsteady motion of a viscous fluid in an infinitely long circular cylinder of radius $a$ is governed by

$$u_t = \nu \left( u_{rr} + \frac{1}{r} u_r - \frac{u}{r^2} \right), \quad 0 \le r \le a, \quad t > 0, \tag{13.4.10}$$

where $u = u(r,t)$ is the tangential fluid velocity and $\nu$ is the constant kinematic viscosity of the fluid.

The cylinder is initially at rest at $t = 0$, and it is then allowed to rotate with constant angular velocity $\Omega$. Thus, the boundary and initial conditions are

$$u(r,t) = a\Omega \qquad \text{on} \quad r = a,\ t > 0, \tag{13.4.11}$$

$$u(r,t) = 0 \qquad \text{at} \quad t = 0 \quad \text{for} \quad 0 < r < a. \tag{13.4.12}$$

We solve the problem by using the joint *Laplace* and the *finite Hankel transform* of order one defined by

$$\tilde{\bar{u}}(k_i, s) = \int_0^\infty e^{-st} dt \int_0^a r J_1(k_i r) u(r,t)\, dr, \tag{13.4.13}$$

where $k_i$ are the positive roots of $J_1(ak_i) = 0$.

Application of the joint transform gives

$$s\, \tilde{\bar{u}}(k_i, s) = -\nu\, k_i^2\, \tilde{\bar{u}}(k_i, s) - \frac{\nu a^2 \Omega k_i}{s} J_1'(ak_i).$$

Or,

$$\tilde{\bar{u}}(k_i, s) = -\frac{\nu a^2 \Omega k_i J_1'(ak_i)}{s(s + \nu k_i^2)}. \tag{13.4.14}$$

The inverse Laplace transform gives

$$\tilde{u}(k_i, t) = -\frac{a^2 \Omega}{k_i} J_1'(ak_i) \left[1 - \exp\left(-\nu t k_i^2\right)\right]. \tag{13.4.15}$$

Thus, the final solution is found from (13.4.15) by using the *inverse Hankel transform* with $J_1'(ak_i) = -J_2(ak_i)$ in the form

$$u(r,t) = 2\Omega \sum_{i=1}^{\infty} \frac{J_1(rk_i)}{k_i J_2(ak_i)} \left[1 - \exp\left(-\nu t k_i^2\right)\right]. \tag{13.4.16}$$

This solution is the sum of the steady-state and the transient fluid velocities.

In view of (13.2.12) for $n = 1$, we can write

$$r = \mathscr{H}_1^{-1}\left\{\frac{a^2}{k_i} J_2(ak_i)\right\} = 2\sum_{i=1}^{\infty} \frac{J_1(rk_i)}{k_i J_2(ak_i)}. \tag{13.4.17}$$

This result is used to simplify (13.4.16) so that the final solution for $u(r,t)$ takes the form

$$u(r,t) = r\Omega - 2\Omega \sum_{i=1}^{\infty} \frac{J_1(rk_i)}{k_i J_2(ak_i)} \exp\left(-\nu t k_i^2\right). \tag{13.4.18}$$

In the limit as $t \to \infty$, the transient velocity component decays to zero, and the ultimate steady state flow is attained in the form

$$u(r,t) = r\Omega. \tag{13.4.19}$$

Physically, this represents the rigid body rotation of the fluid inside the cylinder. □

**Example 13.4.3**
(*Vibrations of a Circular Membrane*). The free symmetric vibration of a thin circular membrane of radius $a$ is governed by the wave equation

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r\right), \qquad 0 < r < a, \qquad t > 0 \tag{13.4.20}$$

with the initial and boundary data

$$u(r,t) = f(r), \quad \frac{\partial u}{\partial t} = g(r) \qquad \text{at} \quad t=0 \quad \text{for} \quad 0 < r < a, \tag{13.4.21ab}$$

$$u(a,t) = 0 \qquad \text{for all} \quad t > 0. \tag{13.4.22}$$

Application of the zero-order *finite Hankel transform* of $u(r,t)$ defined by (13.4.4) to (13.4.20)–(13.4.22) gives

$$\frac{d^2\tilde{u}}{dt^2} + c^2\,k_i^2\,\tilde{u} = 0, \tag{13.4.23}$$

$$\tilde{u} = \tilde{f}(k_i) \quad \text{and} \quad \left(\frac{d\tilde{u}}{dt}\right)_{t=0} = \tilde{g}(k_i). \tag{13.4.24ab}$$

The solution of this system is

$$\tilde{u}(k_i,t) = \tilde{f}(k_i)\cos(ctk_i) + \frac{\tilde{g}(k_i)}{c\,k_i}\sin(ctk_i). \tag{13.4.25}$$

The inverse transform yields the formal solution

$$\begin{aligned} u(r,t) = {} & \frac{2}{a^2}\sum_{i=1}^{\infty} f(k_i)\cos(ctk_i)\,\frac{J_0(rk_i)}{J_1^2(ak_i)} \\ & + \frac{2}{ca^2}\sum_{i=1}^{\infty} g(k_i)\sin(ctk_i)\frac{J_0(rk_i)}{k_i J_1^2(ak_i)}, \end{aligned} \tag{13.4.26}$$

where the summation is taken over all positive roots of $J_0(ak_i) = 0$.

We consider a more general form of the *finite Hankel transform* associated with a more general boundary condition

$$f'(r) + hf(r) = 0 \qquad \text{at } r = a, \tag{13.4.27}$$

where $h$ is a constant.

We define the *finite Hankel transform* of $f(r)$ by

$$\mathscr{H}_n\{f(r)\} = \tilde{f}_n(k_i) = \int_0^a rJ_n(rk_i)f(r)dr, \tag{13.4.28}$$

where $k_i$ are the roots of the equation

$$k_i J_n'(ak_i) + hJ_n(ak_i) = 0. \tag{13.4.29}$$

The corresponding inverse transform is given by

$$f(r) = \mathscr{H}_n^{-1}\{\tilde{f}_n(k_i)\} = 2\sum_{i=1}^{\infty} \frac{k_i^2 \tilde{f}_n(k_i) J_n(rk_i)}{\{(k_i^2 + h^2)a^2 - n^2\} J_n^2(ak_i)}. \tag{13.4.30}$$

This *finite Hankel transform* has the following operational property

$$\mathscr{H}_n\left[\frac{1}{r}\frac{d}{dr}\{rf'(r)\} - \frac{n^2}{r^2}f(r)\right] = -\,k_i^2\tilde{f}_n(k_i) + a\left[f'(a) + hf(a)\right]J_n(ak_i), \tag{13.4.31}$$

which is, by (13.4.29)

$$= -k_i^2\tilde{f}_n(k_i) - \frac{ak_i}{h}\left[f'(a) + hf(a)\right]J_n'(ak_i). \tag{13.4.32}$$

Thus, result (13.4.32) involves $f'(a) + hf(a)$ as the boundary condition. ▯

We apply this more general *finite Hankel transform* pairs (13.4.28) and (13.4.30) to solve the following axisymmetric initial-boundary value problem.

**Example 13.4.4**

(*Temperature Distribution of Cooling of a Circular Cylinder*). Solve the axisymmetric heat conduction problem for an infinitely long circular cylinder of radius $r = a$ with the initial constant temperature $T_0$, and the cylinder is cooling by radiation of heat from its boundary surface at $r = a$ to the outside medium at zero temperature according to Newton's law of cooling, which satisfies the boundary condition

$$\frac{\partial u}{\partial r} + hu = 0 \qquad \text{at } r = a, \qquad t > 0, \tag{13.4.33}$$

where $h$ is a constant.

The problem is governed by the axisymmetric heat conduction equation

$$u_t = \kappa\left(u_{rr} + \frac{1}{r}\,u_r\right), \qquad 0 \le r \le a, \quad t > 0, \tag{13.4.34}$$

with the boundary condition (13.4.33) and the initial condition

$$u(r,0) = T_0 \qquad \text{at } t = 0, \qquad \text{for } 0 < r < a. \tag{13.4.35}$$

Application of the zero-order *Hankel transform* (13.4.28) with (13.4.29) to the system (13.4.33)–(13.4.35) gives

$$\frac{d\tilde{u}}{dt} + \kappa\, k_i^2\, \tilde{u} = 0, \qquad t > 0 \tag{13.4.36}$$

$$\tilde{u}(k_i, 0) = T_0 \int_0^a r J_0(rk_i)\, dr = \frac{aT_0}{k_i} J_1(ak_i). \tag{13.4.37}$$

The solution of (13.4.36)–(13.4.37) is

$$\tilde{u}(k_i, t) = \left(\frac{aT_0}{k_i}\right) J_1(ak_i) \exp(-\kappa\, t\, k_i^2). \tag{13.4.38}$$

The inverse transform (13.4.30) with $n = 0$ and $k_i J_0'(ak_i) + hJ_0(ak_i) = 0$, that is, $k_i J_1(ak_i) = hJ_0(ak_i)$, leads to the formal solution

$$u(r,t) = \left(\frac{2hT_0}{a}\right) \sum_{i=1}^{\infty} \frac{J_0(rk_i) \exp(-\kappa t k_i^2)}{(k_i^2 + h^2) J_0(ak_i)}, \tag{13.4.39}$$

where the summation is taken over all the positive roots of $k_i J_1(ak_i) = hJ_0(ak_i)$. $\square$

## 13.5 Exercises

1. Find the zero-order *finite Hankel transform* of

   (a) $f(r) = r^2$, (b) $f(r) = J_0(\alpha r)$, (c) $f(r) = (a^2 - r^2)$.

2. Show that

$$\mathscr{H}_n \left\{ \frac{J_n(\alpha r)}{J_n(\alpha a)} \right\} = \frac{ak_i}{(\alpha^2 - k_i^2)} J_n'(ak_i)$$

3. If $\mathscr{H}_n\{f(r)\}$ is the *finite Hankel transform* of $f(r)$ defined by (13.2.6), and if $n > 0$, show that

   (a) $\mathscr{H}_n\{r^{-1} f'(r)\} = \frac{1}{2} k_i \left[ \mathscr{H}_{n+1}\{r^{-1} f(r)\} - \mathscr{H}_{n-1}\{r^{-1} f(r)\} \right]$,

   (b) $\mathscr{H}_0\{r^{-1} f'(r)\} = k_i\, \mathscr{H}_1\{r^{-1} f(r)\} - f(a)$.

4. Solve the initial-boundary value problem

$$c^2(u_{rr} + \frac{1}{r}u_r) = u_{tt}, \quad 0 < r < a,\ t > 0,$$
$$u(r,0) = 0, \quad u_t(r,0) = u_0, \quad \text{for all } r, \quad u(a,t) = 0, \quad t > 0,$$

where $u_0$ is a constant.

5. Obtain a solution of the initial-boundary problem

$$\kappa\left(u_{rr} + \frac{1}{r}u_r\right) = u_t, \quad 0 < r < a,\ t > 0,$$
$$u(r,0) = f(r), \quad \text{for} \quad 0 < r < a$$
$$u(a,t) = 0.$$

6. If we define the *finite Hankel transform* of $f(r)$ by

$$\mathscr{H}_n\{f(r)\} = \tilde{f}_n(k_i) = \int_a^b r f(r) A_n(r\,k_i)dr, \qquad b > a,$$

where

$$A_n(rk_i) = J_n(rk_i)Y_n(ak_i) - Y_n(rk_i)J_n(ak_i),$$

and $Y_n(x)$ is the Bessel function of the second kind of order $n$, show that the inverse transform is

$$\mathscr{H}_n^{-1}\{\tilde{f}_n(k_i)\} = f(r) = \frac{\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^2 \tilde{f}_n(k_i) A_n(rk_i) J_n^2(bk_i)}{J_n^2(ak_i) - J_n^2(bk_i)},$$

where $k_i$ are the positive roots of $A_n(bk_i) = 0$.

7. For the transform defined in problem 6, show that

$$\mathscr{H}_n\left[f''(r) + \frac{1}{r}f'(r) - \frac{n^2 f(r)}{r^2}\right] = -k_i^2\tilde{f}_n(k_i) + \frac{2}{\pi}\left[f(b)\frac{J_n(ak_i)}{J_n(bk_i)} - f(a)\right].$$

8. Viscous fluid of kinematic viscosity $\nu$ is bounded between two infinitely long concentric circular cylinders of radii $a$ and $b$. The inner cylinder is stationary and the outer cylinder begins to rotate with uniform angular velocity $\Omega$ at $t = 0$. The axisymmetric flow is governed by (13.4.10) with $v(a,0) = 0$ and $v(b,0) = \Omega b$. show that

$$v(r,t) = (\pi b\Omega)\sum_{i=1}^{\infty}\frac{J_1(ak_i)J_1(bk_i)A_1(rk_i)\left[1 - \exp(-\nu t k_i^2)\right]}{J_1^2(ak_i) - J_1^2(bk_i)},$$

where

$$A_1(rk_i) = J_n(rk_i)Y_n(ak_i) - Y_n(rk_i)J_n(ak_i),$$

and $k_i$ are the positive roots of the equation $A_1(bk_i) = 0$.

9. Find the solution of the forced symmetric vibrations of a thin elastic membrane that satisfy the initial-boundary value problem

$$u_{rr} + \frac{1}{r}\,u_r - \frac{1}{c^2}\,u_{tt} = -\frac{p(r,t)}{T_0},$$

where $p(r,t)$ is the applied pressure which produces vibrations, and the membrane is stretched by a constant tension $T_0$. The membrane is set into motion from rest in its equilibrium position so that

$$u(r,t) = 0 = \left(\frac{\partial u}{dt}\right) \qquad \text{at } t = 0.$$

10. Use the *joint Hankel and Laplace transform* method to the axisymmetric diffusion problem in an infinitely long circular cylinder of radius $a$:

$$\begin{aligned} &u_t = \kappa\left(u_{rr} + \frac{1}{r}\,u_r\right) + Q(r,t), \quad 0 < r < a, \quad t > 0, \\ &u(a,t) = 0 \quad \text{for} \quad t > 0, \\ &u(r,0) = 0 \quad \text{for} \quad 0 < r \leq a, \end{aligned}$$

where $Q(r,t)$ represents a heat source inside the cylinder. Find the explicit solution for two special cases:

(a) $Q(r,t) = \dfrac{\kappa\, Q_0}{k}$, (b) $Q(r,t) = Q_0\,\dfrac{\delta(r)}{r}\,f(t)$,

where $Q_0, \kappa$, and $k$ are constants.

# 14

# *Legendre Transforms*

> "Legendre, who for so many reasons is considered the founder of elliptic functions, greatly smoothed the way for his successors; it is the fact of the double periodicity of the inverse function, immediately discovered by Abel and Jacobi, that is missing and that gave such a restrained analytical character to his treatise."
>
> Charles Hermite

> "First causes are not known to us, but they are subjected to simple and constant laws that can be studied by observation and whose study is the goal of Natural Philosophy. ... Heat penetrates, as does gravity, all the substances of the universe; its rays occupy all regions of space. The aim of our work is to expose the mathematical laws that this element follows. ... The differential equations for the propagation of heat express the most general conditions and reduce physical questions to problems in pure Analysis that is properly the object of the theory."
>
> Clerk Maxwell

## 14.1 Introduction

We consider in this chapter the Legendre transform with a Legendre polynomial as kernel and discuss basic operational properties including the Convolution Theorem. Legendre transforms are then used to solve boundary value problems in potential theory. This chapter is based on papers by Churchill (1954) and Churchill and Dolph (1954) listed in the Bibliography.

## 14.2 Definition of the Legendre Transform and Examples

Churchill (1954) defined the *Legendre transform* of a function $f(x)$ defined in $-1<x<1$ by the integral

$$\mathscr{T}_n\{f(x)\} = \tilde{f}(n) = \int_{-1}^{1} P_n(x) f(x) dx, \tag{14.2.1}$$

provided the integral exists and where $P_n(x)$ is the *Legendre polynomial* of degree $n$ $(\geq 0)$. Obviously $\mathscr{T}_n$ is a linear integral transformation.

When $x = \cos\theta$, (14.2.1) becomes

$$\mathscr{T}_n\{f(\cos\theta)\} = \tilde{f}(n) = \int_{0}^{\pi} P_n(\cos\theta) f(\cos\theta) \sin\theta d\theta. \tag{14.2.2}$$

The *inverse Legendre transform* is given by

$$f(x) = \mathscr{T}_n^{-1}\{\tilde{f}(n)\} = \sum_{n=0}^{\infty} \left(\frac{2n+1}{2}\right) \tilde{f}(n) P_n(x). \tag{14.2.3}$$

This follows from the expansion of any function $f(x)$ in the form

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \tag{14.2.4}$$

where the coefficient $a_n$ can be determined from the orthogonal property of $P_n(x)$. It turns out that

$$a_n = \left(\frac{2n+1}{2}\right) \int_{-1}^{1} P_n(x) f(x) dx = \left(\frac{2n+1}{2}\right) \tilde{f}(n), \tag{14.2.5}$$

and hence, result (14.2.3) follows.

***Example 14.2.1***

$$\mathscr{T}_n\{\exp(i\alpha x)\} = \left(\frac{2\pi}{\alpha}\right)^{1/2} i^n J_{n+1/2}(\alpha), \tag{14.2.6}$$

where $J_\nu(x)$ is the Bessel function.

We have, by definition,

$$\mathscr{T}_n\{\exp(i\alpha x)\} = \int_{-1}^{1} \exp(i\alpha x) P_n(x) dx,$$

which is, by a result in Copson (1935, p. 341),

$$= \sqrt{\frac{2\pi}{\alpha}}\, i^n\, J_{n+1/2}(\alpha).$$

Similarly,

$$\mathscr{T}_n\{\exp(\alpha x)\} = \sqrt{\frac{2\pi}{\alpha}}\, I_{n+1/2}(\alpha), \tag{14.2.7}$$

where $I_\nu(x)$ is the modified Bessel function of the first kind. □

***Example 14.2.2***

$$\text{(a)} \qquad \mathscr{T}_n\{(1-x^2)^{-1/2}\} = \pi\, P_n^2(0) \tag{14.2.8}$$

$$\text{(b)} \qquad \mathscr{T}_n\left\{\frac{1}{2(t-x)}\right\} = Q_n(t), \qquad |t| > 1, \tag{14.2.9}$$

where $Q_n(t)$ is the Legendre function of the second kind given by

$$Q_n(t) = \frac{1}{2}\int_{-1}^{1} (t-x)^{-1} P_n(x) dx.$$

These results are easy to verify with the aid of results given in Copson (1935, p. 292 and p. 310). □

***Example 14.2.3***
If $|r| \le 1$, then

$$\text{(a)} \qquad \mathscr{T}_n\{(1-2rx+r^2)^{-1/2}\} = \frac{2\, r^n}{(2n+1)}, \tag{14.2.10}$$

$$\text{(b)} \qquad \mathscr{T}_n\{1-2rx+r^2)^{-3/2}\} = \frac{2\, r^n}{(1-r^2)}. \tag{14.2.11}$$

We have, from the generating function of $P_n(x)$,

$$(1-2rx+r^2)^{-1/2} = \sum_{n=0}^{\infty} r^n P_n(x), \qquad |r| < 1.$$

Multiplying this result by $P_n(x)$ and using the orthogonality condition of the Legendre polynomial gives

$$\int_{-1}^{1}(1-2rx+r^2)^{-1/2}P_n(x)dx = \frac{2\,r^n}{(2n+1)}. \tag{14.2.12}$$

In particular, when $r=1$, we obtain

$$\mathscr{T}_n\{(1-x)^{-1/2}\} = \frac{2\sqrt{2}}{(2n+1)}. \tag{14.2.13}$$

Differentiating (14.2.12) with respect to $r$ gives

$$\frac{1}{2}\int_{-1}^{1}(1-2rx+r^2)^{-3/2}\,(2rx-2r^2)P_n(x)dx = \frac{2nr^n}{(2n+1)},$$

so that

$$-\mathscr{T}_n\{(1-2rx+r^2)^{-1/2}\}+(1-r^2)\mathscr{T}_n\{(1-2rx+r^2)^{-3/2}\} = \frac{2nr^n}{(2n+1)}.$$

Using (14.2.10), we obtain (14.2.11). □

***Example 14.2.4***
If $|r|<1$ and $\alpha>0$, then

$$\mathscr{T}_n\left\{\int_0^r \frac{t^{\alpha-1}dt}{(1-2xt+t^2)^{1/2}}\right\} = \frac{2r^{n+\alpha}}{(2n+1)(n+\alpha)}. \tag{14.2.14}$$

We replace $r$ by $t$ in (14.2.10) and multiply the result by $t^{\alpha-1}$ to obtain

$$\mathscr{T}_n\{t^{\alpha-1}(1-2xt+t^2)^{-1/2}\} = \frac{2\,t^{n+\alpha-1}}{(2n+1)}.$$

Integrating this result on $(0,r)$ we find (14.2.14). □

***Example 14.2.5***
If $H(x)$ is a Heaviside unit step function, then

$$\mathscr{T}_n\{H(x)\} = \left\{\begin{array}{cc} 1, & n=0 \\ \dfrac{P_{n-1}(0)-P_{n+1}(0)}{(2n+1)}, & n\geq 1 \end{array}\right\}. \tag{14.2.15}$$

Obviously,

$$\mathscr{T}_n\{H(x)\} = \int_0^1 P_n(x)dx = 1 \qquad \text{when } n = 0.$$

However, for $n > 1$, we use the *recurrence relation* for $P_n(x)$ as

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \tag{14.2.16}$$

to derive

$$\begin{aligned}\mathscr{T}_n\{H(x)\} &= \frac{1}{(2n+1)} \int_0^1 [P'_{n+1}(x) - P'_{n-1}(x)]dx \\ &= \frac{1}{2n+1}[P_{n-1}(0) - P_{n+1}(0)].\end{aligned}$$

Debnath and Harrel (1976) introduced the *associated Legendre transform* defined by

$$\mathscr{T}_{n,m}\{f(x)\} = \tilde{f}(n,m) = \int_{-1}^{1} (1-x^2)^{-m/2} P_n^m(x)\, f(x)dx, \tag{14.2.17}$$

where $P_n^m(x)$ is the *associated Legendre function* of the first kind.

The inverse transform is given by

$$\begin{aligned}f(x) = \mathscr{T}_{n,m}^{-1}\{\tilde{f}(n,m)\} &= \sum_{n=0}^{\infty} \frac{(2n+1)}{2} \frac{(n-m)!}{(n+m)!} \\ &\quad \times \tilde{f}(n,m)(1-x^2)^{m/2} P_n^m(x). \end{aligned} \tag{14.2.18}$$

The reader is referred to Debnath and Harrel (1976) for a detailed discussion of this transform. □

## 14.3 Basic Operational Properties of Legendre Transforms

***THEOREM 14.3.1***

If $f'(x)$ is continuous and $f''(x)$ is bounded and integrable in each subinterval of $-1 \le x \le 1$, and if $\mathscr{T}_n\{f(x)\}$ exists and

$$\lim_{|x|\to 1} (1-x^2)f(x) = \lim_{|x|\to 1} (1-x^2)f'(x) = 0, \tag{14.3.1}$$

then

$$\mathscr{T}_n\{R[f(x)]\} = -n(n+1)\tilde{f}(n), \tag{14.3.2}$$

where $R[f(x)]$ is a differential form given by

$$R[f(x)] = \frac{d}{dx}\left[(1-x^2)\,\frac{d}{dx}f(x)\right], \qquad n>0. \tag{14.3.3}$$

**PROOF** We have, by definition,

$$\mathscr{T}_n\{R[f(x)]\} = \int_{-1}^{1} \frac{d}{dx}\left[(1-x^2)\,\frac{d}{dx}f(x)\right] Pn(x)\,dx$$

which is, by integrating by parts together with (14.3.1),

$$= -\int_{-1}^{1} (1-x^2)\,P_n'(x)\,\frac{d}{dx}\,f(x)\,dx.$$

Integrating this result by parts again, we obtain

$$\mathscr{T}_n\{R[f(x)]\} = -[(1-x^2)]P_n'(x)f(x)]_{-1}^{1} + \int_{-1}^{1} \frac{d}{dx}[(1-x^2)]\,P_n'(x)]\,f(x)\,dx.$$

Using (14.3.1) and the differential equation for the Legendre polynomial

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0, \tag{14.3.4}$$

we obtain the desired result

$$\mathscr{T}_n\{R[f(x)]\} = -n(n+1)\tilde{f}(n).$$

We may extend this result to evaluate the Legendre transforms of the differential forms $R^2[f(x)],\ R^3[f(x)],\ldots,R^k[f(x)]$. ∎

Clearly

$$\begin{aligned}\mathscr{T}_n\{R^2[f(x)]\} &= \mathscr{T}_n\{R[R[f(x)]]\}\\ &= -n(n+1)\mathscr{T}_n\{R[f(x)]\} = n^2(n+1)^2\tilde{f}(n),\end{aligned} \tag{14.3.5}$$

provided $f'(x)$ and $f''(x)$ satisfy the conditions of Theorem 14.3.1.

Similarly,

$$\mathscr{T}_n\{R^3[f(x)]\} = (-1)^3 n^3(n+1)^3\tilde{f}(n). \tag{14.3.6}$$

More generally, for a positive integer $k$,

$$\mathscr{T}_n\{R^k[f(x)]\} = (-1)^k n^k (n+1)^k \tilde{f}(n). \tag{14.3.7}$$

***COROLLARY 14.3.1***
If $\mathscr{T}_n\{R[f(x)]\} = -n(n+1)\tilde{f}(n)$, then

$$\mathscr{T}_n\left\{\frac{1}{4}f(x) - R[f(x)]\right\} = \left(n+\frac{1}{2}\right)^2 \tilde{f}(n). \tag{14.3.8}$$

**PROOF** We replace $n(n+1)$ by $\left(n+\dfrac{1}{2}\right)^2 - \dfrac{1}{4}$ in (14.3.2) to obtain

$$\mathscr{T}_n\{R[f(x)]\} = -\left[\left(n+\frac{1}{2}\right)^2 - \frac{1}{4}\right]\tilde{f}(n). \tag{14.3.9}$$

Rearranging the terms in (14.3.9) gives

$$\mathscr{T}_n\left\{\frac{1}{4}\,f(x) - R[f(x)]\right\} = \left(n+\frac{1}{2}\right)^2 \tilde{f}(n).$$

In general, this result can be written as

$$(-1)^k\,\mathscr{T}_n\{R^k[f(x)] - 4^{-k}f(x)\} = \sum_{r=0}^{k-1}(-1)^r\binom{k}{r}\left[4^{-r}\left(n+\frac{1}{2}\right)^{2k-2r}\right]\tilde{f}(n). \tag{14.3.10}$$

The proof of (14.3.10) follows from (14.3.7) by replacing $n(n+1)$ with $\left(n+\frac{1}{2}\right)^2 - \frac{1}{4}$ and using the binomial expansion. ∎

***Example 14.3.1***

$$\mathscr{T}_n\{\log(1-x)\} = \begin{cases} 2(\log 2 - 1), & n = 0 \\ -\dfrac{2}{n(n+1)}, & n > 0 \end{cases}. \tag{14.3.11}$$

Clearly,

$$R[\log(1-x)] = \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\log(1-x)\right] = -1.$$

Although $\dfrac{d}{dx}\log(1-x)$ does not satisfy the conditions of Theorem 14.3.1, we integrate by parts to obtain

$$\mathscr{T}_n\{R[\log(1-x)]\} = \int_{-1}^{1} R[\log(x)]P_n(x)dx$$

$$= [-(1+x)P_n(x)]_{-1}^{1} + \int_{-1}^{1} (1+x)\,P_n'(x)\,dx,$$

which is, since $(1+x) = -(1-x^2)\dfrac{d}{dx}\log(1-x)$, and by integrating by parts,

$$= -2 + \int_{-1}^{1} \log(1-x)\frac{d}{dx}[(1-x^2)]P_n'(x)]dx. \tag{14.3.12}$$

By integrating by parts twice, result (14.3.12) gives

$$\mathscr{T}_n\{R[\log(1-x)]\} = -2 + \int_{-1}^{1} \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\log(1-x)\right] P_n(x)dx,$$

which is, by (14.3.2),

$$= -2 - n(n+1)\,\tilde{f}(n), \tag{14.3.13}$$

where $\tilde{f}(n) = \mathscr{T}_n\{\log(1-x)\}$.

However, $R[\log(1-x)] = -1$ so that $\mathscr{T}_n\{R[\log(1-x)]\} = 0$ for all $n > 0$ and hence, result (14.3.13) gives

$$\mathscr{T}_n[\log(1-x)] = \tilde{f}(n) = -\frac{2}{n(n+1)}.$$

On the other hand, since $P_0(x) = 1$, we have

$$\mathscr{T}_0\{[\log(1-x)]\} = \int_{-1}^{1} \log(1-x)dx,$$

which is, by direct integration,

$$= -[(1-x)\{\log(1-x) - x\}]_{-1}^{1} = 2(\log 2 - 1).$$

□

***THEOREM 14.3.2***
If $f(x)$ and $f'(x)$ are piecewise continuous in $-1<x<1$, $R^{-1}[f(x)]=h(x)$, and $f(0)=\int_{-1}^{1} f(x)dx=0$, then

$$\mathscr{T}_n^{-1}\left\{\frac{\tilde{f}(n)}{n(n+1)}\right\}=A-\int_0^x \frac{ds}{(1-s^2)}\int_{-1}^{s} f(t)dt, \tag{14.3.14}$$

where $A$ is an arbitrary constant of integration.

**PROOF** We have

$$R[h(x)]=f(x)$$

or,

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}h(x)\right]=f(x).$$

Integrating over $(-1,x)$ gives

$$\int_{-1}^{x} f(t)\,dt=(1-x^2)\frac{d}{dx}h(x), \tag{14.3.15}$$

which is a continuous function of $x$ in $|x|<1$ with limit zero as $|x|\to 1$.
Integration of (14.3.15) gives

$$h(x)=\int_0^x \frac{ds}{(1-s^2)}\int_{-1}^{s} f(t)dt-A,$$

where $A$ is an arbitrary constant. Clearly, $h(x)$ satisfies the conditions of Theorem 14.3.1, and there exists a positive real constant $m<1$ such that

$$|h(x)|=O\{(1-x^2)^{-m}\} \qquad \text{as } |x|\to 1.$$

Hence, $\mathscr{T}_n\{R[h(x)]\}$ exists, and by Theorem 14.3.1, it follows that

$$\mathscr{T}_n\{R[h(x)]\}=-n(n+1)\mathscr{T}_n\{h(x)\}=-n(n+1)\mathscr{T}_n\{R^{-1}[f(x)]\}, \tag{14.3.16}$$

whence it turns out that

$$\mathscr{T}_n\{R^{-1}\{f(x)\}\}=-\frac{\tilde{f}(n)}{n(n+1)}. \tag{14.3.17}$$

Inversion leads to the result

$$\mathscr{T}_n^{-1}\left\{\frac{f(n)}{n(n+1)}\right\}=-R^{-1}\{f(x)\}=-h(x)$$

$$=A-\int_0^x \frac{ds}{1-s^2}\int_{-1}^{s} f(t)dt. \tag{14.3.18}$$

This proves the theorem. ▮

***THEOREM 14.3.3***

If $f(x)$ is continuous in each subinterval of $(-1,1)$ and a continuous function $g(x)$ is defined by

$$g(x) = \int_{-1}^{x} f(t)dt, \tag{14.3.19}$$

then

$$\mathscr{T}_n\{g'(x)\} = \tilde{f}(n) = g(1) - \int_{-1}^{1} g(x)P_n'(x)dx. \tag{14.3.20}$$

**PROOF** We have, by definition,

$$\mathscr{T}_n\{g'(x)\} = \int_{-1}^{1} g'(x)\, P_n(x)dx,$$

which is, by integrating by parts,

$$= [P_n(x)g(x)]_{-1}^{1} - \int_{-1}^{1} g(x)\, P_n'(x)dx.$$

Since $P_n(1) = 1$ and $g(-1) = 0$, the preceding result becomes (14.3.20). ▮

***COROLLARY 14.3.2***

If result (14.3.20) is true and $g(x)$ is given by (14.3.19), then

$$\left.\begin{aligned} \mathscr{T}_n\{g(x)\} &= f(0) - f(1) && \text{when } n = 0 \\ &= \frac{\tilde{f}(n-1) - \tilde{f}(n+1)}{(2n+1)} && \text{when } n > 1 \end{aligned}\right\}. \tag{14.3.21}$$

**PROOF** We write $\tilde{f}(n-1)$ and $\tilde{f}(n+1)$ using (14.3.20) and then subtract so that the resulting expression gives (14.3.21) with the help of (14.2.16). ▮

***COROLLARY 14.3.3***

If $g'(x)$ is a sectionally continuous function and $g(x)$ is the continuous function given by (14.3.19), then

$$\left.\begin{aligned}\mathscr{T}_n\{g'(x)\} &= g(1), \quad \text{when } n=0\\ &= g(1) - (2n-1)\,\tilde{g}(n-1) - (2n-5)\,\tilde{g}(n-3) - \cdots - g(0)\\ &\qquad\qquad\qquad\qquad \text{when } n=1,3,5,\ldots\\ &= g(1) - 2(2n-1)\tilde{g}(n-1) - (2n-5)\tilde{g}(n-3) - \cdots - 3g(1)\\ &\qquad\qquad\qquad\qquad \text{when } n=2,4,6,\ldots\end{aligned}\right\}. \tag{14.3.22}$$

These results can readily be verified using (14.3.20) and (14.2.16).

***THEOREM 14.3.4***
(*Convolution*). If $\mathscr{T}_n\{f(x)\}=\tilde{f}(n)$ and $\mathscr{T}_n\{g(x)\}=\tilde{g}(n)$, then

$$\mathscr{T}_n\{f(x)\} * g(x) = \tilde{f}(n)\tilde{g}(n), \tag{14.3.23}$$

where the convolution $f(x) * g(x)$ is given by

$$f(x) * g(x) = h(x) = \frac{1}{\pi}\int_0^\pi f(\cos\mu)\sin\mu\,d\mu \int_0^\pi g(\cos\lambda)d\beta, \tag{14.3.24}$$

with

$$x = \cos v \text{ and } \cos\lambda = \cos\mu\cos v + \sin\mu\sin v\,\cos\beta. \tag{14.3.25}$$

**PROOF** We have, by definition (14.2.2),

$$\begin{aligned}\tilde{f}(n)\tilde{g}(n) &= \int_0^\pi f(\cos\mu)P_n(\cos\mu)\sin\mu\,d\mu \int_0^\pi g(\cos\lambda)P_n(\cos\lambda)\sin\lambda\,d\lambda\\ &= \int_0^\pi f(\cos\mu)\sin\mu\left[\int_0^\pi g(\cos\lambda)P_n(\cos\lambda)P_n(\cos\mu)\sin\lambda\,d\lambda\right]d\mu,\end{aligned} \tag{14.3.26}$$

where $f(x) = f(\cos\mu)$ and $g(x) = g(\cos\lambda)$.

With the aid of an addition formula (see Sansone, 1959, p. 169) given as

$$P_n(\cos\lambda)\,P_n(\cos\mu) = \frac{1}{\pi}\int_0^\pi P_n(\cos v)\,d\alpha, \tag{14.3.27}$$

where $\cos v = \cos\lambda\cos\mu + \sin\lambda\sin\mu\cos\alpha$, the product can be rewritten in the form

$$\tilde{f}(n)\,\tilde{g}(n) = \frac{1}{\pi}\int_0^\pi f(\cos\mu)\sin\mu \times \left[\int_0^\pi\int_0^\pi g(\cos\mu)P_n(\cos\mu)\sin\lambda\;d\alpha\,d\lambda\right]d\mu. \quad (14.3.28)$$

We next use Churchill and Dolph's (1954, pp. 94–96) geometrical arguments to replace the double integral inside the square bracket by

$$\int_0^\pi\int_0^\pi g(\cos\mu\cos v + \sin\mu\sin\,v\cos\beta)P_n(\cos v)\sin v\,dv. \quad (14.3.29)$$

Substituting this result in (14.3.26) and changing the order of integration, we obtain

$$\tilde{f}(n)\,\tilde{g}(n) = \frac{1}{\pi}\int_0^\pi P_n(\cos v)\sin v\left[\int_0^\pi\int_0^\pi f(\cos\mu)\sin\mu\,g(\cos\lambda)d\mu\,d\beta\right]dv$$
$$= \int_0^\pi h(\cos v)P_n(\cos v)\sin v\,dv, \quad (14.3.30)$$

where

$$\cos\lambda = \cos\mu\cos v + \sin\mu\sin\,v\cos\beta, \quad (14.3.31)$$

and

$$h(\cos v) = \frac{1}{\pi}\int_0^\pi f(\cos\mu)\sin\mu\;d\mu\int_0^\pi g(\cos\lambda)\,d\beta.$$

This proves the theorem.

In particular, when $v = 0$, result (14.3.24) becomes

$$h(1) = \int_{-1}^1 f(t)g(-t)dt, \quad (14.3.32)$$

and when $v = \pi$, (14.3.24) gives

$$h(-1) = \int_{-1}^1 f(t)g(-t)dt. \quad (14.3.33)$$

∎

## 14.4 Applications of Legendre Transforms to Boundary Value Problems

We solve the *Dirichlet problem* for the *potential* $u(r,\theta)$ inside a unit sphere $r=1$, which satisfies the Laplace equation

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial u}{\partial r}\right]+\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right]=0, \qquad 0<r<1, \tag{14.4.1}$$

with the boundary condition $(x=\cos\theta)$

$$u(1,x)=f(x), \qquad -1<x<1. \tag{14.4.2}$$

We introduce the *Legendre transform* $\tilde{u}(r,n)=\mathscr{T}_n\{u(r,\theta)\}$ defined by (14.2.1). Application of this transform to (14.4.1)–(14.4.2) gives

$$r^2\,\frac{d^2\tilde{u}(r,n)}{dr^2}+2r\,\frac{d\tilde{u}}{dr}-n(n+1)\tilde{u}(r,n)=0, \tag{14.4.3}$$

$$\tilde{u}(1,n)=\tilde{f}(n), \tag{14.4.4}$$

where $\tilde{u}(r,n)$ is to be continuous function for $r$ for $0\le r<1$.

The bounded solution of (14.4.3)–(14.4.4) is

$$\tilde{u}(r,n)=\tilde{f}(n)\,r^n, \quad 0\le r<1, \qquad \text{for } n=0,1,2,3,\ldots \tag{14.4.5}$$

Thus, the solution for $u(r,x)$ can be found by the inverse transform so that

$$u(r,x)=\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\tilde{f}(n)\,r^n P_n(x) \qquad \text{for } 0<r\le 1,\ |x|<1. \tag{14.4.6}$$

The Convolution Theorem allows us to give another representation of the solutiuon. In view of (14.2.11), we find

$$\mathscr{T}_n^{-1}\{r^n\}=\frac{1}{2}(1-r^2)(1-2rx+r^2)^{-3/2}.$$

Thus, it follows from (14.4.5) that

$$\begin{aligned} u(r,\,\cos\theta) &= \mathscr{T}_n^{-1}\{\tilde{f}(n)\ r^n\} \\ &= \frac{1}{2\pi}\int_0^{\pi} f(\cos\mu)\sin\mu\,d\mu\int_0^{\pi}\frac{(1-r^2)d\lambda}{(1-2r\cos v+r^2)^{3/2}}, \end{aligned} \tag{14.4.7}$$

where

$$\cos v=\cos\mu\cos\theta+\sin\mu\sin\theta\cos\lambda. \tag{14.4.8}$$

Integral (14.4.7) is called the *Poisson integral formula* for the potential inside the unit sphere for the Dirichlet problem.

On the other hand, for the Dirichlet exterior problem, the potential $w(r,\cos\theta)$ outside the unit sphere $(r > 1)$ can be obtained with the boundary condition $w(1,\cos\theta) = f(\cos\theta)$. The solution of the Legendre transformed problem is

$$\tilde{w}(r,n) = \frac{1}{r}\,\tilde{f}(n)\,r^{-n}, \qquad n = 0,1,2,\ldots, \tag{14.4.9}$$

which is, in terms of $w$,

$$w(r,\cos\theta) = \frac{1}{r}\,w\left(\frac{1}{r},\,\cos\theta\right), \quad r > 1 \tag{14.4.10}$$

$$= \frac{1}{2\pi}\int_0^\pi f(\cos\mu)\sin\mu\;d\mu\int_0^\pi \frac{(r^2-1)d\lambda}{(1-2r\cos v + r^2)^{3/2}}, \tag{14.4.11}$$

where $\cos v$ is given by (14.4.8).

## 14.5 Exercises

1. Show that, if $|r| < 1$,

   (a) $\mathscr{T}_n\{x^n\} = \dfrac{2^{n+1}(n!)^2}{(2n+1)!}$.

   (b) $\mathscr{T}_n\left[\log\left\{\dfrac{r - x + (1-2rx+r^2)^{1/2}}{1-x}\right\}\right] = \dfrac{2\,r^{n+1}}{(n+1)(2n+1)}$.

   (c) $\mathscr{T}_n\left[\left\{2\,r(1-rx+r^2)^{-1/2}\right\} - \log\left\{\dfrac{r-x+(1-2rx+r^2)^{1/2}}{1-x}\right\}\right]$
   $= \dfrac{2r^{n+1}}{(n+1)}$.

   (d) $\mathscr{T}_n\left[-\log\dfrac{1}{2}\{1-rx+(1-2rx+r^2)^{1/2}\}\right] = \left\{\begin{array}{ll} 0, & n=0 \\ \dfrac{2\,r^n}{n(2n+1)}, & n>0 \end{array}\right\}$.

   (e) $\mathscr{T}_n\left[(1-2rx+r^2)^{-\frac{1}{2}} - \dfrac{1}{2}\log\left\{\dfrac{1-rx+(1-2rx+r^2)^{1/2}}{2}\right\}\right] = \dfrac{r^n}{n}$.

2. Using the recurrence relation for the Legendre polynomials, show that

   $$\mathscr{T}_n[x\,f(x)] = (2n+1)^{-1}[(n+1)\tilde{f}(n+1) + n\,\tilde{f}(n-1)].$$

   Hence, find $\mathscr{T}_n\{x^2\,f(x)\}$.

3. Use the definition of the *even Legendre-transform pairs* (Tranter, 1966)

$$\mathscr{T}_{2n}\{f(x)\} = \tilde{f}(2n) = \int_0^1 f(x) P_{2n}(x)\, dx, \qquad n = 0, 1, 2, \ldots$$

$$f(x) = \mathscr{T}_{2n}^{-1}\{\tilde{f}(2n)\} = \sum_{n=0}^{\infty} (4n+1)\tilde{f}(2n) P_{2n}(x), \qquad 0 < x < 1,$$

to show that

$$\mathscr{T}_{2n}\left[\frac{d}{dx}\{(1-x^2)f'(x)\}\right] = -2n(2n+1)\tilde{f}(2n) - f'(0)P_{2n}(0).$$

Hence, deduce

$$\mathscr{T}_{2n}\{x\} = -\frac{P_{2n}(0)}{(2n-1)(2n+2)}.$$

4. Use the definition of the *odd Legendre-transform pairs* (Tranter, 1966)

$$\mathscr{T}_{2n+1} = \tilde{f}(2n+1) = \int_0^1 P_{2n+1}(x) f(x) dx, \qquad n = 0, 1, 2, \ldots.$$

$$f(x) = \mathscr{T}_{2n+1}^{-1}\{\tilde{f}(2n+1)\} = \sum_{n=0}^{\infty} (4n+3) P_{2n+1}(x) \tilde{f}(2n+1),$$

to prove the result

$$\mathscr{T}_{2n+1}\left[\frac{d}{dx}\{(1-x^2)f'(x)\}\right] = -(2n+1)(2n+2)\tilde{f}(2n+1) + f(0)P'_{2n+1}(0).$$

Hence, derive

$$\mathscr{T}_{2n+1}\{1\} = \frac{P'_{2n+1}(0)}{(2n+1)(2n+2)}.$$

5. From the definition of the *even Legendre transform*, show that

$$\mathscr{T}_{2n}\{x^{2r}\} = \frac{2^{2n}(2r)!(r+n)!}{(2r+2n+1)!(r-n)!}.$$

6. Show that the *Legendre transform* solution of the Dirichlet boundary value problem for $u(r, \theta)$

$$u_{rr} + \frac{1}{r}u_r + (1-x^2)u_{xx} - 2xu_x = 0, \quad 0 \le r \le a, \quad 0 \le \theta \le \pi$$

$$u(a, \theta) = f(x), \qquad 0 \le \theta \le \pi,$$

where $x = \cos\theta$, is

$$\tilde{u}(r,n) = \left(\frac{r}{a}\right)^n \tilde{f}(n).$$

Obtain the solution for $u(r,\theta)$ with the help of (14.2.11) and the Convolution Theorem 14.3.4.

7. Solve the problem of the electrified disk for the potential $u(\xi,\eta)$ which satisfies the equation (see Tranter, 1966, p. 99)

$$\frac{\partial}{\partial\xi}\left[(1-\xi^2)\frac{\partial u}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial u}{\partial\eta}\right] = 0,$$

and the boundary data

$$u(\xi,\eta) = 0 \quad \text{on } \eta = 0, \quad \text{and} \quad \frac{\partial u}{\partial\xi} = 0 \quad \text{on } \xi = 0,$$

where $(\xi,\eta)$ are the oblate spheroidal corrdinates related to the cylindrical polar coordinates $(r,z)$ by $r = (1-\xi^2)^{1/2}(1-\eta^2)^{1/2}$ and $z = \xi\eta$.

# 15

## *Jacobi and Gegenbauer Transforms*

"The real end of science is the honor of the human mind."

Carl Jacobi

"... Jacobi possessed not only the impulse to acquire pure scientific knowledge, but also the desire to impart it. ..."

Felix Klein

## 15.1 Introduction

This chapter deals with Jacobi and Gegenbauer transforms and their basic operational properties. The former is a fairly general finite integral transform in the sense that both Gegenbauer and Legendre transforms follow as special cases of the Jacobi transform. Some applications of both Jacobi and Gegenbauer transforms are discussed. This chapter is based on papers by Debnath (1963, 1967), Scott (1953), Conte (1955), and Lakshmanarao (1954). In Chapters 12–15, we discussed several special transforms with orthogonal polynomials as kernels. All these special transforms have been unified by Eringen (1954) in his paper on the finite Sturm-Liouville transform.

## 15.2 Definition of the Jacobi Transform and Examples

Debnath (1963) introduced the *Jacobi transform* of a function $F(x)$ defined in $-1 < x < 1$ by the integral

$$J\{F(x)\} = f^{(\alpha,\beta)}(n) = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) F(x)\, dx, \qquad (15.2.1)$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree $n$ and orders $\alpha(> -1)$ and $\beta(> -1)$.

We assume that $F(x)$ admits the following series expansion

$$F(x) = \sum_{n=1}^{\infty} a_n \; P_n^{(\alpha,\beta)}(x). \tag{15.2.2}$$

In view of the orthogonal relation

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \; P_n^{(\alpha,\beta)}(x) \; P_m^{(\alpha,\beta)}(x)\, dx = \delta_n\, \delta_{mn}, \tag{15.2.3}$$

where $\delta_{nm}$ is the Kronecker delta symbol,

$$\delta_n = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(\alpha+\beta+2n+1)\Gamma(n+\alpha+\beta+1)}, \tag{15.2.4}$$

and the coefficients $a_n$ in (15.2.2) are given by

$$a_n = \frac{1}{\delta_n}\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} F(x) P_n^{(\alpha,\beta)}(x)dx = \frac{f^{(\alpha,\beta)}(n)}{\delta_n}. \tag{15.2.5}$$

Thus, the *inverse Jacobi transform* is given by

$$J^{-1}\{f^{(\alpha,\beta)}(n)\} = F(x) = \sum_{n=0}^{\infty} (\delta_n)^{-1} \; f^{(\alpha,\beta)}(n) P_n^{(\alpha,\beta)}(x). \tag{15.2.6}$$

Note that both $J$ and $J^{-1}$ are linear transformations.

***Example 15.2.1***

If $F(x)$ is a polynomial of degree $m < n$, then

$$J\{F(x)\} = 0. \tag{15.2.7}$$

□

***Example 15.2.2***

$$J\{P_m^{(\alpha,\beta)}(x)\} = \delta_{mn}. \tag{15.2.8}$$

□

***Example 15.2.3***

From the uniformly convergent expansion of the generating function for $|z| < 1$

$$2^{\alpha+\beta} Q^{-1}(1-z+Q)^{-\alpha}(1+z+Q)^{-\beta} = \sum_{n=0}^{\infty} z^n P_n^{(\alpha,\beta)}(x), \tag{15.2.9}$$

where $Q = \left(1 - 2xz + z^2\right)^{\frac{1}{2}}$, it turns out that

$$\begin{aligned} &J\{2^{\alpha+\beta}Q^{-1}(1-z+Q)^{-\alpha}(1+z+Q)^{-\beta}\} \\ &\quad = \sum_{n=0}^{\infty} z^n \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)dx \\ &\quad = \sum_{n=0}^{\infty} (\delta_n) z^n. \end{aligned} \tag{15.2.10}$$

□

***Example 15.2.4***

$$\begin{aligned} J\{x^n\} &= \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) x^n dx \\ &= 2^{n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}. \end{aligned} \tag{15.2.11}$$

□

***Example 15.2.5***
If $p > \beta - 1$, then

$$\begin{aligned} J\{(1+x)^{p-\beta}\} &= \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{p} P_n^{(\alpha,\beta)}(x)dx \\ &= \binom{n+\alpha}{n} 2^{\alpha+p+1} \frac{\Gamma(p+1)\Gamma(\alpha+1)\Gamma(p-\beta+1)}{\Gamma(\alpha+p+n+2)\Gamma(p-\beta+n+1)}. \end{aligned} \tag{15.2.12}$$

In particular, when $\alpha = \beta = 0$, the above results reduce to the corresponding results for the *Legendre transform* defined by (14.2.1) so that

$$\begin{aligned} \mathscr{T}_n\{(1+x)^p\} &= \int_{-1}^{1} (1+x)^p P_n(x)dx \\ &= \frac{2^{p+1}\{\Gamma(1+p)\}^2}{\Gamma(p+n+2)\Gamma(p+n+1)}, \quad (p > -1). \end{aligned} \tag{15.2.13}$$

□

***Example 15.2.6***
If Re $\sigma > -1$, then

$$J\{(1-x)^{\sigma-\alpha}\} = \int_{-1}^{1} (1-x)^{\sigma}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x)dx, \quad \text{Re } \sigma > -1,$$
$$= \frac{2^{\sigma+\beta+1}}{n!\,\Gamma(\alpha-\sigma)} \cdot \frac{\Gamma(\sigma+1)\Gamma(n+\beta+1)\Gamma(\alpha-\sigma+n)}{\Gamma(\beta+\sigma+n+2)}, \quad (15.2.14)$$

□

***Example 15.2.7***
If Re $\sigma > -1$, then

$$J\{(1+x)^{\sigma-\beta} P_m^{(\alpha,\sigma)}(x)\} = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\sigma)}(x)dx$$
$$= \frac{2^{\alpha+\sigma+1}\Gamma(n+\alpha+1)\Gamma(\alpha+\beta+m+n+1)\Gamma(\sigma+m+1)}{m!\,(n-m)!\,\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+\sigma+m+n+2)}$$
$$\times \frac{\Gamma(\sigma-\beta+1)}{\Gamma(\alpha-\beta+m+1)}. \quad (15.2.15)$$

□

## 15.3 Basic Operational Properties

***THEOREM 15.3.1***
If $J\{F(x)\} = f^{(\alpha,\beta)}(n)$,

$$\lim_{|x|\to 1} (1-x)^{\alpha+1}(1+x)^{\beta+1} F(x) = 0, \quad (15.3.1a)$$

$$\lim_{|x|\to 1} (1-x)^{\alpha+1}(1+x)^{\beta+1} F'(x) = 0, \quad (15.3.1b)$$

$$R[F(x)] = (1-x)^{-\alpha}(1+x)^{-\beta} \frac{d}{dx}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{d}{dx}F(x)\right], \quad (15.3.2)$$

then $J\{R[F(x)]\}$ exists and is given by

$$J\{R[F(x)]\} = -n(n+\alpha+\beta+1) f^{(\alpha,\beta)}(n), \quad (15.3.3)$$

where $n = 0, 1, 2, 3, \ldots$.

**PROOF** We have, by definition,

$$J\{R[F(x)]\} = \int_{-1}^{1} \frac{d}{dx}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{dF}{dx}\right] P_n^{(\alpha,\beta)}(x)dx,$$

which is, by integrating by parts and using the orthogonal relation (15.2.3),

$$= -n(n+\alpha+\beta+1)\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)F(x)dx$$
$$= -n(n+\alpha+\beta+1)f^{(\alpha,\beta)}(n).$$

This completes the proof.

If $F(x)$ and $R[F(x)]$ satisfy the conditions of Theorem 15.3.1, then $J\{[R[F(x)]]\}$ exists and is given by

$$J\{R^2[F(x)]\} = J\{R[R[F(x)]]\} = (-1)^2 n^2(n+\alpha+\beta+1)^2 f^{(\alpha,\beta)}(n). \quad (15.3.4)$$

More generally, if $F(x)$ and $R^k[F(x)]$ satisfy the conditions of Theorem 15.3.1, where $k = 1, 2, \cdots, m-1$, and $m$ is a positive integer then

$$J\{R^m[F(x)]\} = (-1)^m n^m (n+\alpha+\beta+1)^m f^{(\alpha,\beta)}(n). \quad (15.3.5)$$

When $\alpha = \beta = 0$, $P_n^{(0,0)}(x)$ becomes the Legendre polynomial $P_n(x)$ and the Jacobi transform pairs (15.2.1) and (15.2.5) reduce to the Legendre transform pairs (14.2.1) and (14.2.3). All results for the Jacobi transform also reduce to those given in Chapter 14. ▮

## 15.4 Applications of Jacobi Transforms to the Generalized Heat Conduction Problem

The one-dimensional generalized heat equation for temperature $u(x,t)$ is

$$\frac{\partial}{\partial x}\left[\kappa\frac{\partial u}{\partial x}\right] + Q(x,t) = \rho\, c\,\frac{\partial u}{\partial t}, \quad (15.4.1)$$

where $\kappa$ is the thermal conductivity, $Q(x,t)$ is a continuous heat source within the medium, $\rho$ and $c$ are density and specific heat respectively. If the thermal conductivity is $\kappa = a(1-x^2)$, where $a$ is a real constant, and the source is $Q(x,t) = (\mu x + \nu)\frac{\partial u}{\partial x}$, then the heat equation (15.4.1) reduces to

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right] + \left(\frac{\mu x + \nu}{a}\right)\frac{\partial u}{\partial x} = \left(\frac{\rho\, c}{a}\right)\frac{\partial u}{\partial t}. \quad (15.4.2)$$

We consider a non-homogeneous beam with ends at $x=\pm 1$ whose lateral surface is insulated. Since $\kappa=0$ at the ends, the ends of the beam are also insulated. We assume the initial conditions as

$$u(x,0)=G(x) \qquad \text{for all } -1<x<1, \tag{15.4.3}$$

where $G(x)$ is a suitable function so that $J\{G(x)\}$ exists.

If we write $\dfrac{\mu}{a}=-(\alpha+\beta)$ and $\dfrac{\nu}{a}=\beta-\alpha$ so that $(\alpha,\beta)=-\left(\dfrac{\mu+\nu}{2a},\dfrac{\mu-\nu}{2a}\right)$, the left-hand side of (15.4.2) becomes

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right]+[(\beta-\alpha)-(\beta+\alpha)x]\frac{\partial u}{\partial x}\\
&=\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right]+[(1-x)\beta-(1+x)\alpha]\frac{\partial u}{\partial x}\\
&=(1-x)^{-\alpha}(1+x)^{-\beta}\left\{(1-x)^{\alpha}(1+x)^{\beta}\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right]\right.\\
&\quad\left.+\left[\beta(1+x)^{\beta}(1-x)^{\alpha+1}-\alpha(1-x)^{\alpha}(1+x)^{\beta+1}\right]\frac{\partial u}{\partial x}\right\}\\
&=(1-x)^{-\alpha}(1+x)^{-\beta}\left\{\frac{\partial}{\partial x}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{\partial u}{\partial x}\right]\right\}\\
&=R[u(x,t)].
\end{aligned}$$

Thus, equation (15.4.2) reduces to

$$R[u(x,t)]=\left(\frac{1}{d}\right)\frac{\partial u}{\partial t}, \quad d=\left(\frac{a}{\rho c}\right). \tag{15.4.4}$$

Application of the *Jacobi transform* to (15.4.4) and (15.4.3) gives

$$\frac{d}{dt}u^{(\alpha,\beta)}(n,t)=-d\,n(n+\alpha+\beta+1)u^{(\alpha,\beta)}(n,t), \tag{15.4.5}$$

$$u^{(\alpha,\beta)}(n,0)=g^{(\alpha,\beta)}(n). \tag{15.4.6}$$

The solution of this system is

$$u^{(\alpha,\beta)}(n,t)=g^{(\alpha,\beta)}(n)\exp[-n(n+\alpha+\beta+1)td]. \tag{15.4.7}$$

The *inverse Jacobi transform* gives the formal solution

$$u(x,t)=\sum_{n=0}^{\infty}\delta_n^{-1}g^{(\alpha,\beta)}(n)P_n^{(\alpha,\beta)}(x)\exp[-n(n+\alpha+\beta+1)td], \tag{15.4.8}$$

where $\alpha=-\dfrac{1}{2a}(\mu+\nu)$ and $\beta=\dfrac{1}{2a}(\mu-\nu)$.

## 15.5 The Gegenbauer Transform and Its Basic Operational Properties

When $\alpha = \beta = \nu - \dfrac{1}{2}$, the *Jacobi polynomial* $P_n^{(\alpha,\beta)}(x)$ becomes the *Gegenbauer polynomial* $C_n^{\nu}(x)$ which satisfies the self-adjoint differential form

$$\frac{d}{dx}\left[(1-x^2)^{\nu+\frac{1}{2}}\frac{dy}{dx}\right] + n(n+2\nu)(1-x^2)^{\nu-1}\,y = 0, \tag{15.5.1}$$

and the orthogonal relation

$$\int_{-1}^{1}(1-x^2)^{\nu-\frac{1}{2}}\,C_m^{\nu}(x)\,C_n^{\nu}(x)\,dx = \delta_n\delta_{mn}, \tag{15.5.2}$$

where

$$\delta_n = \frac{2^{1-2\nu}\pi\Gamma(n+2\nu)}{n!\,(n+\nu)[\Gamma(\nu)]^2}. \tag{15.5.3}$$

Thus, when $\alpha = \beta = \nu - \dfrac{1}{2}$, the Jacobi transform pairs (15.2.1) and (15.2.6) reduce to the *Gegenbauer transform* pairs, in the form

$$G\{F(x)\} = f^{(\nu)}(n) = \int_{-1}^{1}(1-x^2)^{\nu-\frac{1}{2}}\,C_n^{\nu}(x)\,F(x)dx, \tag{15.5.4}$$

$$G^{-1}\{f^{(\nu)}(n)\} = F(x) = \sum_{n=0}^{\infty}\delta_n^{-1}\,C_n^{\nu}(x)f^{(\nu)}(n), \qquad -1<x<1. \tag{15.5.5}$$

Obviously, $G$ and $G^{-1}$ stand for the Gegenbauer transformation and its inverse respectively. They are linear integral transformations.

When $\alpha = \beta = \nu - \dfrac{1}{2}$, the differential form (15.3.2) becomes

$$R[F(x)] = (1-x^2)\frac{d^2F}{dx^2} - (2\nu+1)\,x\,\frac{dF}{dx}, \tag{15.5.6}$$

which can be expressed as

$$R[F(x)] = (1-x^2)^{\frac{1}{2}-\nu}\frac{d}{dx}\left[(1-x^2)^{\nu+\frac{1}{2}}\frac{dF}{dx}\right]. \tag{15.5.7}$$

Under the Gegenbauer transformation $G$, the differential form (15.5.6) is reduced to the algebraic form

$$G\{R[F(x)]\} = -n(n+2\nu)f^{(\nu)}(n). \tag{15.5.8}$$

This follows directly from the relation (15.3.3).

Similarly, we obtain

$$G\{R^2[F(x)]\} = (-1)^2 n^2 (n+2\nu)^2 f^{(\nu)}(n). \tag{15.5.9}$$

More generally,

$$G\{R^k[F(x)]\} = (-1)^k n^k (n+2\nu)^k f^{(\nu)}(n), \tag{15.5.10}$$

where $k = 1, 2, \ldots$.

***Convolution Theorem 15.5.1*** If $G\{F(x)\} = f^{(\nu)}(n)$ and $G\{G(x)\} = g^{(\nu)}(n)$, then

$$f^{(\nu)}(n) g^{(\nu)}(n) = G\{H(x)\} = h^{(\nu)}(n), \tag{15.5.11}$$

where

$$H(x) = G^{-1}\{h^{(\nu)}(n)\} = G^{-1}\{f^{(\nu)}(n) g^{(\nu)}(n)\} = F(x) * G(x), \tag{15.5.12}$$

and $H(x)$ is given by

$$H(\cos\psi) = A(\sin\psi)^{1-2\nu} \int_0^\pi \int_0^\pi F(\cos\theta) G(\cos\phi)(\sin\theta)^{2\nu} \times (\sin\phi)^{2\nu-1} (\sin\lambda)^{2\nu-1} d\theta\, d\alpha, \tag{15.5.13}$$

where $\alpha$ is defined by (15.5.19).

**PROOF** We have, by definition,

$$\begin{aligned} f^{(\nu)}(n)\, g^{(\nu)}(n) &= \int_{-1}^{1} F(x)(1-x^2)^{v-\frac{1}{2}} C_n^\nu(x)\, dx \\ &\quad \times \int_{-1}^{1} G(x)(1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x)\, dx \\ &= \int_0^\pi F(\cos\theta)(\sin\theta)^{2\nu} C_n^\nu(\cos\theta)\, d\theta \\ &\quad \times \int_0^\pi G(\cos\phi)(\sin\phi)^{2\nu} C_n^\nu(\cos\phi)\, d\phi \\ &= \int_0^\pi F(\cos\theta)(\sin\theta)^{2\nu} [\int_0^\pi G(\cos\phi) C_n^\nu(\cos\theta) \\ &\quad \times C_n^\nu(\cos\phi)(\sin\phi)^{2\nu} d\phi]\, d\theta. \end{aligned} \tag{15.5.14}$$

The addition formula for the Gegenbauer polynomial (see Erdélyi, 1953, p. 177) is

$$C_n^\nu(\cos\theta)C_n^\nu(\cos\phi) = A\int_0^\pi C_n^\nu(\cos\psi)(\sin\lambda)^{2\nu-1}\,d\lambda, \tag{15.5.15}$$

where

$$A = \{\Gamma(n+2\nu)/n!\,2^{2\nu-1}\,\Gamma^2(\nu)\}, \tag{15.5.16}$$

and

$$\cos\psi = \cos\theta\cos\phi + \sin\theta\sin\phi\cos\lambda. \tag{15.5.17}$$

In view of this formula, result (15.5.14) assumes the form

$$\begin{aligned} f^{(\nu)}(n)g^{(\nu)}(n) &= A\int_0^\pi F(\cos\,\theta)(\sin\,\theta)^{2\nu}[\int_0^\pi\int_0^\pi G(\cos\,\phi)C_n^\nu(\cos\,\psi) \\ &\quad\times(\sin\phi)^{2\nu}(\sin\lambda)^{2\nu-1}d\lambda\,d\phi]\,d\theta. \end{aligned} \tag{15.5.18}$$

We next introduce a new variable $\alpha$ defined by the relation

$$\cos\phi = \cos\theta\cos\psi + \sin\theta\sin\psi\cos\alpha. \tag{15.5.19}$$

Thus, under transformation of coordinates defined by (15.5.17) and (15.5.19), the elementary area $d\lambda\,d\phi = (\sin\psi/\sin\phi)d\psi\,d\alpha$, where $(\sin\psi/\sin\phi)$ is the Jacobian of the transformation. In view of this transformation, the square region of the $\phi$-$\lambda$ plane given by $(0\le\phi\le\pi,\ 0\le\lambda\le\pi)$ transforms into a square region of the same dimension in the $\psi$-$\alpha$ plane. Consequently, the double integral inside the square bracket in (15.5.18) reduces to

$$\int_0^\pi\int_0^\pi G(\cos\phi)\,C_n^\nu(\cos\psi)(\sin\phi)^{2\nu-1}(\sin\lambda)^{2\nu-1}\sin\psi\,d\psi\,d\alpha, \tag{15.5.20}$$

where $\cos\psi$ is defined by (15.5.17) and $\cos\phi$ is defined by (15.5.19). If the double integral (15.5.20) is substituted into (15.5.18), and if the order of integration is interchanged, (15.5.18) becomes

$$f^{(\nu)}(n)g^{(\nu)}(n) = \int_0^\pi(\sin\psi)^{2\nu}{C_n}^\nu(\cos\psi)H(\cos\psi)\,d\psi = G\{H(\cos\psi)\}, \tag{15.5.21}$$

where

$$\begin{aligned} H(\cos\psi) &= A(\sin\psi)^{1-2\nu}\int_0^\pi\int_0^\pi F(\cos\theta)G(\cos\phi)(\sin\theta)^{2\nu} \\ &\quad\times(\sin\phi)^{2\nu-1}(\sin\lambda)^{2\nu-1}\,d\theta\,d\alpha. \end{aligned} \tag{15.5.22}$$

When $\nu = \frac{1}{2}$, $C_n^{\frac{1}{2}}(x)$ becomes the Legendre polynomial, the Gegenbauer transform pairs (15.5.4) and (15.5.5) reduce to the Legendre transform pairs (14.2.1) and (14.2.3), and the Convolution Theorem 15.5.1 reduces to the corresponding Convolution Theorem 14.3.4 for the Legendre transform. ∎

## 15.6 Application of the Gegenbauer Transform

The generalized one-dimensional heat equation in a non-homogeneous solid beam for the temperature $u(x,t)$ is

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right] - (2\nu+1)\, x\frac{\partial u}{\partial x} = \frac{1}{d}\frac{\partial u}{\partial t}, \tag{15.6.1}$$

where $\kappa = (1-x^2)$ is the thermal conductivity, $d = \left(\frac{a}{\rho c}\right)$, and the second term on the left hand side represents the continuous source of heat within the solid beam. We assume that the beam is bounded by the planes at $x = \pm 1$ and its lateral surfaces are insulated. The initial condition is

$$u(x,0) = G(x) \qquad \text{for } -1 < x < 1, \tag{15.6.2}$$

where $G(x)$ is a given function so that its Gegenbauer transform exists.

Application of the Gegenbauer transform to (15.6.1) and (15.6.2) and the use of (15.5.8) gives

$$\frac{d}{dt}u^{(\nu)}(n,t) = -d\, n(n+2\nu)\, u^{(\nu)}(n,t), \tag{15.6.3}$$

$$u^{(\nu)}(n,0) = g^{(\nu)}(n). \tag{15.6.4}$$

This solution of this system is

$$u^{(\nu)}(n,t) = g^{(\nu)}(n)\, \exp[-n(n+2\nu)td]. \tag{15.6.5}$$

The inverse transform gives the formal solution

$$u(x,t) = \sum_{n=0}^{\infty} \delta_n^{-1}\, C_n^{\nu}(x)\, g^{(\nu)}(n)\, \exp[-n(n+2\nu)td], \tag{15.6.6}$$

where $\delta_n$ is given by (15.5.3).

# 16

# *Laguerre Transforms*

"The search for truth is more precious than its possession."
Albert Einstein

"Nature is an infinite sphere of which the center is everywhere and the circumference nowhere."
Blaise Pascal

"Mathematics is the tool specially suited for dealing with abstract concepts of any kind and there is no limit to its power in this field."
Paul A. M. Dirac

## 16.1 Introduction

This chapter is devoted to the study of the Laguerre transform and its basic operational properties. It is shown that the Laguerre transform can be used effectively to solve the heat conduction problem in a semi-infinite medium with variable thermal conductivity in the presence of a heat source within the medium. This chapter is based on a series of papers by Debnath (1960–1962) and McCully (1960) listed in the Bibliography.

## 16.2 Definition of the Laguerre Transform and Examples

Debnath (1960) introduced the *Laguerre transform* of a function $f(x)$ defined in $0 \leq x < \infty$ by means of the integral

$$L\{f(x)\} = \tilde{f}_\alpha(n) = \int_0^\infty e^{-x}\, x^\alpha\, L_n^\alpha(x) f(x)\, dx, \tag{16.2.1}$$

where $L_n^\alpha(x)$ is the *Laguerre polynomial* of degree $n(\geq 0)$ and order $\alpha(>-1)$, which satisfies the ordinary differential equation expressed in the self-adjoint form

$$\frac{d}{dx}\left[e^{-x}\,x^{\alpha+1}\frac{d}{dx}L_n^\alpha(x)\right]+ne^{-x}\,x^\alpha\,L_n^\alpha(x)=0. \tag{16.2.2}$$

In view of the orthogonal property of the Laguerre polynomials

$$\int_0^\infty e^{-x}\,x^\alpha\,L_n^\alpha(x)\,L_m^\alpha(x)\,dx=\binom{n+\alpha}{n}\Gamma(\alpha+1)\delta_{mn}=\delta_n\,\delta_{nm}, \tag{16.2.3}$$

where $\delta_{mn}$ is the Kronecker delta symbol, and $\delta_n$ is given by

$$\delta_n=\binom{n+\alpha}{n}\Gamma(\alpha+1). \tag{16.2.4}$$

The *inverse Laguerre transform* is given by

$$f(x)=L^{-1}\{\tilde{f}_\alpha(n)\}=\sum_{n=0}^\infty(\delta_n)^{-1}\tilde{f}_\alpha(n)\,L_n^\alpha(x). \tag{16.2.5}$$

When $\alpha=0$, the Laguerre transform pairs due to McCully (1960) follow from (16.2.1) and (16.2.5) in the form

$$L\{f(x)\}=\tilde{f}_0(n)=\int_0^\infty e^{-x}\,L_n(x)f(x)\,dx, \tag{16.2.6}$$

$$L^{-1}\{\tilde{f}_0(n)\}=f(x)=\sum_{n=0}^\infty\tilde{f}_0(n)\,L_n(x), \tag{16.2.7}$$

where $L_n(x)$ is the Laguerre polynomial of degree $n$ and order zero.

Obviously, $L$ and $L^{-1}$ are linear integral transformations. The following examples (Debnath, 1960) illustrate the Laguerre transform of some simple functions.

***Example 16.2.1***

$$\text{If}\quad f(x)=L_m^\alpha(x)\quad\text{then}\quad L\{L_m^\alpha(x)\}=\delta_n\,\delta_{nm}. \tag{16.2.8}$$

This follows directly from the definitions, (16.2.1) and (16.2.3). ▯

***Example 16.2.2***

If $f(x)=x^{s-1}$ where $s$ is a positive real number, then

$$L\{x^{s-1}\}=\int_0^\infty e^{-x}\,x^{\alpha+s-1}\,L_n^\alpha(x)\,dx=\frac{\Gamma(s+\alpha)\Gamma(n-s+1)}{n!\;\Gamma(1-s)}, \tag{16.2.9}$$

in which a result due to Howell (1938) is used. ▯

***Example 16.2.3***
If $a > -1$, and $f(x) = e^{-ax}$, then

$$L\{e^{-ax}\} = \int_0^\infty e^{-x(1+a)} x^\alpha L_n^\alpha(x)\, dx = \frac{\Gamma(n+\alpha+1)a^n}{n!\,(a+1)^{n+\alpha+1}}, \qquad (16.2.10)$$

where result in Erdélyi *et al.* (1954, vol. 2, p 191) is used. ▯

***Example 16.2.4***
If $f(x) = e^{-ax} L_m^\alpha(x)$, then

$$L\{e^{-ax} L_m^\alpha(x)\} = \int_0^\infty e^{-x(a+1)} x^\alpha L_n^\alpha(x) L_m^\alpha(x)\, dx,$$

which is, due to Howell (1938),

$$= \frac{1}{n!\,m!} \frac{\Gamma(n+\alpha+1)\Gamma(m+\alpha+1)}{\Gamma(1+\alpha)} \cdot \frac{(a-1)^{n-m+\alpha+1}}{a^{n+m+2\alpha+2}} \times {}_2F_1\left(n+\alpha+1,\, \frac{m+a+1}{a+1},\, \frac{1}{a^2}\right), \qquad (16.2.11)$$

where ${}_2F_1(x, \alpha, \beta)$ is the hypergeometric function. ▯

***Example 16.2.5***

$$L\{f(x)x^{\beta-\alpha}\} = \int_0^\infty e^{-x}\, x^\beta\, L_n^\alpha(x) f(x)\, dx.$$

We use a result from Erdélyi (1953, vol. 2, p. 192) as

$$L_n^\alpha(x) = \sum_{m=0}^{n} (m!)^{-1} (\alpha-\beta)_m\, L_{n-m}^\beta(x) \qquad (16.2.12)$$

to obtain the following result:

$$L\{f(x)x^{\beta-\alpha}\} = \sum_{m=0}^{n} (m!)^{-1} (\alpha-\beta)_m\, \tilde{f}_\beta(n-m). \qquad (16.2.13)$$

In particular, when $\beta = \alpha - 1$, we obtain

$$L\left\{\frac{f(x)}{x}\right\} = \sum_{m=0}^{n} (m!)^{-1}\, \tilde{f}_{\alpha-1}(n-m).$$

▯

***Example 16.2.6***

$$L\{e^x x^{-\alpha}\Gamma(\alpha,x)\} = \sum_{n=0}^{\infty} \frac{\delta_n}{(n+1)}, \quad -1 < \alpha < 0. \tag{16.2.14}$$

We use a result from Erdélyi (1953, vol. 2, p. 215) as

$$e^x x^{-\alpha}\,\Gamma(\alpha,x) = \sum_{n=0}^{\infty}(n+1)^{-1} L_n^{\alpha}(x), \quad (\alpha > -1, x > 0),$$

in the definition (16.2.1) to derive (16.2.14). ▯

***Example 16.2.7***

If $\beta > 0$, then

$$L\{x^{\beta}\} = \Gamma(\alpha+\beta+1) \sum_{n=0}^{\infty} \frac{(-\beta)_n\,\delta_n}{\Gamma(n+\alpha+1)}. \tag{16.2.15}$$

Using the result from Erdélyi (1953, vol. 2, p. 214)

$$x^{\beta} = \Gamma(\alpha+\beta+1) \sum_{n=0}^{\infty} \frac{(-\beta)_n}{\Gamma(n+\alpha+1)} L_n^{\alpha}(x),$$

where

$$-\beta < 1 + \min\left(\alpha, \frac{\alpha}{2} - \frac{1}{4}\right), \quad x > 0,\ \alpha > -1,$$

we can easily obtain (16.2.15). ▯

***Example 16.2.8***

If $|z| < 1$ and $\alpha \geq 0$, then

$$\text{(a)} \quad L\left\{(1-z)^{-(\alpha+1)} \exp\left(\frac{xz}{z-1}\right)\right\} = \sum_{n=0}^{\infty} \delta_n\, z^n, \tag{16.2.16}$$

$$\text{(b)} \quad L\left\{(xz)^{-\frac{\alpha}{2}} e^z J_{\alpha}\left[2(xz)^{\frac{1}{2}}\right]\right\} = \sum_{n=0}^{\infty} \frac{\delta_n z^n}{\Gamma(n+\alpha+1)}. \tag{16.2.17}$$

We have the following generating functions (Erdélyi, 1953, vol. 2, p. 189)

$$(1-z)^{-(\alpha+1)} \exp\left(\frac{xz}{z-1}\right) = \sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n, \ |z| < 1,$$

$$(xz)^{-\alpha/2}\, e^z J_{\alpha}[2\sqrt{xz}] = \sum_{n=0}^{\infty} \frac{z^n L_n^{\alpha}(x)}{\Gamma(n+\alpha+1)}, \ |z| < 1.$$

In view of these results combined with the orthogonality relation (16.2.3), we obtain (16.2.16) and (16.2.17). ▯

***Example 16.2.9***
(*Recurrence Relations*).

$$\text{(a)}\quad \tilde{f}_{\alpha+1}(n) = (n+\alpha+1)\tilde{f}_{\alpha}(n) - (n+1)\tilde{f}_{\alpha}(n+1), \tag{16.2.18}$$

$$\text{(b)}\quad n!\,\tilde{f}_{m-n}(n) = (-1)^{n-m}\,m!\sum_{k=0}^{m}(k!)^{-1}(2n-2m)_k\,\tilde{f}_{m-n}(m-k). \tag{16.2.19}$$

We have

$$\tilde{f}_{\alpha+1}(n) = \int_0^\infty e^{-x}\,x^{\alpha+1}\,L_n^{\alpha+1}(x) f(x)\,dx,$$

which is, by using the recurrence relation for the Laguerre polynomial,

$$= \int_0^\infty e^{-x} x^{\alpha}[(n+\alpha+1)L_n^{\alpha}(x) - (n+1)L_{n+1}^{\alpha}(x)] f(x)\,dx$$
$$= (n+\alpha+1)\,\tilde{f}_{\alpha}(n) - (n+1)\,\tilde{f}_{\alpha}(n+1).$$

Similarly, we find

$$n!\,\tilde{f}_{m-n}(n) = \int_0^\infty e^{-x} x^{m-n} n!\,L_n^{m-n}(x) f(x)\,dx.$$

We next use the following result due to Howell (1938)

$$n!\,L_n^{m-n}(x) = (-1)^{n-m}\,m! L_m^{n-m}(x)$$

to obtain

$$\begin{aligned} n!\,\tilde{f}_{m-n}(n) &= (-1)^{n-m} m! \int_0^\infty e^{-x} x^{m-n} L_m^{n-m}(x) f(x)\,dx \\ &= (-1)^{n-m} m! \sum_{k=0}^{m} (k!)^{-1}(2n-2m)_k\,\tilde{f}_{m-n}(m-k). \end{aligned}$$

□

## 16.3 Basic Operational Properties

We obtain the Laguerre transform of derivatives of $f(x)$ as

$$L\{f'(x)\} = \tilde{f}_\alpha(n) - \alpha \sum_{k=0}^{n} f_{\alpha-1}(k) + \sum_{k=0}^{n-1} f_\alpha(k), \tag{16.3.1}$$

$$L\{f''(x)\} = \tilde{f}_\alpha(n) - 2\alpha \sum_{m=0}^{n} \tilde{f}_{\alpha-1}(n-m) + 2\sum_{m=0}^{n-1} \tilde{f}_\alpha(n-m-1)$$
$$- 2\alpha \sum_{m=0}^{n-1} (m+1)\tilde{f}_{\alpha+1}(n-m-1) + \alpha(\alpha-1)\sum_{m=0}^{n}(m+1) f_{\alpha-2}(n-m)$$
$$+ \sum_{m=0}^{n-2}(m+1)\tilde{f}_\alpha(n-m-2), \tag{16.3.2}$$

and so on for the Laguerre transforms of higher derivatives.

We have, by definition,

$$L\{f'(x)\} = \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) f'(x)\,dx$$
$$= \left[e^{-x} n^\alpha L_n^\alpha(x) f(x)\right]_0^\infty + \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) f(x)\,dx$$
$$-\alpha \int_0^\infty e^{-x} x^{\alpha-1} L_n^\alpha(x) f(x)\,dx - \int_0^\infty e^{-x} x^\alpha \left[\frac{d}{dx} L_n^\alpha(x)\right] f(x)\,dx,$$

which is, due to Erdélyi (1954, vol. 2, p. 192),

$$= \tilde{f}_\alpha(n) - \alpha \sum_{k=0}^{n} \tilde{f}_{\alpha-1}(k) + \sum_{k=0}^{n-1} f_\alpha(k).$$

Similarly, we can derive (16.3.2).

***THEOREM 16.3.1***

If $g(x) = \int_0^x f(t)\,dt$ so that $g(x)$ is absolutely continuous and $g'(x)$ exists, and if $g'(x)$ is bounded and integrable, then

$$\tilde{f}_\alpha(n) - \tilde{f}_\alpha(n-1) = \tilde{g}_\alpha(n) - \alpha\,\tilde{g}_{\alpha-1}(n), \tag{16.3.3}$$

and

$$L\left\{\int_0^x f(t)\,dt\right\} = \tilde{f}_0(n) - \tilde{f}_0(n-1), \tag{16.3.4}$$

where $L$ stands for the zero-order Laguerre transform defined by (16.2.6).

**PROOF** We have

$$\tilde{f}_\alpha(n) = \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) g'(x)\,dx,$$

which is, by integrating by parts,

$$= \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) g(x)\,dx - \alpha \int_0^\alpha e^{-x} x^{\alpha-1} L_n^\alpha(x) g(x)\,dx$$
$$- \int_0^\infty e^{-x} x^\alpha \left[\frac{d}{dx} L_n^\alpha(x)\right] g(x)\,dx.$$

Thus,

$$\tilde{f}_\alpha(n) - \tilde{f}_\alpha(n+1) = \int_0^\infty e^{-x} x^\alpha \left[L_n^\alpha(x) - L_{n+1}^\alpha(x)\right] g(x)\,dx$$
$$+\alpha \int_0^\infty e^{-x} x^\alpha \left[L_{n+1}^\alpha(x) - L_n^\alpha(x)\right] g(x)\,dx$$
$$- \int_0^\infty e^{-x} x^\alpha \frac{d}{dx} \left[L_n^\alpha(x) - L_{n+1}^\alpha(x)\right] g(x)\,dx.$$

Thus,

$$\tilde{f}_\alpha(n) - \tilde{f}_\alpha(n+1) = \int_0^\infty e^{-x} x^\alpha \left[L_n^\alpha(x) - L_{n+1}^\alpha(x)\right] g(x)\,dx$$
$$+\alpha \int_0^\infty e^{-x} x^\alpha L_{n+1}^{\alpha-1}(x) g(x)\,dx - \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) g(x)\,dx$$
$$= -\tilde{g}_\alpha(n+1) + \alpha\,\tilde{g}_{\alpha-1}(n+1).$$

This proves (16.3.3).

Putting $\alpha = 0$, and replacing $n$ by $n-1$ gives

$$\tilde{g}_0(n) = \tilde{f}_0(n) - \tilde{f}_0(n-1).$$

Or,

$$L\left\{\int_0^x f(t)\,dt\right\} = \tilde{f}_0(n) - \tilde{f}_0(n-1).$$

∎

***THEOREM 16.3.2***

If $L\{f(x)\} = \tilde{f}_\alpha(n)$ exists, then

$$L\{R[f(x)]\} = -n\tilde{f}_\alpha(n), \tag{16.3.5}$$

where $R[f(x)]$ is the differential operator given by

$$R[f(x)] = e^x x^{-\alpha}\frac{d}{dx}\left[e^{-x}x^{\alpha+1}\frac{d}{dx}f(x)\right]. \tag{16.3.6}$$

**PROOF** We have, by definition,

$$L\{R[f(x)]\} = \int_0^\infty L_n^\alpha(x)\frac{d}{dx}\left[e^{-x}x^{\alpha+1}\frac{df}{dx}\right]dx,$$

which is, by integrating by parts and using (16.2.2),

$$= -n\int_0^\infty e^{-x}x^\alpha L_n^\alpha(x)f(x)\,dx = -n\tilde{f}_\alpha(n).$$

This completes the proof of the basic operational property. This result can easily be extended as follows:

$$L\{R^2[f(x)]\} = L\{R[R[f(x)]]\} = (-1)^2\,n^2\,\tilde{f}_\alpha(n). \tag{16.3.7}$$

More generally,

$$L\{R^m[f(x)]\} = (-1)^m\,n^m\,\tilde{f}_\alpha(n), \tag{16.3.8}$$

where $m$ is a non-negative integer. ∎

The Convolution Theorem for the Laguerre transform can be stated as follows:

***THEOREM 16.3.3***

(*Convolution Theorem*). If $L\{f(x)\} = \tilde{f}_\alpha(n)$ and $L\{g(x)\} = \tilde{g}_\alpha(n)$, then

$$L^{-1}\{\tilde{f}_\alpha(n)\tilde{g}_\alpha(n)\} = h(x), \tag{16.3.9}$$

where $h(x)$ is given by the following repeated integral

$$h(x) = \frac{\Gamma(n+\alpha+1)}{\sqrt{\pi}\,\Gamma(n+1)} \int_0^\infty e^{-t}\, t^\alpha f(t)\, dt \int_0^\pi \exp(-\sqrt{xt}\cos\phi)$$
$$\times \sin^{2\alpha}\phi\, g(x+t+2\sqrt{xt}\cos\phi)\, \frac{J_{\alpha-\frac{1}{2}}(\sqrt{xt}\sin\phi)\, d\phi}{\left[\frac{1}{2}(\sqrt{xt}\sin\phi)\right]^{\alpha-\frac{1}{2}}}. \quad (16.3.10)$$

In order to avoid long proof of this Convolution Theorem 16.3.3, we will not present the proof here, but refer the reader to the article of Debnath (1969). However, when $\alpha = 0$ and $\phi$ is replaced by $(\pi - \theta)$, and the standard result

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, \cos x \quad (16.3.11)$$

is used, the Convolution Theorem 16.3.3 reduces to that of McCully's (1960). We now state and prove *McCully's Convolution Theorem* as follows:

***THEOREM 16.3.4***
(McCully's Theorem). If $L\{f(x)\} = \tilde{f}_0(n)$ and $L\{g(x)\} = \tilde{g}_0(n)$, then

$$L^{-1}\{\tilde{f}_0(n)\tilde{g}_0(n)\} = h(x), \quad (16.3.12)$$

where $h(x)$ is given by the formula

$$h(x) = \frac{1}{\pi} \int_0^\infty e^{-t} f(t)\, dt \int_0^\pi \exp(\sqrt{xt}\cos\theta)\cos(\sqrt{xt}\sin\,\theta)$$
$$\times g(x+t-2\sqrt{xt}\cos\theta)\, d\theta. \quad (16.3.13)$$

**PROOF** We have, by definition,

$$\tilde{f}_0(n)\tilde{g}_0(n) = \int_0^\infty e^{-x} L_n(x) f(x)\, dx \int_0^\infty e^{-y} L_n(y) g(y)\, dy$$
$$= \int_0^\infty e^{-x} f(x)\, dx \int_0^\infty e^{-y} L_n(x) L_n(y)\, g(y)\, dy. \quad (16.3.14)$$

This can be written in the form

$$\tilde{f}_0(n)\tilde{g}_0(n) = L\{h(t)\} = \int_0^\infty e^{-t} L_n(t) h(t)\, dt.$$

This shows that $h$ is the convolution of $f$ and $g$ and has the representation

$$h(x) = f(x) * g(x). \tag{16.3.15}$$

It follows from a formula of Bateman (1944, p. 457) that

$$L_n(x)L_n(y) = \frac{1}{\pi}\int_0^\pi e^{\sqrt{xy}\cos\theta}\cos(\sqrt{xy}\sin\theta)L_n(x+y-2\sqrt{xy}\cos\theta)\,d\theta. \tag{16.3.16}$$

In view of this result, (16.3.14) becomes

$$\pi\tilde{f}_0(n)\tilde{g}_0(n) = \int_0^\infty e^{-x}f(x)\,dx\left[\int_0^\infty e^{-y}g(y)\int_0^\pi \exp(\sqrt{xy}\cos\theta)\right.$$
$$\left.\times\cos(\sqrt{xy}\sin\theta)L_n(x+y-2\sqrt{xy}\cos\theta)d\theta dy\right]. \tag{16.3.17}$$

Using $\sqrt{y}$ as the variable of integration combined with polar coordinates, the integral inside the square bracket in (16.3.17) can be reduced to the form

$$\int_0^\infty e^{-t}L_n(t)\,dt\int_0^\pi \exp(\sqrt{xt}\cos\phi)\cos(\sqrt{xt}\sin\phi)$$
$$\times\, g(x+t-2\sqrt{xt}\cos\phi)\,d\phi, \tag{16.3.18}$$

so that (16.3.17) becomes

$$\tilde{f}_0(n)\tilde{g}_0(n) = L\{h(t)\} = \int_0^\infty e^{-t}L_n(t)h(t)\,dt,$$

where $h(x)$ is given by

$$h(x) = \frac{1}{\pi}\int_0^\infty e^{-t}f(t)\,dt\int_0^\pi \exp(\sqrt{xt}\cos\theta)\cos(\sqrt{xt}\sin\theta)$$
$$\times\, g(x+t-2\sqrt{xt}\cos\theta)\,d\theta. \tag{16.3.19}$$

This proves the McCully's Theorem for the Laguerre transform (16.2.6).

■

## 16.4 Applications of Laguerre Transforms

***Example 16.4.1***
(*Heat Conduction Problem*). The diffusion equation for one-dimensional linear flow of heat in a semi-infinite medium $0 \le x < \infty$ with a source $Q(x,t)$ in the

medium is

$$\frac{\partial}{\partial x}\left[\kappa\,\frac{\partial u}{\partial x}\right]+Q(x,t)=\rho c\,\frac{\partial u}{\partial t},\quad t>0, \tag{16.4.1}$$

where $\kappa=\kappa(x)=\lambda\,e^{-x}\,x^{\beta}$ is the variable thermal conductivity; $Q(x,t)=\mu\,e^{-x}\,x^{\beta'}\,\frac{\partial u}{\partial x}$; $\rho=\nu\,e^{-x}x^{\beta'}$; $\lambda$, $\mu$, $\nu$, and $c$ are constants; and $\beta\geq 1$ and $\beta-\beta'=1$. Thus, the above equation reduces to

$$\frac{\partial}{\partial x}\left[e^{-x}x^{\beta}\,\frac{\partial u}{\partial x}\right]+\frac{\mu}{\lambda}\,e^{-x}x^{\beta'}\,\frac{\partial u}{\partial x}=\frac{\nu c}{\lambda}\,e^{-x}\,x^{\beta'}\,\frac{\partial u}{\partial t}. \tag{16.4.2}$$

The initial condition is

$$u(x,0)=g(x),\quad 0\leq x<\infty. \tag{16.4.3}$$

Clearly, equation (16.4.2) assumes the form

$$e^{x}x^{-\alpha}\frac{\partial}{\partial x}\left(e^{-x}x^{\alpha+1}\frac{\partial u}{\partial x}\right)=\gamma\frac{\partial u}{\partial t}, \tag{16.4.4}$$

where $\alpha=\dfrac{\mu}{\lambda}+\beta-1$ and $\gamma=\dfrac{\nu c}{\lambda}$.

Application of the Laguerre transform to (16.4.4) gives

$$\frac{d}{dt}\,u_{\alpha}(n,t)=-\frac{n}{\gamma}\,u_{\alpha}(n,t),\qquad\qquad u_{\alpha}(n,0)=g_{\alpha}(n).$$

Thus, the solution of this system is

$$u_{\alpha}(n,t)=g_{\alpha}(n)\exp\left(-\frac{nt}{\gamma}\right). \tag{16.4.5}$$

The inverse transform (16.2.5) gives the formal solution

$$u(x,t)=\sum_{n=0}^{\infty}(\delta_n)^{-1}g_{\alpha}(n)L_n^{\alpha}(x)\exp\left(-\frac{nt}{\gamma}\right), \tag{16.4.6}$$

where $\delta_n$ is given by (16.2.4). $\square$

***Example 16.4.2***

(*Diffusion Equation*). Solve equation (16.4.1) with

$$\kappa=x\,e^{-x},\quad Q(x,t)=e^{-x}\,f(t),\ \text{and}\ \rho c=e^{-x}.$$

In this case, the diffusion equation (16.4.1) becomes

$$\frac{\partial u}{\partial t}=e^{x}\frac{\partial}{\partial x}\left(x\,e^{-x}\,\frac{\partial u}{\partial x}\right)+f(t),\quad 0\leq x<\infty,\ t>0, \tag{16.4.7}$$

has to be solved with the initial-boundary data

$$\left.\begin{aligned} &u(x,0)=g(x), && 0\le x<\infty \\ &\frac{\partial}{\partial t}u(x,t)=f(x), \quad \text{at } t=0, && \text{for } x>0 \end{aligned}\right\}. \tag{16.4.8}$$

Application of the Laguerre transform $L\{u(x,t)\}=\tilde{u}_0(n,t)$ to (16.4.7)–(16.4.8) gives

$$\tilde{u}_0(n,t) = g_0(n)\, e^{-nt}, \quad n=1,2,3,\ldots \tag{16.4.9}$$

$$\tilde{u}_0(0,t) = g_0(0) + \int_0^t f(\tau)\, d\tau. \tag{16.4.10}$$

The inverse Laguerre transform (16.2.5) leads to the formal solution

$$\begin{aligned} u(x,t) &= g_0(0) + \int_0^t f(\tau)\, d\tau + \sum_{n=1}^{\infty} g_0(n) e^{-nt} L_n(x) \\ &= \int_0^t f(\tau) d\tau + \sum_{n=0}^{\infty} g_0(n) e^{-nt} L_n(x). \end{aligned} \tag{16.4.11}$$

In view of the Convolution Theorem 16.3.4, this result takes the form

$$\begin{aligned} u(x,t) = \int_0^t f(\tau)\, d\tau + \frac{1}{\pi}\int_0^{\infty} e^{-\tau}(e^{\tau}-1)^{-1} \exp\left(\frac{-\tau}{e^t-1}\right) \\ \times \int_0^{\pi} \exp(\sqrt{x\tau}\cos\theta)\cos(\sqrt{x\tau}\sin\theta) g(x+\tau-2\sqrt{x\tau}\cos\theta)\, d\theta d\tau. \end{aligned} \tag{16.4.12}$$

This result is obtained by McCully (1960).

Another application of the Laguerre transform to the problem of oscillations of a very long and heavy chain with variable tension was discussed by Debnath (1961).

We conclude this chapter by adding references of recent work on the Laguerre-Pinney transformation and the Wiener-Laguerre transformation by Glaeske (1981, 1986). For more details, the reader is referred to these papers. □

## 16.5 Exercises

1. Find the zero-order Laguerre transform of each of the following functions:

   (a) $H(x-a)$ for constant $a \geq 0$, (b) $e^{-ax}$ $(a > -1)$,

   (c) $AL_m(x)$, (d) $x^m$, (e) $L_n(x)$.

2. If $L\{f(x)\} = f_0(n) = \int\limits_0^\infty e^{-x} L_n(x) f(x)\, dx$, and $a > 0$, show that

   (a) $L\{\sin ax\} = \dfrac{a^n}{(1+a^2)^{\frac{n+1}{2}}} \sin\left[n \tan^{-1}\left(\dfrac{1}{a}\right) + \tan^{-1}(-a)\right]$,

   (b) $L\{\cos ax\} = \dfrac{a^n}{(1+a^2)^{\frac{n+1}{2}}} \cos\left[n \tan^{-1}\left(\dfrac{1}{a}\right) + \tan^{-1}(-a)\right]$.

3. If $L\{f(x)\} = \tilde{f}_0(n) = \int\limits_0^\infty e^{-x} L_n(x) f(x)\, dx$, prove the following properties:

   (a) $L\{xf'(x)\} = -(n+1)\tilde{f}_0(n+1) + n\tilde{f}_0(n)$,

   (b) $L\left[e^x \dfrac{d}{dx}\{x\, e^{-x} f'(x)\}\right] = -n\tilde{f}_0(n)$,

   (c) $L\left[e^{-x} \dfrac{d}{dx}\{x\, e^{x} f'(x)\}\right] = n\tilde{f}_0(n) - 2(n+1)\tilde{f}_0(n+1)$,

   (d) $L\left[\dfrac{d}{dx}\{xf'(x)\}\right] = -(n+1)\tilde{f}_0(n+1)$.

4. Show that

   (a) $\tilde{f}_\alpha(n) = L\{L_n^\alpha(x)\} = \dfrac{\Gamma(n+\alpha+1)}{n!}$ for $\alpha > -1$.

   (b) $\tilde{f}_\alpha(n) = L\{xL_n^\alpha(x)\} = \dfrac{\Gamma(n+\alpha+1)}{n!}(2n+\alpha+1)$ for $\alpha > -1$.

# 17

# *Hermite Transforms*

> "We are servants rather than masters in mathematics."
>
> Charles Hermite

> "Success [in teaching] depends ... to a great extent upon the teacher's leading the student continually to some research. This however does not occur by chance ... but chiefly as follows ... through his arrangement of the material and emphasis, the teacher's presentation of lectures on a discipline lets the student discern leading ideas appropriately. In these ways, the fully conversant thinker logically advances from mature and previous research and attains new results or better foundations than exist. Next the teacher should not fail to designate boundaries not yet crossed by science and to point out some positions from which further advances would then be possible. A university teacher should also not deny the student a deeper insight into the progress of his own investigations, nor should he remain silent about his own past errors and disappointments."
>
> Karl Weierstrass

## 17.1 Introduction

In this chapter we introduce the Hermite transform with a kernel involving a Hermite polynomial and discuss its basic operational properties, including the convolution theorem. Debnath (1964) first introduced this transform and proved some of its basic operational properties. This chapter is based on papers by Debnath (1964, 1968) and Dimovski and Kalla (1988).

## 17.2 Definition of the Hermite Transform and Examples

Debnath (1964) defined the *Hermite transform* of a function $F(x)$ defined in $-\infty < x < \infty$ by the integral

$$H\{F(x)\} = f_H(n) = \int_{-\infty}^{\infty} \exp(-x^2) H_n(x)\, F(x) dx, \tag{17.2.1}$$

where $H_n(x)$ is the well-known *Hermite polynomial* of degree $n$.

The *inverse Hermite transform* is given by

$$H^{-1}\{f_H(n)\} = F(x) = \sum_{n=0}^{\infty} (\delta_n)^{-1} f_H(n)\, H_n(x), \tag{17.2.2}$$

where $\delta_n$ is given by

$$\delta_n = \sqrt{\pi}\, n!\, 2^n. \tag{17.2.3}$$

This follows from the expansion of any function $F(x)$ in the form

$$F(x) = \sum_{n=0}^{\infty} a_n\, H_n(x), \tag{17.2.4}$$

where the coefficients $a_n$ can be determined from the orthogonal relation of the Hermite polynomial $H_n(x)$ as

$$\int_{-\infty}^{\infty} \exp(-x^2) H_n(x) H_n(x) dx = \delta_{nm}\, \delta_n. \tag{17.2.5}$$

Multiplying (17.2.4) by $\exp(-x^2) H_m(x)$ and integrating over $(-\infty, \infty)$ and using (17.2.4), we obtain

$$a_n = \delta_n^{-1} f_H(n) \tag{17.2.6}$$

so that (17.2.2) follows immediately.

***Example 17.2.1***

If $F(x)$ is a polynomial of degree $m$, then

$$f_H(n) = 0 \qquad \text{for } n > m. \tag{17.2.7}$$

$\square$

***Example 17.2.2***
If $F(x)=H_m(x)$, then

$$H\{H_m(x)\} = \int_{-\infty}^{\infty} \exp(-x^2)H_n(x)H_m(x)dx = \delta_n\delta_{nm}. \tag{17.2.8}$$

□

***Example 17.2.3***
If

$$\exp(2\,xt - t^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,H_n(x) \tag{17.2.9}$$

is the generating function of $H_n(x)$, then

$$H\{\exp(2xt - t^2)\} = \sqrt{\pi}\sum_{n=0}^{\infty}(2t)^n, \qquad |t| < \frac{1}{2}. \tag{17.2.10}$$

We have, by definition,

$$\begin{aligned} H\{\exp(2xt - t^2)\} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{\infty} \exp(-x^2)H_n^2(x)ds \\ &= \sum_{n-0}^{\infty} \delta_n \frac{t^n}{n!} = \sqrt{\pi}\sum_{n=0}^{\infty}(2t)^n, \qquad |t| < \frac{1}{2}. \end{aligned}$$

□

***Example 17.2.4***
If $F(x)=H_m(x)H_p(x)$, then

$$H\{H_m(x)H_p(x)\} = \left\{ \begin{array}{ll} \dfrac{\sqrt{\pi}\,2^k\,m!\,n!\,p!}{(k-m)!(k-n)!(k-p)!}, & \begin{array}{c} m+n+p=2k, \\ k \geq m,\,n,\,p \end{array} \\ 0, & \text{otherwise} \end{array} \right\}. \tag{17.2.11}$$

This follows from a result proved by Bailey (1939). □

***Example 17.2.5***
If $F(x)=H_m^2(x)H_n(x)$, then

$$H\left\{{H_m}^2(x)H_n(x)\right\} = 2^m\delta_n\sum_{k=0}^{n}\binom{m}{k}\binom{n}{k}\binom{2k}{k}, \qquad \text{if } m > n. \tag{17.2.12}$$

Using a result proved by Feldheim (1938), (17.2.12) follows immediately. □

***Example 17.2.6***
If $F(x) = H_{n+p+q}(x)H_p(x)H_q(x)$, then

$$H\{F(x)\} = \delta_{n+p+q}. \tag{17.2.13}$$

We have, by definition,

$$H\{F(x)\} = \int_{-\infty}^{\infty} \exp(-x^2)H_{n+p+q}(x)H_p(x)H_q(x)\,dx = \delta_{n+p+q},$$

where a result due to Bailey (1939) is used and $\delta_n$ is given by (17.2.3). □

***Example 17.2.7***
If $F(x) = \exp(ax)$, then

$$H\{\exp(ax)\} = \sqrt{\pi}\sum a^n \exp\left(\frac{1}{4}a^2\right). \tag{17.2.14}$$

This result follows from the standard result

$$\int_{-\infty}^{\infty} \exp(-x^2 + 2bx)\; H_n(x)dx = \sqrt{\pi}(2b)^n \exp(b^2).$$

□

***Example 17.2.8***
If $|2z| < 1$, show that

$$H\{\exp(z^2)\sin(\sqrt{2}\,xz)\} = \left\{\begin{matrix} 0, & n \neq 2m+1 \\ \sqrt{\pi}\sum_{m=0}^{\infty}(-1)^m(2z)^{2m+1}, & n = 2m+1 \end{matrix}\right\}. \tag{17.2.15}$$

We have, by definition,

$$H\{\exp(z^2)\sin(\sqrt{2}\,xz)\} = \int_{-\infty}^{\infty} \exp(z^2 - x^2)H_n(x)\sin(\sqrt{2}\,xz)dx.$$

We use a result (see Erdélyi et al., 1954, vol. 2, p. 194)

$$\exp(z^2)\sin(\sqrt{2}\,xz) = \sum_{m=0}^{\infty}(-1)^m H_{2m+1}(x)\frac{z^{2m+1}}{(2m+1)!}, \tag{17.2.16}$$

to derive

$$H\{\exp(z^2)\sin(\sqrt{2}\,xz)\}$$
$$=\sum_{m=0}^{\infty}(-1)^m\frac{z^{2m+1}}{(2m+1)!}\int_{-\infty}^{\infty}\exp(-x^2)H_n(x)H_{2m+1}(x)dx$$
$$=\left\{\begin{matrix}\sqrt{\pi}\sum_{m=0}^{\infty}(-1)^m(2z)^{2m+1}, & n=2m+1\\ 0, & n\neq 2m+1\end{matrix}\right\}.$$

□

***Example 17.2.9***

$$H\left[(1-z^2)^{-\frac{1}{2}}\exp\left\{\frac{2xyz-(x^2+y^2)z^2}{(1-z^2)}\right\}\right]$$
$$=\sqrt{\pi}\sum_{m=0}^{\infty}z^m\,H_m(y)\,\delta_{mn}. \tag{17.2.17}$$

We use a result (see Erdélyi et al., 1954, vol. 2, p. 194)

$$(1-z^2)^{-\frac{1}{2}}\exp\left\{\frac{2xyz-(x^2+y^2)z^2}{(1-z^2)}\right\}=\sum_{m=0}^{\infty}\left(\frac{1}{2}\,z\right)^m\frac{1}{m!}\,H_m(x)H_m(y)$$

to derive

$$H\left[(1-z^2)^{-\frac{1}{2}}\exp\left\{\frac{2xyz-(x^2+y^2)z^2}{(1-z^2)}\right\}\right]$$
$$=\sum_{m=0}^{\infty}\left(\frac{1}{2}\,z\right)^m\frac{1}{m!}\,H_m(y)\int_{-\infty}^{\infty}\exp(-x^2)\,H_n(x)\,H_m(x)\,dx$$
$$=\sum_{m=0}^{\infty}\left(\frac{1}{2}\,z\right)^m\frac{1}{m!}\,H_m(y)\,\delta_m\,\delta_{mn}\;=\sqrt{\pi}\sum_{m=0}^{\infty}z^m\,H_m(y)\,\delta_{mn}.$$

□

## 17.3 Basic Operational Properties

***THEOREM 17.3.1***

If $F'(x)$ is continuous and $F''(x)$ is bounded and locally integrable in the interval $-\infty < x < \infty$, and if $H\{F(x)\} = f_H(n)$, then

$$H\{R[F(x)]\} = -2n\, f_H(n), \tag{17.3.1}$$

where $R[F(x)]$ is the differential form given by

$$R[F(x)] = \exp(x^2)\frac{d}{dx}\left[\exp(-x^2)\frac{dF}{dx}\right]. \tag{17.3.2}$$

**PROOF** We have, by definition,

$$H\{R[F(x)]\} = \int_{-\infty}^{\infty} \frac{d}{dx}\left[\exp(-x^2)\,\frac{dF}{dx}\right] H_n(x)dx$$

which is, by integrating by parts and using the orthogonal relation (17.2.8),

$$= -2n \int_{-\infty}^{\infty} \exp(-x^2)\, H_n(x)\, F(x)dx \;=\; -2n\, f_H(n).$$

Thus, the theorem is proved.

If $F(x)$ and $R[F(x)]$ satisfy the conditions of Theorem 17.3.1, then

$$H\{R^2[F(x)]\} = H\{R[R[F(x)]]\} = (-1)^2(2n)^2 f_H(n). \tag{17.3.3}$$

$$H\{R^3[F(x)]\} = (-1)^3(2n)^3 f_H(n). \tag{17.3.4}$$

More generally,

$$H\{R^m[F(x)]\} = (-1)^m(2n)^m f_H(n), \tag{17.3.5}$$

where $m = 1, 2, \ldots, m-1$. ∎

***THEOREM 17.3.2***

If $F(x)$ is bounded and locally integrable in $-\infty < x < \infty$, and $f_H(0) = 0$, then $H\{F(x)\} = f_H(n)$ exists and for each constant $C$,

$$H^{-1}\left\{-\frac{f_H(n)}{2n}\right\} = R^{-1}[F(x)]$$

$$= \int_0^x \exp(s^2) \int_{-\infty}^{s} \exp(-t^2)\, F(t)dt\, ds + C, \tag{17.3.6}$$

where $R^{-1}$ is the inverse of the differential operator $R$ and $n$ is a positive integer.

**PROOF** We write

$$R^{-1}[F(x)] = Y(x)$$

so that $Y(x)$ is a solution of the differential equation

$$R[Y(x)] = F(x). \tag{17.3.7}$$

Since $f_H(0) = 0$, and $H_0(x) = 1$, then

$$\int_{-\infty}^{\infty} \exp(-x^2)F(x)dx = 0.$$

The first integral of (17.3.7) is

$$\exp(-x^2)Y'(x) = \int_{-\infty}^{x} \exp(-t^2)F(t)dt,$$

which is a continuous function of $x$ and tends to zero as $|x| \to \infty$. The second integral

$$Y(x) = \int_{0}^{x} \exp(s^2) \int_{-\infty}^{s} \exp(-t^2)\, F(t)dt\, ds + C,$$

where $C$ is an arbitrary constant, is also continuous. Evidently,

$$\lim_{|x|\to\infty} \exp(-x^2)Y(x) = 0$$

provided $Y(x)$ is bounded.

Then $H\{Y(x)\}$ exists and

$$H\{R[Y(x)]\} = -2n\, H\{Y(x)\}.$$

Or,

$$H[F(x)] = -2n\, H\{Y(x)\}.$$

Hence,

$$f_H(n) = -2n\, H\{R^{-1}[F(x)]\}.$$

Thus, for any positive integer $n$,

$$H\{R^{-1}[F(x)]\} = -\frac{f_H(n)}{2n}.$$

∎

***THEOREM 17.3.3***

If $F(x)$ has bounded derivatives of order $m$ and if $H\{F(x)\} = f_H(n)$ exists, then

$$H\{F^{(m)}(x)\} = f_H(n+m). \tag{17.3.8}$$

**PROOF** We have, by definition,

$$H\{F'(x)\} = \int_{-\infty}^{\infty} \exp(-x^2) H_n(x) F'(x)\, dx,$$

which is, by integrating by parts,

$$= [\exp(-x^2)F(x)H_n(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} F(x)\frac{d}{dx}\left[e^{-x^2}\, H_n(x)\right] dx$$

$$= 2\int_{-\infty}^{\infty} x \exp(-x^2) H_n(x) F(x) - \int_{-\infty}^{\infty} F(x) \exp(-x^2) H_n'(x) dx. \quad (17.3.9)$$

We use *recurrence relations* (A-6.5)–(A-6.6) for the Hermite polynomial to rewrite (17.3.9) in the form

$$H\{F'(x)\} = \int_{-\infty}^{\infty} \exp(-x^2)[H_{n+1}(x) + 2nH_{n-1}(x)]F(x)dx$$

$$-2n\int_{-\infty}^{\infty} \exp(-x^2) H_{n-1}(x) F(x) dx$$

$$= \int_{-\infty}^{\infty} \exp(-x^2) H_{n+1}(x) F(x) dx = f_H(n+1).$$

Proceeding in a similar manner, we can prove

$$H\{F^{(m)}(x)\} = f_H(n+m).$$

Thus, the theorem is proved. ▮

***THEOREM 17.3.4***

If the Hermite transforms of $F(x)$ and $xF^{(m-1)}(x)$ exist, then

$$H\{xF^{(m)}(x)\} = nf_H(m+n-1) + \frac{1}{2} f_H(m+n+1). \quad (17.3.10)$$

**PROOF** We have, by definition,

$$H\{xF^{(m)}(x)\} = \int_{-\infty}^{\infty} \exp(-x^2) H_n(x) \left\{x \frac{d^m F(x)}{dx^m}\right\} dx$$

$$= \left[x \exp(-x^2) H_n(x) F^{(m-1)}(x)\right]_{-\infty}^{\infty}$$

$$- \int_{-\infty}^{\infty} \frac{d}{dx}\left[x \exp(-x^2) H_n(x)\right] F^{(m-1)}(x) dx.$$

Thus,

$$H\{xF^{(m)}(x)\} = \int_{-\infty}^{\infty} 2x^2 \exp(-x^2) H_n(x) F^{(m-1)}(x)dx$$
$$- \int_{-\infty}^{\infty} \exp(-x^2) H_n(x) F^{(m-1)}(x)dx$$
$$-n \int_{-\infty}^{\infty} 2x \exp(-x^2) H_{n-1}(x) F^{(m-1)}(x)dx,$$

which is, by the recurrence relations (17.3.10)–(17.3.11), and (17.3.8),

$$= \int_{-\infty}^{\infty} x \exp(-x^2)[H_{n+1}(x) + 2nH_{n-1}(x)]F^{(m-1)}(x)dx$$
$$- n \int_{-\infty}^{\infty} \exp(-x^2)[H_n(x) + 2(n-1)H_{n-2}(x)]F^{(m-1)}(x)dx$$
$$- f_H(n+m+1)$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} \exp(-x^2)[H_{n+2}(x) + 2(n+1)H_n(x)]F^{(m-1)}(x)dx$$
$$+ n \int_{-\infty}^{\infty} \exp(-x^2)[H_n(x) + 2(n-1)H_{n-2}(x)]F^{(m-1)}(x)dx$$
$$- nf_H(n+m-1) - 2n(n-1)f_H(n+m-3) - f_H(n+m+1)$$
$$= \frac{1}{2} f_H(n+m+1) + (n+1)f_H(n+m-1)$$
$$+ n[f_H(n+m-1) + 2(n-1)f_H(n+m-3)]$$
$$- nf_H(n+m-1) - 2n(n-1)f_H(n+m-3) - f_H(n+m+1)$$
$$= nf_H(n+m-1) + \frac{1}{2} f_H(n+m+1).$$

In particular, when $m=1$ and $m=2$, we obtain

$$H\{xF'(x)\} = nf_H(n) + \frac{1}{2} f_H(n+2), \tag{17.3.11}$$

$$H\{xF''(x)\} = nf_H(n+1) + \frac{1}{2} f_H(n+3). \tag{17.3.12}$$

The reader is referred to a paper by Debnath (1968) for other results similar to those of (17.3.11)–(17.3.12). ∎

***DEFINITION 17.3.1*** *(Generalized Convolution). The generalized convolution of $F(x)$ and $G(x)$ for the Hermite transform defined by*

$$H\{F(x)*G(x)\} = \mu_n H\{F(x)\}H\{G(x)\} = \mu_n\, f_H(n)g_H(n), \tag{17.3.13}$$

*where $\mu_n$ is a non-zero quantity given by*

$$\mu_n = \sqrt{\pi}(-1)^n \left\{2^{2n+1}\,\Gamma\left(n+\frac{3}{2}\right)\right\}^{-1}. \tag{17.3.14}$$

Debnath (1968) first proved the convolution theorem of the Hermite transform for odd functions. However, Dimovski and Kalla (1988) extended the theorem for both odd and even functions. We follow Dimovski and Kalla to state and prove the convolution theorem of the Hermite transform. Before we discuss the theorem, it is observed that, if $F(x)$ is an odd function, then

$$H\{F(x);\ 2n\} = f_H(2n) = \int_{-\infty}^{\infty} \exp(-x^2)H_{2n}(x)F(x)dx = 0, \tag{17.3.15}$$

but

$$H\{F(x);\ 2n+1\} = f_H(2n+1) \neq 0. \tag{17.3.16}$$

On the other hand, if $F(x)$ is an even function, then

$$H\{F(x);\ 2n+1\} = f_H(2n+1) = 0, \tag{17.3.17}$$

but

$$H\{F(x);\ 2n\} = f_H(2n) \neq 0. \tag{17.3.18}$$

***THEOREM 17.3.5***

(*Convolution of the Hermite Transform for Odd Functions*). If $F(x)$ and $G(x)$ are odd functions and $n$ is an odd positive integer, then

$$H\left\{F(x)\overset{\circ}{*}G(x);\ 2n+1\right\} = \mu_n f_H(2n+1)g_H(2n+1), \tag{17.3.19}$$

where $\overset{\circ}{*}$ denotes the convolution operation for odd functions and is given by

$$F(x)\overset{\circ}{*}G(x) = \frac{x}{\pi}\int_{-\infty}^{\infty} \exp(-t^2)t\ F(t)dt \int_0^{\pi} \exp(-xt\ \cos\,\phi)\sin\phi$$
$$\times \int_0^{\pi} \frac{G[(x^2+t^2+2xt\cos\phi)]^{\frac{1}{2}}}{(x^2+t^2+2xt\cos\phi)^{\frac{1}{2}}}\, J_0(xt\sin\phi)\, d\phi, \tag{17.3.20}$$

and $J_0(z)$ is the Bessel function of the first kind of order zero.

**PROOF** We have, by definition,

$$f_H(2n+1) = \int_{-\infty}^{\infty} \exp(-x^2)\, H_{2n+1}(x)\, F(x)\, dx. \tag{17.3.21}$$

We replace $H_{2n+1}(x)$ by using a result for Erdélyi (1953, vol. 2, p. 1993)

$$H_{2n+1}(x) = (-1)^n\, 2^{2n+1}\, n!\, x\, L_n^{\frac{1}{2}}(x^2), \tag{17.3.22}$$

where $L_n^{\alpha}(x)$ is the Laguerre polynomial of degree $n$ and order $\alpha$ so that (17.3.21) reduces to the form

$$f_H(2n+1) = (-1)^n 2^{2n+2}\, n! \int_0^{\infty} x\, \exp(-x^2) L_n^{\frac{1}{2}}(x^2)\, F(x)\, dx. \tag{17.3.23}$$

Invoking the change of variable $x^2 = t$, we obtain

$$H\{F(x);\, 2n+1\} = (-1)^n 2^{2n+1} n! \int_0^{\infty} \sqrt{t}\, \exp(-t) L_n^{\frac{1}{2}}(t)\, \frac{F(\sqrt{t})}{\sqrt{t}}\, dt. \tag{17.3.24}$$

It is convenient to introduce the transformation $T$ by

$$(T\ F)(t) = \frac{F(\sqrt{t})}{\sqrt{t}}, \quad 0 \le t < \infty \tag{17.3.25}$$

so that the inverse of $T$ is given by

$$T^{-1}(\Phi)(x) = x\, \Phi(x^2). \tag{17.3.26}$$

Consequently, (17.3.24) takes the form

$$H\{F(x);\ 2n+1\} = (-1)^n\, 2^{2n+1}\, n!\, L\{T\ F(x)\}, \tag{17.3.27}$$

where $L$ is the Laguerre transformation of degree $n$ and order $\alpha = \frac{1}{2}$ defined by (16.2.1) in Chapter 16.

The use of (17.3.27) allows us to write the product of two Hermite transforms as the product of two Laguerre transforms as

$$f_H(2n+1) g_H(2n+1) = 2^{4n+2} (n!)^2 L\{T\ F(x)\}\, L\{T\ G(x)\}. \tag{17.3.28}$$

We now apply the Convolution Theorem for the Laguerre transform (when $\alpha = 0$) proved by Debnath (1969) in the form

$$L\{F \tilde{*} G(x)\} = \frac{n!\sqrt{\pi}}{\Gamma\left(n + \frac{3}{2}\right)}\ L\{F(x)\}\ L\{G(x)\}, \tag{17.3.29}$$

where $F \tilde{*} G$ is given by

$$F \tilde{*} G(x) = \int_0^{\infty} \exp(-\tau)\sqrt{\tau}\, F(\tau) d\tau \int_0^{\pi} \exp(-\sqrt{t\tau}\, \cos\phi) \sin\phi$$
$$\times G(t+\tau+2\sqrt{t\tau}\, \cos\phi) J_0(\sqrt{t\tau}\, \sin\phi)\, d\phi. \quad (17.3.30)$$

Substituting (17.3.29) into (17.3.28), we obtain

$$f_H(2n+1) g_H(2n+1) = \pi^{-\frac{1}{2}} 2^{4n+2} n!\, \Gamma\left(n+\frac{3}{2}\right) L\{T\ F \tilde{*} T\ G\},$$

which is, by (17.3.27),

$$= \frac{2^{2n+1}\Gamma\left(n+\frac{3}{2}\right)}{(-1)^n\sqrt{\pi}} H\{T^{-1}\,(T\ F \tilde{*} T\ G)\}. \quad (17.3.31)$$

Or, equivalently,

$$H\left\{F \overset{\circ}{*} G(x);\ 2n+1\right\} = \mu_n\ H\{F(x)\} H\{G(x)\}, \quad (17.3.32)$$

where

$$F \overset{\circ}{*} G(x) = T^{-1}\{T\ F \overset{\circ}{*} T\ G(x)\}. \quad (17.3.33)$$

This coincides with (17.3.20). Thus, the proof is complete. ∎

***THEOREM 17.3.6***

(*Convolution of the Hermite Transform for Even Functions*). If $F(x)$ and $G(x)$ are even functions and $n$ is an even positive integer, then

$$H\left\{F(x) \overset{e}{*} G(x);\ 2n\right\} = \mu_n H\{F(x);\ 2n\} H\{G(x);\ 2n\}. \quad (17.3.34)$$

**PROOF** We use result (17.3.8), that is,

$$H\{F'(x); n\} = H\{F(x);\ n+1\}$$

so that

$$H\{I\ F(x);\ 2n+1\} = H\{F(x),\, 2n\}, \quad (17.3.35)$$

where

$$I\, F(x) = \int_0^x F(t) dt \quad \text{and} \quad [I\, F(x)]' = F(x).$$

Obviously,

$$\begin{aligned}
H\Big\{F(x) \overset{e}{*} G(x);\ 2n\Big\} &= H\Big\{\big[I\,F(x) \overset{e}{*} I\,G(x)\big]';\ 2n\Big\} \\
&= H\big\{I\,F(x) \overset{\circ}{*} I\,G(x);\ 2n+1\big\} \\
&= \mu_n\,H\{I\,F(x);\ 2n+1\}H\{I\,G(x);\ 2n+1\} \\
&= \mu_n\,H\{F(x);\ 2n\}\,H\{G(x);\ 2n\}.
\end{aligned}$$

This proves the theorem. ∎

***THEOREM 17.3.7***

If $F(x)$ and $G(x)$ are two arbitrary functions such that their Hermite transforms exist, then

$$H\{F(x) * G(x);\ n\} = \mu_{[n/2]} H\{F(x);\ n\}\,H\{G(x);\ n\}, \tag{17.3.36}$$

where

$$F(x) * G(x) = F_0(x) \overset{\circ}{*} G_0(x) + F_e(x) \overset{e}{*} G_e(x), \tag{17.3.37}$$

and

$$F_0(x) = \frac{1}{2}[F(x) - F(-x)] \quad \text{and} \quad F_e(x) = \frac{1}{2}[F(x) + F(-x)]. \tag{17.3.38}$$

**PROOF** We first note that arbitrary functions $F(x)$ and $G(x)$ can be expressed as sums of even and odd functions, that is, $F(x) = F_0(x) + F_e(x)$ and $G(x) = G_0(x) + G_e(x)$ so that result (17.3.38) follows.

Suppose $n$ is odd. Then

$$H\{F(x);\ n\} = H\{F_0(x);\ n\},\ \ H\{G(x);\ n\} = H\{G_0(x);\ n\},$$

and

$$H\{F(x) + G(x);\ n\} = H\{F_0(x) + G_0(x);\ n\}.$$

Clearly,

$$\begin{aligned}
&H\{F(x) * G(x);\ 2n+1\} \\
&= H\ \{F_0(x) \overset{\circ}{*} G_0(x);\ 2n+1\} + H\ \{F_e(x) \overset{e}{*} G_e(x);\ 2n+1\} \\
&= \mu_n\,H\{F_0(x)\}H\{G_0(x)\}\ =\ \mu_n\,H\{F(x)\}H\{G(x)\}.
\end{aligned}$$

Similarly, the case for even $n$ can be handled without any difficulty.

We conclude this chapter by citing some recent work on the generalized Hermite transformation by Glaeske (1983, 1986, 1987). These papers include some interesting discussion on operational properties and convolution structure of the generalized Hermite transformations. For more details, the reader is referred to these papers. ∎

## 17.4 Exercises

1. Find the Hermite transform of the following functions:

   (a) $\exp(-x^2)H_n(x)$, (b) $x^m$, (c) $x^2\, H_n(x)$.

2. Show that
$$H\{x^n\} = \sqrt{\pi}\, n! P_n(1),$$
   where $P_n(x)$ is the Legendre polynomial.

3. Show that
$$H\left\{H_n^2(x)\right\} = \sqrt{\pi} \sum_{r=0}^{n} \binom{n}{r} 2^{r+n}\, (2r)!\, n!\,.$$

# 18

## *The Radon Transform and Its Applications*

> "This struck me as a typical nineteenth century piece of mathematics which a Cauchy or a Riemann might have dashed off in a light moment, but a diligent search of standard texts on analysis failed to reveal it, so I had to solve the problem myself. I still felt that the problem must have been solved, so I contacted mathematicians on three continents to see if they knew about it, but to no avail."
>
> Allan MacLeod Cormack

> "We live in an age in which mathematics plays a more and more important role, to the extent that it is hard to think of an aspect of human life to which it either has not provided, or does not have the potential to provide, crucial insights. Mathematics is the language in which quantitative models of the world around us are described. As subjects become more understood, they become more mathematical. A good example is medicine, where the Radon transform is what makes X-ray tomography work, where statistics form the basis of evaluating the success or failure of treatments, and where mathematical models of organs such as the heart, of tumor growth, and of nerve impulses are of key importance."
>
> John Ball

### 18.1 Introduction

The origin of the *Radon transform* can be traced to *Johann Radon*'s 1917 celebrated work "On the determination of functions from integrals along certain manifolds." In his seminal work, Radon demonstrated how to construct a function of two variables from its integrals over all straight lines in the plane. He also made other generalizations of this transform involving the reconstruction of a function from its integrals over other smooth curves as well as the reconstruction of a function of $n$ variables from its integrals over all hyperplanes. Although the Radon transform had some direct ramifications on solutions of hyperbolic partial differential equations with constant coefficients, it did not receive much attention from mathematicians and scientists.

In the 1960s, the Radon transform played a major role in tomography which is widely used method to reconstruct cross-sections of the interior structure of an object without having to cut or damage the object. Through the interaction of a physical organ or "probe" — varying from X-rays, gamma rays, visible light, electrons, or neutrons to ultrasound waves — with the object, we usually obtain line or (hyper) plane integrals of the internal distribution to be reconstructed. There is a close relation between the Radon transform and the development of X-ray scans (or *CAT scans*) in medical imaging. In practice, X-ray scans provide a picture of an internal organ of a human or animal body, and hence help detect and locate many types of abnormalities. Thus, one of the most prominent examples of applications of computer assisted tomography occurs in diagnostic medicine, where the method is employed to generate images of the interior of human organs. The central problem of reconstruction and the introduction of new algorithms and faster electronic computers led to a rapid development of computerized tomography.

More than fifty years later, *Allan Cormack*, a young South African physicist, became interested in finding a set of maps of absorption coefficients for different sections of the human body. In order to make X-ray radiotherapy more effective, he quickly recognized the importance of the Radon transform which is similar to measurements of the absorption of X-rays along lines in their sections of the human body. Since the logarithm of the ratio of incident to reflected X-ray intensities along a given straight line is just the line integral of the absorption coefficient along that line, the problem is mathematically equivalent to finding a function from the values of its integrals along all or some lines in the plane. As early as 1963, Cormack already obtained three alternative solutions of this major problem. At the same time, *Godfrey Hounsfield*, a young British biomedical engineer, realized the unique importance of the major ideas of Radon and Cormack and then used them to develop a new X-ray machine that totally revolutionized the field of medical imaging. Soon after that Cormack and Hounsfield joined together to work on the refinement of the solution of medical imaging. Their joint work led to the major discovery of the CT-scanning technique and then culminated in winning the 1979 Nobel Prize in Physiology and Medicine. In their Nobel Prize addresses Cormack and Hounsfield acknowledged the pioneering work of Radon in 1917.

The Radon transform is found to be very useful in many diverse fields of science and engineering including medical imaging, astronomy, crystallography, electron microscopy, geophysics, material science, and optics. It is important to mention that the Radon transform has been used in *computer assisted tomography* (CAT) heavily. The problem of determining internal structure of an object by observations or projections is closely associated with the Radon transform. In this chapter, we introduce the Radon transform, its basic properties, its inverse and the relationship between the Radon transform and Fourier transform. Included are Parseval theorems and applications of the Radon transform.

## 18.2 The Radon Transform

***DEFINITION 18.2.1*** *(The Radon Transform). If L is any straight line in the $x$-$y$ plane (or, in $\mathbb{R}^2$) and ds is the arc length along L (see Figure 18.1),the Radon transform of a function $f(x,y)$ of two real variables is defined by its integral along L as*

$$\hat{f}(p,\phi) = \mathscr{R}\{f(x,y)\} = \int_L f(x,y)\,ds. \tag{18.2.1}$$

In other words, the totality of all these line integrals constitutes the Radon transform of $f(x,y)$ and each line integral is called a sample of the Radon transform of $f(x,y)$. Thus, the Radon transform $\hat{f}$ of $f$ can be viewed as a function defined on all straight lines in the plane and the value of $\hat{f}(p,\phi)$ at a given $L$ is the integral of $f(x,y)$ over that line.

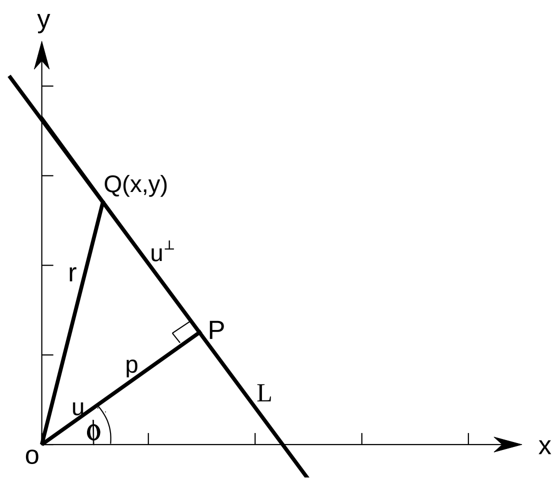


**Figure 18.1** Graph of the line $L$.

Making references to Figure 18.1, we write the equation of the line $L$ in the form $p = x\cos\phi + y\sin\phi$, where $p$ is the length of the perpendicular from the origin to $L$ and $\phi$ is the angle that the perpendicular makes with the positive $x$-axis. If we rotate the coordinate system by an angle $\phi$, and label the new axes by $p$ and $s$, then $x = p\cos\phi - s\sin\phi$, $y = p\sin\phi + s\cos\phi$. Consequently, the Radon transform (18.2.1) can be defined by

$$\mathscr{R}\{f(x,y)\} = \hat{f}(p,\phi) = \int_{-\infty}^{\infty} f\left(p\cos\phi - s\sin\phi,\ p\sin\phi + s\cos\phi\right)\, ds. \tag{18.2.2}$$

This definition is very practical in two dimensions. However, it does not lend itself readily to higher dimensions.

In order to generalize the above definition in higher dimensions, we introduce the unit vectors $\mathbf{u} = (\cos\phi, \sin\phi)$ and $\mathbf{u}^{\perp} = (-\sin\phi, \cos\phi)$, so that $\mathbf{x} = (x, y) = (r, \theta) = p\,\mathbf{u} + t\,\mathbf{u}^{\perp}$ for some scalar parameter $t$, where $r$ and $\theta$ are the usual polar coordinates. The equation of the line $L$ can now be written in terms of the unit vector $\mathbf{u}$ as $p = \mathbf{x}.\mathbf{u} = x\cos\phi + y\sin\phi$. Using the definition of Dirac delta function, we express (18.2.1) in the form

$$\hat{f}(p,\phi) = \int_{-\infty}^{\infty} f\left(p\,\mathbf{u} + t\,\mathbf{u}^{\perp}\right)\, dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\,\delta(p - \mathbf{x}.\mathbf{u})\, d\mathbf{x}. \tag{18.2.3}$$

It is noted that the integral is taken over a line orthogonal to the line $\theta = \phi$ and that $\hat{f}(-p,\phi) = \hat{f}(p,\phi+\pi)$ so that negative values for $p$ can be assigned and $\phi$ may be restricted to $[0, \pi]$.

***DEFINITION 18.2.2*** *(The Radon Transform in Higher Dimensions). In $n$ dimensional Euclidean space $\mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, ...., x_n)$ and $f(\mathbf{x}) = f(x_1, x_2, ...., x_n)$. Let us introduce a unit vector $\mathbf{u} = (u_1, u_2, ..., u_n)$ in $\mathbb{R}^n$ that defines the orientation of a hyperplane with the equation*

$$p = \mathbf{x}.\mathbf{u} = x_1u_1 + ...... + x_nu_n. \tag{18.2.4}$$

*Then the Radon transform of a function $f(\mathbf{x})$ is defined by*

$$\hat{f}(p,\mathbf{u}) = \mathscr{R}\{f(\mathbf{x})\}(p,\mathbf{u}) = \int_{-\infty}^{\infty} f(\mathbf{x})\,\delta(p - \mathbf{x}.\mathbf{u})\, d\mathbf{x}, \tag{18.2.5}$$

*where the integration is taken over $d\mathbf{x} = dx_1dx_2...dx_n$.*

Therefore, the Radon transform of a function of $n$ variables is the totality of all integrals of $f$ over all hyperplanes in $\mathbb{R}^n$. In other words, the Radon transform $\hat{f}(p,\mathbf{u})$ of $f(\mathbf{x})$ is a function defined on all hyperplanes in $\mathbb{R}^n$, and the value of that function at any hyperplane is the integral of $f(\mathbf{x})$ over that hyperplane. The inverse Radon transform is equivalent to finding $f(\mathbf{x})$ from the values of its integrals over all hyperplanes.

The Radon transform has been further generalized to use this idea in the field of medical imaging, where the integral along a line represents a measurement of the intensity of the X-ray beam at the detector after passing through the object to be radiographed. This is essentially the idea of the X-ray transform where a two-dimensional object lying in the plane $\mathbb{R}^2$ which may be considered as a planer cross section of a human or animal organ. After sending the X-ray beam through the object along a line $L$, we calculate the integral in (18.2.1), which is also called the *X-ray transform* (or the Radon transform). Thus, this transform assigns to each suitable function $f$ on $\mathbb{R}^2$ another unique function $\hat{f} = \mathscr{R}\{f\}$ which domain is the set of lines in $\mathbb{R}^2$. From a practical point of view, the major interest lies essentially in internal structure of the object, and hence, the central problem is to reconstruct $f$ (or, to find the inverse) from the given $\hat{f}$. This is called the reconstruction problem which has a definite solution based on the general mathematical theory. However, in practice, this problem can be solved using sampling procedures, numerical approximations, or computer algorithms.

To explain this idea, we observe that in $\mathbb{R}^2$, if a function $f(x, y)$ is integrated over a line $L_\theta$ with direction $\theta$, and then if the line is moved parallel to itself, we obtain a function $L_\theta\, f = P^1\, f$ defined on a line $L_{\theta^\perp}$ orthogonal to $L_\theta$. The value of this function at any point $\mathbf{x}$ on $L_{\theta^\perp}$ is equal to the integral of $f(\mathbf{x})$ over the line with direction $\theta$ which intersects $L_{\theta^\perp}$ at the point $\mathbf{x}$.

Similarly, in $\mathbb{R}^3$, if a function $f(\mathbf{x})$ is integrated over a plane $P^2$, and then if the plane is moved parallel to itself, we obtain a function $P^2\, f$ defined on a straight line orthogonal to the plane $P^2$. The value of this function at any point $\mathbf{x}$ on this line is equal to the integral over the plane passing through $\mathbf{x}$ and parallel to $P^2$. If $f(\mathbf{x})$ represents the density at $\mathbf{x}$, then an X-ray taken in the $\theta$ direction generates a function $L_\theta\, f = P^1\, f$ defined on the plane orthogonal to $\theta$. The value of $(L_\theta\, f)$ at the point $\mathbf{x}$ on this plane is equal to the integral of $f$ along a line through $\mathbf{x}$ in the direction $\theta$, that is,

$$[L_\theta\, f]\,(\mathbf{x}) = \int_{-\infty}^{\infty} f\,(\mathbf{x} + t\,\theta)\; dt, \tag{18.2.6}$$

where $\mathbf{x} \in \theta^\perp$ that represents the plane orthogonal to $\theta$.

***DEFINITION 18.2.3*** *(The k-Plane Transform). If $\Pi$ is a k-dimensional subspace of $\mathbb{R}^n$ determined by the direction $\theta$, then the k-plane transform of $f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$ in the direction $\theta$ at the point $\boldsymbol{\xi} \in \Pi^\perp$ is defined by*

$$\left(P^k\, f\right)(\Pi,\, \boldsymbol{\xi}) = \int_\Pi f(\boldsymbol{\xi},\, \boldsymbol{\eta})d\boldsymbol{\eta}, \tag{18.2.7}$$

*where $\mathbf{x} = (\boldsymbol{\xi},\, \boldsymbol{\eta})$, $\boldsymbol{\eta} \in \Pi$, and $\boldsymbol{\xi} \in \Pi^\perp$ .*

*Or, equivalently,*

$$\left(P^k\, f\right)(\theta,\, \boldsymbol{\xi}) = \int f(\boldsymbol{\xi} + \mathbf{u}.\theta)d\mathbf{u}, \tag{18.2.8}$$

*where* $\mathbf{u}.\theta = (u_1\,\theta_1 + u_2\,\theta_k + ... + u_k\,\theta_k)$ *and* $\boldsymbol{\xi} \in \theta^{\perp}$.

The 1-plane transform $\left(P^1\, f\right)$ is called the *X-ray transform* and $(n-1)$-plane transform is the Radon transform. The $k$-plane transformation is a linear transformation.

***Example 18.2.1***
Show that

$$\mathscr{R}\left[\exp\left\{-a^2(x^2+y^2)\right\}\right] = \frac{\sqrt{\pi}}{a}\exp\left(-a^2p^2\right), \qquad a > 0. \tag{18.2.9}$$

Using $m = u_1x + u_2y$ , $n = -u_2x + u_1y$, we have

$$\begin{aligned} x^2 + y^2 &= m^2 + n^2 \\ f(m,n) &= \exp\left\{-a^2(m^2+n^2)\right\}. \end{aligned}$$

So it turns out that

$$\begin{aligned} \hat{f}(p,\mathbf{u}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-a^2\left(m^2+n^2\right)\right\}\delta(p-m)\,dm\,dn \\ &= \int_{-\infty}^{\infty} \exp(-a^2m^2)\,\delta(m-p)dm \int_{-\infty}^{\infty} \exp(-a^2n^2)dn \\ &= \frac{\sqrt{\pi}}{a}\exp(-a^2p^2). \end{aligned}$$

In other words,

$$\mathscr{R}\left[\exp\left\{-a^2(x^2+y^2)\right\}\right] = \frac{\sqrt{\pi}}{a}\exp\left(-a^2p^2\right).$$

Alternatively, we can use (18.2.2) to obtain

$$\hat{f}(p,\,\phi) = \int_{-\infty}^{\infty} \exp\left[-a(p^2+s^2)\right] ds = e^{-ap^2}\int_{-\infty}^{\infty} e^{-as^2}\,ds = \sqrt{\frac{\pi}{a}}\,e^{-ap^2}.$$

When $a = 1$, the above result yields

$$\mathscr{R}\left[\exp(-x^2-y^2)\right] = \sqrt{\pi}\,e^{-p^2}. \tag{18.2.10}$$

□

***Example 18.2.2***
Find the Radon transform of the following functions:

(a) $f(x,y) = x\,\exp\left[-a(x^2+y^2)\right], \quad a > 0,$
(b) $g(x,y) = y\,\exp\left[-a(x^2+y^2)\right], \quad a > 0.$

(a) We have, by definition (18.2.2),

$$\hat{f}(p,\,\phi) = \int_{-\infty}^{\infty} (p\cos\phi - s\sin\phi)\exp\left[-a(p^2+s^2)\right] ds$$
$$= \sqrt{\frac{\pi}{a}}\, p\, \cos\phi\, \exp(-ap^2). \tag{18.2.11}$$

(b) Similarly,

$$\hat{g}(p,\,\phi) = \int_{-\infty}^{\infty} (p\cos\phi + s\sin\phi)\exp\left[-a(p^2+s^2)\right] ds$$
$$= \sqrt{\frac{\pi}{a}}\, p\, \sin\phi\, \exp(-ap^2). \tag{18.2.12}$$

Combining these two results gives

$$\mathscr{R}\left\{(x+y)\,\exp\left[-a(x^2+y^2)\right]\right\} = \hat{f}(p,\,\phi) + i\,\hat{g}(p,\,\phi) = \sqrt{\frac{\pi}{a}}\, p\, e^{i\phi}\,\exp(-ap^2). \tag{18.2.13}$$

▯

## 18.3 Properties of the Radon Transform

(*Relation between the Fourier Transform and the Radon Transform*).
Consider two-dimensional Fourier transform defined by

$$\tilde{f}(\mathbf{k}) = \mathscr{F}\{f(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\mathbf{k}.\mathbf{x})} f(x,y)dxdy \tag{18.3.1}$$

where $\mathbf{k} = (k, l)$ and $\mathbf{x} = (x, y)$.

The kernel $\exp[-i(\mathbf{k}.\mathbf{x})]$ of the Fourier transform can be written as

$$e^{-i(\mathbf{k}.\mathbf{x})} = \int_{-\infty}^{\infty} e^{-it}\delta(t - \mathbf{k}.\mathbf{x})dt \tag{18.3.2}$$

so that (18.3.1) becomes

$$\tilde{f}(\mathbf{k}) = \int_{-\infty}^{\infty} e^{-it}dt \int_{-\infty}^{\infty} f(\mathbf{x})\,\delta(t - \mathbf{k}.\mathbf{x})\,d\mathbf{x}. \tag{18.3.3}$$

Substituting $\mathbf{k} = s\mathbf{u}$ and $t = sp$ in (18.3.3) where $s$ is real and $\mathbf{u}$ is a unit vector yields

$$\tilde{f}(s\mathbf{u}) = \int_{-\infty}^{\infty} e^{-isp} dp \int_{-\infty}^{\infty} f(\mathbf{x})\, \delta(p - \mathbf{u}.\mathbf{x}) d\mathbf{x}$$
$$= \int_{-\infty}^{\infty} e^{-isp} \hat{f}(p, \mathbf{u}) dp$$
$$= \mathscr{F}\left\{\hat{f}(p, \mathbf{u})\right\}. \tag{18.3.4}$$

In other words,

$$\tilde{f} = \mathscr{F}(\mathscr{R} f).$$

This means that $\tilde{f}$ is the two-dimensional Fourier transform of $f$, whereas $\mathscr{F}(\mathscr{R} f)$ is the one-dimensional Fourier transform of $\mathscr{R} f$. Thus,

$$\hat{f}(p, \mathbf{u}) = \mathscr{R}\{f(\mathbf{x})\}.$$

Or, equivalently,

$$\mathscr{F}^{-1}\left\{\tilde{f}(s\mathbf{u})\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ips}\, \tilde{f}(s\mathbf{u})\, ds. \tag{18.3.5}$$

***THEOREM 18.3.1***

Linearity:
$$\mathscr{R}\left[af(\mathbf{x}) + bg(\mathbf{x})\right] = a\mathscr{R}\left[f(\mathbf{x})\right] + b\mathscr{R}\left[g(\mathbf{x})\right]. \tag{18.3.6}$$

Shifting: (a) If $\mathscr{R}\{f(x, y)\} = \hat{f}(p, u_1, u_2)$, then
$$\mathscr{R}\{f(x - a, y - b)\} = \hat{f}(p - au_1 - bu_2, \mathbf{u}). \tag{18.3.6a}$$

(b) In general,
$$\hat{f}\left(\mathbf{x} - \mathbf{a}\right) = \hat{f}\left(p - \mathbf{a}.\mathbf{u},\ \mathbf{u}\right). \tag{18.3.6b}$$

Scaling: (a) If $\mathscr{R}\{f(x, y)\} = \hat{f}(p, u_1, u_2)$, then
$$\mathscr{R}\{f(ax, by)\} = \frac{1}{|ab|} \hat{f}\left(p, \frac{u_1}{a}, \frac{u_2}{b}\right). \tag{18.3.7a}$$

(b)
$$\hat{f}\left(a\mathbf{x}\right) = \frac{1}{a^n} \hat{f}\left(p,\ \frac{\mathbf{x}}{a}\right) = \frac{1}{a^{n-1}} \hat{f}\left(ap,\ \mathbf{x}\right). \tag{18.3.7b}$$

Symmetry: (a) If $\hat{f}(p, \mathbf{u}) = \mathscr{R}\left[f(x, y)\right]$, then, if $a \neq 0$,
$$\hat{f}(ap, a\mathbf{u}) = |a|^{-1} \hat{f}(p, \mathbf{u}). \tag{18.3.8a}$$

(b)
$$\hat{f}\left(p,\ a\mathbf{u}\right) = |a|^{-1} \hat{f}\left(\frac{p}{a}, \mathbf{u}\right). \tag{18.3.8b}$$

Here $a$ and $b$ are two constants.

**PROOF Linearity:**

$$\begin{aligned}\mathscr{R}\left[af(\mathbf{x})+bg(\mathbf{x})\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[af(\mathbf{x})+bg(\mathbf{x})\right]\delta(p-\mathbf{x}.\mathbf{u})\,d\mathbf{x}\\ &= a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\delta(p-\mathbf{x}.\mathbf{u})\,d\mathbf{x}\\ &\quad + b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x})\delta(p-\mathbf{x}.\mathbf{u})\,d\mathbf{x}\\ &= a\mathscr{R}\left[f(\mathbf{x})\right]+b\mathscr{R}\left[g(\mathbf{x})\right].\end{aligned}$$

This property is also true for any $\mathbf{x}=(x_1, x_2,,\ldots ,x_n)\in\mathbb{R}^n$, $n\geq 2$.

**Shifting:**

(a) Putting $x-a=\xi$ and $y-b=\eta$, we can write

$$\begin{aligned}\mathscr{R}\left\{f(x-a,y-b)\right\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-a,y-b)\,\delta(p-xu_1-yu_2)\,dxdy\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,\delta(p-(a+\xi)u_1-(b+\eta)u_2)\,d\xi d\eta\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,\delta(p-au_1-bu_2-\xi u_1-\eta u_2)\,d\xi d\eta\\ &= \hat{f}(p-au_1-bu_2,\mathbf{u}).\end{aligned}$$

Similarly, results (18.3.6b) can be proved.

**Scaling:**

(a) Let $a>0$, $b>0$. Putting $ax=\xi$, $by=\eta$, we have

$$\begin{aligned}\mathscr{R}\left[f(ax\,,by)\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(ax,by)\,\delta(p-xu_1-yu_2)\,dxdy\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,\delta\left(p-\frac{u_1}{a}\,\xi-\frac{u_2}{b}\,\eta\right)\frac{d\xi d\eta}{ab}\\ &= \frac{1}{ab}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,\delta\left(p-\frac{u_1}{a}\,\xi-\frac{u_2}{b}\,\eta\right)\\ &= \frac{1}{ab}\,\hat{f}\left(p,\frac{u_1}{a},\frac{u_2}{b}\right).\end{aligned}$$

If $a$ or $b<0$, then

$$\mathscr{R}\left[f(ax\,,by)\right] = -\frac{1}{ab}\,\hat{f}\left(p,\frac{u_1}{a},\frac{u_2}{b}\right).$$

Or,

$$\mathscr{R}\left[f(ax\,,by)\right] = \frac{1}{|ab|}\,\hat{f}\left(p,\frac{u_1}{a},\frac{u_2}{b}\right).$$

Similarly, results (18.3.7b) can be proved.

**Symmetry:**
(a) By definition, we have

$$\hat{f}\,(ap\,,a\mathbf{u}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(ap-axu_1-ayu_2)\,dxdy,$$

$$\text{which is, by using}\quad \delta(ap-axu_1-ayu_2)=\frac{1}{|a|}\,\delta(p-xu_1-yu_2),$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(p-xu_1-yu_2)\,dxdy$$

$$= \frac{1}{|a|}\,\hat{f}(p,\mathbf{u}).$$

This implies that the Radon transform is an even homogeneous function of degree $-1$. In particular if $a=-1$, we have

$$\hat{f}(-p,-\mathbf{u})=\hat{f}(p,\mathbf{u}),$$

that is, $\hat{f}$ is an even function.
(b) Another form of symmetry property is

$$\hat{f}(p,a\mathbf{u})=|a|^{-1}\hat{f}\left(\frac{p}{a},\mathbf{u}\right).$$

We have, by definition,

$$\begin{aligned}\hat{f}\,(p,a\mathbf{u}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(p-axu_1-ayu_2)\,dxdy\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(\frac{p}{a}\,.a-axu_1-ayu_2)\,dxdy\\ &= \frac{1}{|a|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(\frac{p}{a}-xu_1-yu_2)\,dxdy\\ &= \frac{1}{|a|}\hat{f}\left(\frac{p}{a},\mathbf{u}\right).\end{aligned}$$

∎

In general, there is an important relation between the $n$-dimensional Fourier transform and the Radon transform given by

$$\tilde{f}(s\mathbf{u})=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty} e^{-ips}\,\tilde{f}(p,\,\mathbf{u})\,dp. \tag{18.3.9}$$

The $n$-dimensional Fourier transform of $f(\mathbf{x})$ is

$$\tilde{f}(\mathbf{k})=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty} e^{-i(\mathbf{k}.\mathbf{x})}\,\tilde{f}(\mathbf{x})\,d\mathbf{x}. \tag{18.3.10}$$

Invoking the hyperspherical polar coordinates allows us to write $\mathbf{k}=s\mathbf{u}$ where $\mathbf{u}\in S^{n-1}$ which is the generalized unit sphere $\sum_{k=1}^{n} x_k^2=1$. Thus, (18.3.10) becomes

$$\tilde{f}(\mathbf{k})=\tilde{f}(s\mathbf{u})==\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty} e^{-is(\mathbf{u}.\mathbf{x})}\,\tilde{f}(\mathbf{x})\,d\mathbf{x}. \tag{18.3.11}$$

We put $g(\mathbf{x})=\exp\left[-is(\mathbf{u}.\mathbf{x})\right]f(\mathbf{x})$ so that

$$\hat{g}(p,\boldsymbol{\eta})=\int_L \exp\left[-is(\mathbf{u}.\mathbf{x})\right]f(\mathbf{x})\,ds, \tag{18.3.12}$$

where $L$ is the hyperplane $\mathbf{x}.\boldsymbol{\eta}=p$ and $ds$ is the $(n-1)$-dimensional surface area in $\mathbb{R}^n$. Hence,

$$\hat{g}(p,\mathbf{u})=\exp(-ips)\int_{\mathbf{u}.\mathbf{x}=p} f(\mathbf{x})ds=e^{-ips}\hat{f}(p,\mathbf{u}). \tag{18.3.13}$$

Using the result (18.4.12), we have

$$\begin{aligned}\int_{-\infty}^{\infty}\hat{g}(p,\mathbf{u})\,dp &= \int_{-\infty}^{\infty} e^{-ips}\hat{f}(p,\mathbf{u})\,dp=\int_{-\infty}^{\infty} g(\mathbf{x})d\mathbf{x}\\ &=\int_{-\infty}^{\infty} e^{-is(\mathbf{u}.\mathbf{x})}f(\mathbf{x})\,d\mathbf{x}=(2\pi)^{\frac{n}{2}}\tilde{f}(s\mathbf{u})\\ &=\frac{1}{(2\pi)^{\frac{n-1}{2}}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ips}\hat{f}(p,\mathbf{u})\,dp.\end{aligned}$$

Consequently, this yields the following result

$$\tilde{f}(s\mathbf{u})=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty} e^{-ips}\hat{f}(p,\mathbf{u})dp. \tag{18.3.14}$$

Denoting the one-dimensional Fourier transform along the radial direction by $\mathscr{F}_r$, equation (18.3.14) can be written as

$$\mathscr{F}\{f\}=\frac{1}{(2\pi)^{\frac{n-1}{2}}}\mathscr{F}_r\{\hat{f}\} \tag{18.3.15}$$

In view of (18.3.14) and the inverse Fourier transform, we find the relation

$$\begin{aligned}\hat{f}(p,\mathbf{u}) &= (2\pi)^{\frac{n-1}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ips}\tilde{f}(s\mathbf{u})ds\\ &=(2\pi)^{\frac{n-2}{2}}\int_{-\infty}^{\infty}\tilde{f}(s\mathbf{u})\,e^{isp}\,ds.\end{aligned} \tag{18.3.16}$$

## 18.4 The Radon Transform of Derivatives

Since

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\left[\frac{f\left(x+\frac{h}{u_1},y\right)-f(x,y)}{\frac{h}{u_1}}\right],$$

we can write

$$\begin{aligned}\mathscr{R}\left[\frac{\partial f}{\partial x}\right] &= u_1 \lim_{h\to 0}\left[\frac{\mathscr{R}\left\{f\left(x+\frac{h}{u_1},y\right)\right\}-\mathscr{R}\left\{f(x,y)\right\}}{h}\right] \\ &= u_1 \lim_{h\to 0}\frac{\hat{f}(p+h,\mathbf{u})-\hat{f}(p,\mathbf{u})}{h} \\ &= u_1 \frac{\partial}{\partial p}\hat{f}(p,\mathbf{u}).\end{aligned}$$

Similarly partial derivative with respect to $y$

$$\mathscr{R}\left[\frac{\partial f}{\partial y}\right] = u_2 \frac{\partial}{\partial p}\hat{f}(p,\mathbf{u}).$$

The Radon transform of the first derivatives, that is,

$$\mathscr{R}\left[\sum_{k=1}^{n} a_k \frac{\partial f}{\partial x_k}\right](p,\mathbf{u}) = (\mathbf{a}.\mathbf{u})\frac{\partial}{\partial p}\hat{f}(p,\mathbf{u}), \tag{18.4.1}$$

where $\mathbf{a}=(a_1,a_2,...,a_n)$.

Or, equivalently,

$$\mathscr{R}\left[(\mathbf{a}.\nabla)f\right](p,\mathbf{u}) = (\mathbf{a}.\mathbf{u})\frac{\partial}{\partial p}\hat{f}(p,\mathbf{u}), \tag{18.4.2}$$

where $\nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},...,\frac{\partial}{\partial x_n}\right)$ is the gradient operator.

In particular,

$$\mathscr{R}\left[\frac{\partial f}{\partial x_k}\right](p,\mathbf{u}) = u_k \frac{\partial}{\partial p}\hat{f}(p,\mathbf{u}). \tag{18.4.3}$$

We leave proofs of the above results to the reader.

The Radon transform of the second order derivatives are given by

$$\begin{aligned}\mathscr{R}\left[\frac{\partial^2 f}{\partial x^2}\right] &= \mathscr{R}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right] = u_1\frac{\partial}{\partial p}\mathscr{R}\left[\frac{\partial f}{\partial x}\right] = u_1^2\frac{\partial^2}{\partial p^2}\hat{f}(p,\mathbf{u}), \\ \mathscr{R}\left[\frac{\partial^2 f}{\partial x\partial y}\right] &= u_1u_2\frac{\partial^2}{\partial p^2}\hat{f}(p,\mathbf{u}), \\ \mathscr{R}\left[\frac{\partial^2 f}{\partial y^2}\right] &= u_2^2\frac{\partial^2}{\partial p^2}\hat{f}(p,\mathbf{u}).\end{aligned}$$

We state more general results:

If $L=L\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},...,\frac{\partial}{\partial x_n}\right)$ is a linear differential operator with constant coefficients, then

$$\mathscr{R}\left[L\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},...,\frac{\partial}{\partial x_n}\right)f(\mathbf{x})\right](p,\mathbf{u})$$
$$=L\left(u_1\frac{\partial}{\partial p},u_2\frac{\partial}{\partial p},...,u_n\frac{\partial}{\partial p}\right)\hat{f}(p,\mathbf{u}). \tag{18.4.4}$$

In particular,

$$\mathscr{R}\left[\sum_{k=1}^{n}\sum_{l=1}^{n}(a_k b_l)\frac{\partial^2 f}{\partial x_k \partial x_l}\right](p,\mathbf{u})=(\mathbf{a}.\mathbf{u})(\mathbf{b}.\mathbf{u})\frac{\partial^2\hat{f}(p,\mathbf{u})}{\partial p^2}, \tag{18.4.5}$$

$$\mathscr{R}\left[\frac{\partial^2 f(\mathbf{x})}{\partial x_k\partial x_l}\right](p,\mathbf{u})=(u_k u_l)\frac{\partial^2\hat{f}(p,\mathbf{u})}{\partial p^2}. \tag{18.4.6}$$

If $a_k b_l=\delta_{kl}$ (the Kronecker delta), then the operator involved is the Laplace operator

$$\nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+....+\frac{\partial^2}{\partial x_n^2},$$

and

$$\mathscr{R}\left\{\nabla^2 f(\mathbf{x})\right\}=|\mathbf{u}|^2\,\frac{\partial^2\hat{f}(p,\mathbf{u})}{\partial p^2}=\frac{\partial^2\hat{f}(p,\mathbf{u})}{\partial p^2}, \tag{18.4.7}$$

where $|\mathbf{u}|^2=1$ is used to obtain the last result. This is a very important result that is employed to solve partial differential equations.

## 18.5 Derivatives of the Radon Transform

In order to calculate the derivative of the radon transform, the following formulas of the derivative of the Dirac delta function are needed and stated as

$$\frac{\partial}{\partial x}\,\delta(x-y)=-\frac{\partial}{\partial y}\,\delta(x-y). \tag{18.5.1}$$

If $y$ is replaced by $by$, then

$$\frac{\partial}{\partial(by)}\,\delta(x-by)=\frac{1}{b}\frac{\partial}{\partial y}\,\delta(x-by)=-\frac{\partial}{\partial x}\,\delta(x-by). \tag{18.5.2}$$

Similarly, an $n$-dimensional result is

$$\frac{\partial}{\partial y_j}\,\delta(\mathbf{x}-\mathbf{y})=-\frac{\partial}{\partial x_j}\,\delta(\mathbf{x}-\mathbf{y}). \tag{18.5.3}$$

It follows from the above results that

$$\frac{\partial}{\partial \xi_j}\,\delta(p-\boldsymbol{\xi}.\mathbf{x}) = -x_j\,\frac{\partial}{\partial p}\,\delta((p-\boldsymbol{\xi}.\mathbf{x}), \tag{18.5.4}$$

where $\boldsymbol{\xi}$ is not necessarily a unit vector in the term $\boldsymbol{\xi}.\mathbf{x}$.

Formula (18.5.4) reveals that the derivatives are calculated with respect to components of $\boldsymbol{\xi}$ and then evaluated at $\boldsymbol{\xi}=\mathbf{u}$. Using the definition of (18.2.5), we obtain

$$\begin{aligned}\frac{\partial \hat{f}}{\partial u_k} &= \left[\frac{\partial \hat{f}(p,\boldsymbol{\xi})}{\partial \xi_k}\right]_{\boldsymbol{\xi}=\mathbf{u}} = \left[\int f(\mathbf{x})\,\frac{\partial}{\partial \xi_k}\,\delta(p-\boldsymbol{\xi}.\mathbf{x})\,d\mathbf{x}\right]_{\boldsymbol{\xi}=\mathbf{u}} \\ &= -\frac{\partial}{\partial p}\int x_k f(\mathbf{x})\,\delta(p-\mathbf{u}.\mathbf{x})\,d\mathbf{x}.\end{aligned} \tag{18.5.5}$$

Consequently, we obtain the formula for the derivative of the Radon transform

$$\frac{\partial \hat{f}(p,\mathbf{u})}{\partial u_k} = \left[\frac{\partial}{\partial \xi_k}\mathscr{R}\left\{f(\mathbf{x})\right\}\right]_{\boldsymbol{\xi}=\mathbf{u}} = -\frac{\partial}{\partial p}\mathscr{R}\left\{x_k\,f(\mathbf{x})\right\}. \tag{18.5.6}$$

More generally, we state the derivatives of the Radon transform as

$$\left(\mathbf{a}.\frac{\partial}{\partial \mathbf{u}}\right)\mathscr{R}\left\{f(\mathbf{x})\right\}(p,\mathbf{u}) = -\frac{\partial}{\partial p}\mathscr{R}\left[(\mathbf{a}.\mathbf{x})f(\mathbf{x})\right](p,\mathbf{u}), \tag{18.5.7}$$

where $\mathbf{a}.\frac{\partial}{\partial \mathbf{u}} = \sum_{k=1}^{n} a_k\,\frac{\partial}{\partial u_k}$.

Formula (18.5.6) can be generalized for higher order derivatives as follows:

$$\frac{\partial^2 \hat{f}(p,\mathbf{u})}{\partial u_l\,\partial u_k} = (-1)^2\,\frac{\partial^2}{\partial p^2}\mathscr{R}\left\{x_l\,x_k f(\mathbf{x})\right\}, \tag{18.5.8}$$

$$\frac{\partial^3 \hat{f}(p,\mathbf{u})}{\partial u_l\,\partial^2 u_k} = (-1)^3\,\frac{\partial^3}{\partial p^3}\mathscr{R}\left\{x_l\,x_k^2 f(\mathbf{x})\right\}, \tag{18.5.9}$$

where + or − sign is used for even or odd order derivatives, respectively.

More generally,

$$\sum_{k,l=1}^{n}(a_k\,b_l)\,\frac{\partial^2 \hat{f}(p,\mathbf{u})}{\partial u_k\,\partial u_l} = \frac{\partial^2}{\partial p^2}\mathscr{R}\left\{(\mathbf{a}.\mathbf{x})\,(\mathbf{b}.\mathbf{x})f(\mathbf{x})\right\}. \tag{18.5.10}$$

For a two dimensional function, $f(\mathbf{x})=f(x,y)$ we obtain

$$\frac{\partial^{k+l}\hat{f}(p,\mathbf{u})}{\partial u_1^k\,\partial u_2^l} = \left(-\frac{\partial}{\partial p}\right)^{k+l}\mathscr{R}\left\{x^k\,x^l\,f(\mathbf{x})\right\}(p,\mathbf{u}). \tag{18.5.11}$$

Finally, the property involving the integration of the radon transform with respect to $p$ can be stated as follows:

$$\int_{-\infty}^{\infty}\hat{f}(p,\mathbf{u})\,dp = \int_{-\infty}^{\infty} f(\mathbf{x})\,d\mathbf{x}. \tag{18.5.12}$$

***Example 18.5.1***
The Radon transform of Hermite polynomials is given by

$$\mathscr{R}\left\{H_l(x)H_k(y)\exp(-(x^2+y^2)\right\} = \sqrt{\pi}(\cos\phi)^l\,(\sin\phi)^k\,e^{-p^2}H_{l+k}(p), \tag{18.5.13}$$

where $H_n(x)$ is the Hermite polynomials of degree defined by Rodrigues formulas

$$e^{-x^2}H_n(x) = (-1)^n\left(\frac{\partial}{\partial x}\right)^n e^{-x^2}. \tag{18.5.14}$$

Obviously,

$$\exp\left[-(x^2+y^2)\right]\,H_l(x)H_k(x) = (-1)^{l+k}\left(\frac{\partial}{\partial x}\right)^l\left(\frac{\partial}{\partial y}\right)^k\exp\left[-(x^2+y^2)\right]. \tag{18.5.15}$$

It follows from the above formulas

$$\mathscr{R}\left[\left(\frac{\partial}{\partial x}\right)^l\left(\frac{\partial}{\partial y}\right)^k f(x,y)\right] = (\cos\phi)^l(\sin\phi)^k\left(\frac{\partial}{\partial p}\right)^{l+k}\hat{f}(p,\mathbf{u}), \tag{18.5.16}$$

that the Radon transform of (18.5.15) is given by

$$\mathscr{R}\left[\exp\{-(x^2+y^2)\}\,H_l(x)H_k(x)\right] = (\cos\phi)^l(\sin\phi)^k\left(\frac{\partial}{\partial p}\right)^{l+k}\sqrt{\pi}\,e^{-p^2}, \tag{18.5.17}$$

where $\hat{f}(p,\mathbf{u}) = \mathscr{R}\left[\exp\{-(x^2+y^2)\}\right] = \sqrt{\pi}\,e^{-p^2}$ is used. $\square$

## 18.6 Convolution Theorem for the Radon Transform

***THEOREM 18.6.1***
*(Convolution).* If $\hat{f}(p,\,\mathbf{u}) = \mathscr{R}\{f(\mathbf{x})\}$ and $\hat{g}(p,\,\mathbf{u}) = \mathscr{R}\{g(\mathbf{x})\}$, then

$$\mathscr{R}\{(f\star g)(\mathbf{x})\} = \left(\hat{f}\star\hat{g}\right)(p,\,\mathbf{u}). \tag{18.6.1}$$

**PROOF** Let $h(\mathbf{x})$ be the convolution of $f(\mathbf{x})$ and $g(\mathbf{x})$. So, we have

$$h(\mathbf{x}) = (f \star g)(\mathbf{x}) = \int_{-\infty}^{\infty} f(\mathbf{x})g(\mathbf{x}-\mathbf{y})d\mathbf{y}, \tag{18.6.2}$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$. Taking the Radon transform of (18.5.3) yields the following

$$\begin{aligned}
\hat{h}(p, \mathbf{u}) &= \mathscr{R}\{h(\mathbf{x})\} = \mathscr{R}\{(f \star g)(\mathbf{x})\} \\
&= \int_{-\infty}^{\infty} h(\mathbf{x})\, \delta(p - \mathbf{u}.\mathbf{x})\, d\mathbf{x} \\
&= \int_{-\infty}^{\infty} f(\mathbf{y})\, d\mathbf{y} \int_{-\infty}^{\infty} g(\mathbf{x}-\mathbf{y})\, \delta(p - \mathbf{u}.\mathbf{x})\, d\mathbf{x} \\
&= \int_{-\infty}^{\infty} f(\mathbf{y})\, d\mathbf{y} \int_{-\infty}^{\infty} g(\mathbf{z})\, \delta(p - \mathbf{u}.\mathbf{y} - \mathbf{u}.\mathbf{z})\, d\mathbf{z}, \qquad (\mathbf{z} = \mathbf{x} - \mathbf{y}) \\
&= \int_{-\infty}^{\infty} f(\mathbf{y})\, \hat{g}(p - \mathbf{u}.\mathbf{y},\, \mathbf{u})\, d\mathbf{y} \\
&= \int_{-\infty}^{\infty} f(\mathbf{y})\, d\mathbf{y} \int_{-\infty}^{\infty} \hat{g}(p - s,\, \mathbf{u})\, \delta(s - \mathbf{u}.\mathbf{y})\, ds \\
&= \int_{-\infty}^{\infty} \hat{g}(p - s,\, \mathbf{u})\, ds \int_{-\infty}^{\infty} f(\mathbf{y})\, \delta(s - \mathbf{u}.\mathbf{y})\, d\mathbf{y} \\
&= \int_{-\infty}^{\infty} \hat{f}(s,\, \mathbf{u})\, \hat{g}(p - s,\, \mathbf{u})\, ds \\
&= \left[\left(\hat{f} * \hat{g}\right)(p,\, \mathbf{u})\right],
\end{aligned}$$

where

$$\hat{g}(p - \mathbf{u}.\mathbf{y},\, \mathbf{u}) = \int_{-\infty}^{\infty} \hat{g}(p - s,\, \mathbf{u})\, \delta(s - \mathbf{u}.\mathbf{y})\, ds.$$

Or,

$$\hat{h}(p,\, \mathbf{u}) = \mathscr{R}\{f \star g\}(\mathbf{x}) = \hat{f}(p,\, \mathbf{u}) \star \hat{g}(p,\, \mathbf{u}).$$

∎

## 18.7 Inverse of the Radon Transform and the Parseval Relation

We consider the $n$-dimensional Fourier transform $\tilde{f}(\mathbf{k})$ of $f(\mathbf{x})$ defined by

$$\tilde{f}(\mathbf{k}) = \mathscr{F}\{f(\mathbf{x})\} = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{-i\mathbf{k}.\mathbf{x}}\, f(\mathbf{x})d\mathbf{x}. \tag{18.7.1}$$

Using the hyperspherical polar coordinates, we can put $\mathbf{k} = \rho\mathbf{u}$ where $\mathbf{u} \in S^{n-1}$ which is the generalized unit sphere $\sum_{r=1}^{n} x_r^2 = 1$. Consequently,

$$\tilde{f}(\mathbf{k}) = \tilde{f}(\rho\mathbf{u}) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{-i\rho(\mathbf{u}.\mathbf{x})} f(\mathbf{x}) d\mathbf{x}. \tag{18.7.2}$$

For fixed $\rho$ and $\mathbf{u}$, we set $F(\mathbf{x})$ =exp$[-i\rho(\mathbf{u}.\mathbf{x})]\, f(\mathbf{x})$, so that

$$\hat{F}(p, \mathbf{u}) = \int_L e^{-i\rho(\mathbf{u}.\mathbf{x})} f(\mathbf{x})\, ds, \tag{18.7.3}$$

where $L$ is the hyperplane $\mathbf{u}'.\mathbf{x} = p$ and $ds$ is $(n-1)$-dimensional surface area measure in $\mathbb{R}^n$. Thus,

$$\hat{F}(p, \mathbf{u}) = e^{-i\rho p} \int_{\mathbf{u}'.\mathbf{x}=p} f(\mathbf{x})\, ds = e^{-i\rho p} \hat{f}(p, \mathbf{u}), \tag{18.7.4}$$

which is integrated by using (18.5.12) so that

$$\int_{-\infty}^{\infty} \hat{F}(p, \mathbf{u})\, dp = \int_{-\infty}^{\infty} e^{-i\rho p} \hat{f}(p, \mathbf{u})\, dp. \tag{18.7.5}$$

Therefore, result (18.7.2) becomes

$$\tilde{f}(\rho\mathbf{u}) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{-i\rho p} \hat{f}(p, \mathbf{u}) dp. \tag{18.7.6}$$

We next denote the one-dimensional Fourier transform along the radial direction by $\mathscr{F}_r$, so that (18.7.6) can be written as

$$\mathscr{F}\{f(\rho\mathbf{u})\} = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \mathscr{F}_r \left[\hat{f}(p, \mathbf{u})\right]. \tag{18.7.7}$$

Invoking the inverse Fourier transform in (18.7.6)

$$\hat{f}(p, \mathbf{u}) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{-\infty}^{\infty} \tilde{f}(\rho\mathbf{u})\, e^{i\rho p}\, d\rho. \tag{18.7.8}$$

We next consider the inverse Fourier transform with (18.7.6) to obtain

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{i\mathbf{k}.\mathbf{x}} \tilde{f}(\mathbf{k})\, d\mathbf{k} \qquad (18.7.9)\\ &= \frac{1}{(2\pi)^{n/2}} \int_0^{\infty} \rho^{n-1} d\rho \int_{|\mathbf{u}|=1} e^{i\rho(\mathbf{x}.\mathbf{u})} \tilde{f}(\rho\mathbf{u})\, d\mathbf{u} \\ &= \frac{1}{(2\pi)^{n/2}} \int_0^{\infty} \rho^{n-1} d\rho \int_{|\mathbf{u}|=1} d\mathbf{u} \int_{-\infty}^{\infty} e^{i\rho(\mathbf{x}.\mathbf{u})} \hat{f}(p, \mathbf{u}) e^{-i\rho p}\, dp \\ &= \int_{|\mathbf{u}|=1} h(\mathbf{x}.\mathbf{u},\, \mathbf{u})\, d\mathbf{u}, \end{aligned}$$

where

$$h(t,\mathbf{u}) = \frac{1}{(2\pi)^{n/2}} \int_0^\infty d\rho \int_{-\infty}^\infty \rho^{n-1}\, e^{-i\rho(p-t)}\, \hat{f}(p,\mathbf{u})dp. \tag{18.7.10}$$

We thus obtain a theorem for the inverse Radon transform:

***THEOREM 18.7.1***
*(Inversion Theorem).* If $\hat{f}$ is the Radon transform of $f(\mathbf{x})$, then

$$f(\mathbf{x}) = \int_{|\mathbf{u}|=1} h(\mathbf{x}.\mathbf{u},\mathbf{u})d\mathbf{u}, \tag{18.7.11}$$

where

$$h(t,\mathbf{u}) = \left\{ \begin{array}{ll} a_n \dfrac{\partial^{n-1}}{\partial t^{n-1}} \hat{f}(t,\mathbf{u}), & \text{for odd } n \\ a_n \boldsymbol{H} \left[ \dfrac{\partial^{n-1}}{\partial p^{n-1}} \hat{f}(p,\mathbf{u}) \right](t), & \text{for even } n \end{array} \right\}, \tag{18.7.12ab}$$

where $\boldsymbol{H}$ stands for the Hilbert transform with respect to $p$,

$$a_n = \left\{ \begin{array}{ll} \dfrac{i^{n-1}}{2\,(2\pi)^{n-1}}, & \text{for odd } n \\ \dfrac{i^n}{2\,(2\pi)^{n-1}}, & \text{for even } n \end{array} \right\}. \tag{18.7.13ab}$$

***THEOREM 18.7.2***
*(Two Dimensional Inversion Theorem).* If $\hat{f}(p,\mathbf{u})$ is the Radon transform of $f(\mathbf{x}) = f(x,y)$, then

$$f(x,y) = -\frac{1}{4\pi^2} \int_{|\mathbf{u}|=1} d\mathbf{u} \int_{-\infty}^\infty \frac{\hat{f}_p(p,\mathbf{u})}{p - \mathbf{x}.\mathbf{u}}\, dp. \tag{18.7.14}$$

In this case $n = 2$, it follows from (18.7.12ab)–(18.7.13ab) that

$$h(t,\mathbf{u}) = \frac{-1}{2\,(2\pi)} \frac{1}{\pi} \int_{-\infty}^\infty \frac{\hat{f}_p(p,\mathbf{u})}{p-t}\, dp, \tag{18.7.15}$$

where

$$\hat{f}_p(p,\mathbf{u}) = \frac{\partial}{\partial p} \hat{f}(p,\mathbf{u}).$$

Consequently, the formula (18.7.11) reduces to (18.7.14).
On the other hand, using $\mathbf{u} = (\cos\phi,\, \sin\phi)$, (18.7.14) gives

$$f(x,y) = -\frac{1}{\pi} \int_0^\pi d\phi \int_{-\infty}^\infty \frac{\hat{f}_p(p,\mathbf{u})}{p - \mathbf{x}.\mathbf{u}}\, dp. \tag{18.7.16}$$

A simple change of variables $x = r\cos\theta$, $y = r\sin\theta$ leads to the inversion formula in the polar form

$$f(r,\theta) = -\frac{1}{4\pi^2}\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} \frac{\hat{f}_p(p,\,\phi)}{p - r\cos(\phi-\theta)}\,dp. \tag{18.7.17}$$

We discuss the three-dimensional inverse Radon transform independently because it does not involve the Hilbert transform.

***THEOREM 18.7.3***

*(Three Dimensional Inversion Theorem).* If $\hat{f}(p,\mathbf{u}) = \mathscr{R}\{f(x,\,y,\,z)\}$, then

$$f(\mathbf{x}) = -\nabla^2 \int_{|\mathbf{u}|=1} \hat{f}(p,\,\mathbf{x}.\mathbf{u})\,d\mathbf{u}, \tag{18.7.18}$$

where $\mathbf{x} = (x,\,y,\,z) \in \mathbb{R}^3$.

**PROOF** We begin with the inverse of the three-dimensional Fourier transform in the form

$$f(\mathbf{x}) = \mathscr{F}_3^{-1}\tilde{f}(s\mathbf{u}) = \int_0^{\infty} s^2\,ds \int_{|\mathbf{u}|=1} \tilde{f}(q\mathbf{u})\,e^{is(\mathbf{x}.\mathbf{u})}\,d\mathbf{u}$$

where the integral over the unit sphere is stated as follows:

$$\int_{|\mathbf{u}|=1} d\mathbf{u} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta,$$

where $\mathbf{u} = (\sin\theta\cos\phi,\,\sin\theta\sin\phi,\,\cos\theta)$, $\theta$ is the polar angle and $\phi$ is the azimuthal angle.

Invoking the symmetry of the Radon transform $\hat{f}$, where $\mathscr{F}\{\hat{f}\} = \tilde{f}$, the integral over $q$ from 0 to $\infty$ can be replaced by one-half the integral from $-\infty$ to $\infty$ and hence,

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{2}\int_{|\mathbf{u}|=1} d\mathbf{u}\left[\int_{-\infty}^{\infty} s^2\,\tilde{f}(s\mathbf{u})e^{isp}\right]_{p=\mathbf{x}.\mathbf{u}} ds \\ &= \frac{1}{2}\int_{|\mathbf{u}|=1} \mathscr{F}^{-1}\left[\{s^2\tilde{f}(s\mathbf{u})\}\right]_{p=\mathbf{x}.\mathbf{u}} d\mathbf{u} \end{aligned}$$

which is, by the Fourier transform of the second derivatives, or,

$$= -\frac{1}{2}\int_{|\mathbf{u}|=1}\left[\hat{f}_{pp}(p,\mathbf{u})\right]_{p=\mathbf{x}.\mathbf{u}} d\mathbf{u}. \tag{18.7.19}$$

This is an inversion formula for the three-dimensional Radon transform. In view of the fact that, for any $f(\mathbf{x},\mathbf{u})$,

$$\nabla^2 f(\mathbf{x},\mathbf{u}) = |\mathbf{u}|^2\left[f_{pp}(p)\right]_{p=\mathbf{x}.\mathbf{u}} = \left[f_{pp}(p)\right]_{p=\mathbf{x}.\mathbf{u}}, \tag{18.7.20}$$

we obtain another form of the inversion formula for the Radon transform

$$f(\mathbf{x}) = -\frac{1}{2}\nabla^2 \int_{|\mathbf{u}|=1} \hat{f}(\mathbf{x}.\mathbf{u},\ \mathbf{u})\, d\mathbf{u}. \tag{18.7.21}$$

$\blacksquare$

We next introduce the *adjoint Radon transform* from the definition of the inner product as

$$\begin{aligned}
\langle \phi,\ \mathscr{R}[f]\rangle &= \int_{-\infty}^{\infty} dp \int_{|\mathbf{u}|=1} \phi(p,\mathbf{u})\, \overline{(\mathscr{R}f)(p,\mathbf{u})}\ d\mathbf{u} \\
&= \int_{-\infty}^{\infty} dp \int_{|\mathbf{u}|=1} \phi(p,\mathbf{u}) d\mathbf{u} \int_{-\infty}^{\infty} \bar{f}(\mathbf{x})\, \delta(p-\mathbf{x}.\mathbf{u})\, d\mathbf{x} \\
&= \int_{-\infty}^{\infty} \bar{f}(\mathbf{x})\, d\mathbf{x} \left[\int_{|\mathbf{u}|=1} d\mathbf{u} \int_{-\infty}^{\infty} \phi(p,\mathbf{u})\, \delta(p-\mathbf{x}.\mathbf{u})\, dp\right] \\
&= \int_{-\infty}^{\infty} \left[\int_{|\mathbf{u}|=1} \phi(\mathbf{x}.\mathbf{u},\mathbf{u})\, d\mathbf{u}\right] \bar{f}(\mathbf{x})\, d\mathbf{x} \\
&= \int_{-\infty}^{\infty} (R^*[\phi])\, \bar{f}(\mathbf{x})\, d\mathbf{x} = \langle R^*[\phi],\ f\rangle
\end{aligned} \tag{18.7.22}$$

where the adjoint $R^*$ is defined by

$$R^*[\phi](\mathbf{x}) = \int_{|\mathbf{u}|=1} \phi(\mathbf{x}.\mathbf{u},\ \mathbf{u})\, d\mathbf{u}. \tag{18.7.23}$$

This means that the action of the adjoint $R^*$ on $\phi$ corresponds to the integration of $\phi$ over all hyperplanes passing through a given point.

We use (18.7.12ab) to introduce the operator $K$ as follows:

$$K\phi(p,\mathbf{u}) = \left\{\begin{array}{ll} a_n \dfrac{\partial^{n-1}}{\partial p^{n-1}}\phi(p,\mathbf{u}), & \text{for odd } n \\ a_n \boldsymbol{H}\left[\dfrac{\partial^{n-1}}{\partial p^{n-1}}\phi(p,\mathbf{u})\right], & \text{for even } n \end{array}\right\}, \tag{18.7.24}$$

where $a_n$ is defined by (18.7.13ab) and $\boldsymbol{H}$ stands for the Hilbert transform. Clearly, it follows from (18.7.12ab) that

$$K\hat{f}(\mathbf{x}.\mathbf{u},\ \mathbf{u}) = h(\mathbf{x}.\mathbf{u},\ \mathbf{u}) \tag{18.7.25}$$

and hence, by (18.7.23),

$$R^*[K\hat{f}(\mathbf{x}.\mathbf{u},\mathbf{u})] = R^*[h(\mathbf{x}.\mathbf{u},\mathbf{u})] = \int_{|\mathbf{u}|=1} h(\mathbf{x}.\mathbf{u},\mathbf{u})\, d\mathbf{u} = f(\mathbf{x}). \tag{18.7.26}$$

This means that the inversion formula (18.7.11) can be written as

$$f = R^* K[\hat{f}]. \tag{18.7.27}$$

***THEOREM 18.7.4***

*(Parseval's Theorem).* If $\mathscr{R}\{f(\mathbf{x})\} = \hat{f}(p, \mathbf{u})$ and $\mathscr{R}\{g(\mathbf{x})\} = \hat{g}(p, \mathbf{u})$, then

(a) for even $n$

$$\begin{aligned}\langle f, g\rangle &= \int_{-\infty}^{\infty} f(\mathbf{x})\bar{g}(\mathbf{x})\,d\mathbf{x}\\ &= a_n \int_{|\mathbf{u}|=1} d\mathbf{u} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(p, \mathbf{u})\,\bar{\hat{g}}(q, \mathbf{u})(p-q)^{-n}\,dp\,dq, \end{aligned}\tag{18.7.28}$$

where $a_n = (-1)^{\frac{n}{2}}(2\pi)^{-n}\,(n-1)!$,

(b) for odd $n$

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(\mathbf{x})\bar{g}(\mathbf{x})\,d\mathbf{x}$$

$$= \frac{(-1)^{\frac{n-1}{2}}}{2(2\pi)^{n-1}} \int_{|\mathbf{u}|=1} d\mathbf{u} \int_{-\infty}^{\infty} \hat{f}(p, \mathbf{u})\,\bar{\hat{g}}_p^{(n-1)}(p, \mathbf{u})\,dp \tag{18.7.29}$$

$$= \frac{1}{2(2\pi)^{n-1}} \int_{|\mathbf{u}|=1} d\mathbf{u} \int_{-\infty}^{\infty} \hat{f}_p^{(m)}(p, \mathbf{u})\,\bar{\hat{g}}_p^{(m)}(p, \mathbf{u})\,dp, \tag{18.7.30}$$

where $m = \frac{n-1}{2}$.

We next introduce an operator $H$ due to Ludwig (1966) by

$$H[f](p) = \frac{1}{\sqrt{2}}\,\frac{1}{(2\pi)^m}\,\frac{\partial^m f(p)}{\partial p^m}, \tag{18.7.31}$$

where $m = \frac{n-1}{2}$. Consequently, the Parseval's formula (18.7.30) reduces to the form

$$\langle f, g\rangle = \langle H\hat{f}, H\hat{g}\rangle. \tag{18.7.32}$$

Ludwig (1966) proved that $H\mathscr{R}$ is a unitary transformation from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R} \times S^{n-1})$. For a proof of the above Parseval's formulas, the reader is referred to Ludwig (1966) and Zayed (1996).

## 18.8 Applications of the Radon Transform

We prove a remarkable relation between the Radon transform and the solution of the Cauchy problem involving the solution of the wave equation

$$u_{tt} = c^2 \nabla^2 u, \qquad \mathbf{x} \in \mathbb{R}^3,\ t > 0, \tag{18.8.1}$$

$$u(\mathbf{x},\, 0) = f(\mathbf{x}) \quad u_t(\mathbf{x},\, 0) = g(\mathbf{x}), \tag{18.8.2ab}$$

where $c$ is a constant and $\nabla^2$ is the three-dimensional Laplacian.

We apply the radon transform of $u(\mathbf{x},\, t)$ by

$$\hat{u}(p, \boldsymbol{\xi}, t) = \mathscr{R}\{u(\mathbf{x},\, t)\} = \int_{-\infty}^{\infty} u(\mathbf{x},\, t)\, \delta(p - \mathbf{x}.\boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{18.8.3}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ is the three-dimensional unit vector in $\mathbb{R}^3$ so that $|\boldsymbol{\xi}| = \xi_1^2 + \xi_2^2 + \xi_3^2 = 1$.

Application of (18.8.3) to (18.8.1)-(18.8.2ab) gives

$$\hat{u}_{tt} = c^2 \left(\xi_1^2 + \xi_2^2 + \xi_3^2\right) \hat{u}_{pp} = c^2\, \hat{u}_{pp}, \tag{18.8.4}$$

$$\hat{u}(p,\, \boldsymbol{\xi},\, 0) = \hat{f}(p,\, \boldsymbol{\xi}), \qquad \left[\frac{d\hat{u}(p,\, \boldsymbol{\xi},\, t)}{dt}\right]_{t=0} = \hat{g}(p, \boldsymbol{\xi}). \tag{18.8.5}$$

Thus, the radon transform $\hat{u}(p, \boldsymbol{\xi}, t)$ satisfies the Cauchy problem (18.8.4)–(18.8.5). We solve this problem by the application of the Fourier transform of $\hat{u}(p, \boldsymbol{\xi}, t)$ so that

$$\frac{d^2\hat{U}}{dt^2} = -c^2 k^2\, \hat{U}, \tag{18.8.6}$$

$$\hat{U}(k,\, \boldsymbol{\xi},\, 0) = \hat{F}(k,\, \boldsymbol{\xi}), \qquad \left(\frac{d\hat{U}}{dt}\right)_{t=0} = \hat{G}(k,\, \boldsymbol{\xi}), \tag{18.8.7}$$

where

$$\hat{U}(k, \boldsymbol{\xi}, t) = \mathscr{F}\{\hat{u}(p,\, \boldsymbol{\xi},\, t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikp}\, \hat{u}(p,\, \boldsymbol{\xi},\, t)\, dp \tag{18.8.8}$$

The solution of this transformed problem is obtained in Chapter 2, and the solution of (18.8.6)-(18.8.7) gives

$$\hat{U}(k,\, \boldsymbol{\xi},\, t) = \hat{F}(k,\, \boldsymbol{\xi}) \cos(ckt) + \frac{\hat{G}(k,\, \boldsymbol{\xi})}{2ick} \sin(ckt).$$

Following the method presented in Section 2.12, the D'Alembert solution is obtained in the form

$$\hat{u}(p, \boldsymbol{\xi}, t) = \frac{1}{2}\left[\hat{f}(p-ct) + \hat{f}(p+ct)\right] + \frac{1}{2c}\int_{p-ct}^{p+ct} \hat{g}(\alpha, \boldsymbol{\xi})\, d\alpha. \quad (18.8.9)$$

The inverse Radon transform yields the solution of the Cauchy problem in the form

$$u(\mathbf{x}, t) = \mathscr{R}^{-1}\left\{\hat{u}(p, \boldsymbol{\xi}, t)\right\} = -\nabla^2 \int_{|\boldsymbol{\xi}|=1} \hat{u}(\mathbf{x}.\boldsymbol{\xi}, \boldsymbol{\xi}, t)\, d\boldsymbol{\xi}. \quad (18.8.10)$$

It is noted that the Radon transform transformed the (1+3)-dimensional wave equation (18.8.1) to the (1+1)-dimensional wave equation (18.8.4) which can be solved by using standard methods. In general, the Radon transform reduces problems with $(n+1)$ independent variables to problems with two independent variables. In other words, as stated in equation (18.4.4), if $L$ is a differential operator of $(n+1)$-dimensions with constant coefficients, then its Radon transform is

$$\mathscr{R}\left[L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}; \frac{\partial}{\partial t}\right) f(\mathbf{x}, t)\right]$$
$$= L\left(u_1\frac{\partial}{\partial p}, u_2\frac{\partial}{\partial p}, \ldots, u_n\frac{\partial}{\partial p}; \frac{\partial}{\partial t}\right) \hat{f}(p, \boldsymbol{\xi}, t). \quad (18.8.11)$$

This fundamental property help solve hyperbolic partial differential equations with constant coefficients.

## 18.9 Exercises

1. Show that

   (a) $\mathscr{R}\left\{x^2 \exp(-x^2-y^2)\right\} = \frac{\sqrt{\pi}}{2}\left(2\,p^2\cos^2\phi + \sin^2\phi\right) e^{-p^2}.$

   (b) $\mathscr{R}\left\{y^2 \exp(-x^2-y^2)\right\} = \frac{\sqrt{\pi}}{2}\left(2\,p^2\sin^2\phi + \cos^2\phi\right) e^{-p^2}.$

   (c) $\mathscr{R}\left\{(x^2+y^2) \exp(-x^2-y^2)\right\} = \frac{\sqrt{\pi}}{2}\left(2\,p^2 + 1\right) e^{-p^2}.$

2. Verify that

$$\frac{\partial f}{\partial u_1} = -\frac{\partial}{\partial p}\left[\mathscr{R}\left\{x\, e^{-x^2-y^2}\right\}\right].$$

3. If $f(x,y) = \exp\left(-x^2 - y^2\right)$, show that

$$\frac{\partial \hat{f}}{\partial u_k} = \sqrt{\pi}\, u_k \left(2p^2 - 1\right) e^{-p^2}.$$

4. Prove result (18.5.16).

5. If $A$ is nonsingular $m \times n$ matrix, show that

   (a) $\mathscr{R}\{f(A\mathbf{x})\} = \hat{f}(p, A\mathbf{u})$,

   where $\mathbf{x} = (x_1, x_2, ..., x_n)$, and $A\mathbf{u}$ is a unit vector.

   (b) $\mathscr{R}\{f(c\mathbf{x})\} = c^{-n} \hat{f}\left(p, \frac{\mathbf{u}}{c}\right) = c^{1-n} \hat{f}\left(cp, \mathbf{u}\right), \quad A = cI.$

6. If $f(x,y) = \exp(-x^2 - y^2)$, $n = 2$, $c = (\sigma\sqrt{2})^{-1}$, use 5(b) and

$$f(A\mathbf{x}) = \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \quad \text{and} \quad \frac{1}{\mathrm{c}}\, \hat{\mathrm{f}}(\mathrm{cp}, \mathbf{u}) = \sigma\sqrt{2\pi}\, \exp\left(-\frac{\mathrm{p}^2}{2\sigma^2}\right),$$

show that the Radon transform of the symmetric Gaussian probability density function is given by

$$\mathscr{R}\left\{\frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)\right\} = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{p^2}{2\sigma^2}\right).$$

# 19

# *Wavelets and Wavelet Transforms*

> "Wavelets are without doubt an exciting and intuitive concept. The concept brings with it a new way of thinking, which is absolutely essential and was entirely missing in previously existing algorithms."
>
> Yves Meyer

> "Today the boundaries between mathematics and signal and image processing have faded, and mathematics has benefited from the rediscovery of wavelets by experts from other disciplines. The detour through signal and image processing was the most direct path leading from Haar basis to Daubechies's wavelets."
>
> Yves Meyer

## 19.1 Brief Historical Remarks

The concept of "wavelets" or "ondelettes" started to appear in the literature only in the early 1980s. This new concept can be viewed as a synthesis of various ideas which originated from different disciplines including mathematics, physics and engineering. In 1982 Jean Morlet, a French geophysical engineer, first introduced the idea of wavelet transform as a new mathematical tool for seismic signal analysis. It was Alex Grossmann, a French theoretical physicist, who quickly recognized the importance of the Morlet wavelet transform which is something similar to coherent states formalism in quantum mechanics, and developed an exact inversion formula for the wavelet transform. In 1984 the joint venture of Morlet and Grossmann led to a detailed mathematical study of the continuous wavelet transforms and their various applications. It has become clear from their work that, analogous to the Fourier expansions, the wavelet theory has provided a new method for decomposing a function or a signal.

In 1985 Yves Meyer, a French pure mathematician, recognized immediately the deep connection between the Calderón formula in harmonic analysis and the new algorithm discovered by Morlet and Grossmann. Using the knowledge of the Calderón-Zygmund operators and the Littlewood-Paley theory, Meyer was able to give a mathematical foundation for wavelet theory. The first major achievement of wavelet analysis was Daubechies, Grossmann and Meyer's (1986) construction of a "painless" non-orthogonal wavelet expansion. During 1985-1986, further work of Meyer and Lemarié on the first construction of a smooth orthonormal wavelet basis on $\mathbb{R}$ and $\mathbb{R}^N$ marked the beginning of their famous contributions to the wavelet theory. At the same time, Stéphans Mallat recognized that some quadratic mirror filters play an important role for the construction of orthogonal wavelet bases generalizing the Haar system. Meyer (1986) and Mallat (1988) realized that the orthogonal wavelet bases could be constructed systematically from a general formalism. Their collaboration culminated with the remarkable discovery by Mallat (1989 a,b) of a new formalism, which is the so called *multiresolution analysis*. It was also Mallat who constructed the wavelet decomposition and reconstruction algorithms using the multiresolution analysis. Mallat's brilliant work was the major source of many new developments in wavelets. A few months later, G. Battle (1987) and Lamarié (1988) independently proposed the construction of spline orthogonal wavelets with exponential decay.

Inspired by the work of Meyer, Ingrid Daubechies (1988) made a new remarkable contribution to *wavelet theory* by constructing families of compactly supported orthonormal wavelets with some degree of smoothness. Her 1988 paper had a tremendous positive impact on the study of wavelets and their diverse applications. This work significantly explained the connection between the continuous wavelets on $\mathbb{R}$, and the discrete wavelets on $\mathbb{Z}$ or $\mathbb{Z}_{\mathbb{N}}$, where the latter has become useful for digital signal analysis. The idea of frames was introduced by Duffin and Schaeffer (1952) and subsequently studied in some detail by Daubechies (1990, 1992). In spite of tremendous success, experts in wavelet theory recognized that it is difficult to construct wavelets that are symmetric, orthogonal and compactly supported. In order to overcome this difficulty, Cohen et al. (1992a,b) studied bi-orthogonal wavelets in some detail. Chui and Wang (1991, 1992) introduced compactly supported spline wavelets, and semi-orthogonal wavelet analysis. On the other hand, Beylkin, Coifman and Rokhlin (1991), and Beylkin (1992) have successfully applied the multiresolution analysis generated by a completely orthogonal scaling function to study a wide variety of integral operators on $L^2(\mathbb{R})$ by a matrix in a wavelet basis. This work culminated with the remarkable discovery of new algorithms in numerical analysis. Consequently, some significant progress has been made in boundary element methods, finite element methods, and numerical solutions of partial differential equations using wavelet analysis. For more detailed historical introduction, the reader is referred to Debnath (2002).

We close this historical introduction by citing some of the applications which include addressing problems in signal processing, computer vision, seismology, turbulence, computer graphics, image processing, structures of the galaxies in the Universe, digital communication, pattern recognition, approximation theory, quantum optics, biomedical engineering, sampling theory, matrix theory, operator theory, differential equations, numerical analysis, statistics and multiscale segmentation of well logs, natural scenes and mammalian visual systems. Wavelets allow complex information such as music, speech, images, patterns, etc., to be decomposed into elementary form, called building blocks (wavelets).

## 19.2 Continuous Wavelet Transforms

An integral transform is an operator $T$ on a space of functions on some $\Omega \subset \mathbb{R}^N$ which is defined by

$$(Tf)(y) = \int_{\Omega} K(x,y) f(x) dx.$$

The properties of the transform depend on the function $K$, which is called the *kernel* of the transform. For example, in the case of the Fourier transform $K(x,y) = e^{-ixy}$. Note that $y$ can be interpreted as a scaling factor. We take the exponential function $\varphi(x) = e^{ix}$ and then generate a one parameter family of functions by taking scaled copies of $\varphi$, that is $\varphi_\alpha(x) = e^{-i\alpha x}$, for all $\alpha \in \mathbb{R}$. The continuous wavelet transform is similar to the Fourier transform in the sense that it is based on a single function $\psi$ and that this function is scaled. But unlike the Fourier transform, we also shift the function, thus, generating a two parameter family of functions $\psi_{a,b}$. It is convenient to define $\psi_{a,b}$ as follows:

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}} \psi\left(\frac{x-b}{a}\right).$$

Then the *continuous wavelet* transform is defined by

$$(W_\psi f)(a,b) = \int_{-\infty}^{\infty} f(t)\, \overline{\psi_{a,b}(t)}\, dt = |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t)\, \overline{\psi\left(\frac{t-b}{a}\right)}\, dt.$$

The continuous wavelet transform is not a single transform as the Fourier transform, but any transform obtained in this way. Properties of a particular transform will depend on the choice of $\psi$. One of the first properties we expect of any integral transform is that the original function can be reconstructed from the transform. We will prove a theorem which gives conditions on $\psi$ that guarantee invertibility of the transform. First we need to define the object of our study more precisely.

***DEFINITION 19.2.1*** *(Wavelet) By a wavelet we mean a function* $\psi \in$

*$L^2(\mathbb{R})$ satisfying the admissibility condition*

$$\int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty, \tag{19.2.1}$$

*where $\widehat{\psi}$ is the Fourier transform $\psi$, i.e.,*

$$\widehat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} \psi(x) dx.$$

If $\psi \in L^2(\mathbb{R})$, then $\psi_{a,b}(x) \in L^2(\mathbb{R})$ for all $a, b$. Indeed,

$$\|\psi_{a,b}(t)\|^2 = |a|^{-1} \int_{-\infty}^{\infty} \left|\psi\left(\frac{x-b}{a}\right)\right|^2 dt = \int_{-\infty}^{\infty} |\psi(t)|^2 dt = \|\psi\|^2. \tag{19.2.2}$$

The Fourier transform of $\psi_{a,b}(x)$ is given by

$$\hat{\psi}_{a,b}(\omega) = |a|^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} \psi\left(\frac{x-b}{a}\right) dx = \sqrt{|a|}\, e^{-ib\omega} \hat{\psi}(a\omega). \tag{19.2.3}$$

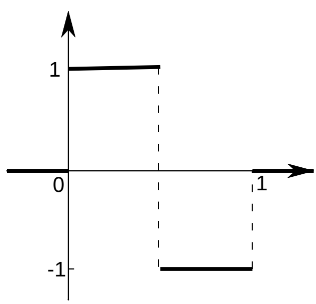


**Figure 19.1** The Haar wavelet.

***Example 19.2.1***
(*The Haar Wavelet*) Let

$$\psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The Haar wavelet is shown in Figure 19.1.

Then the Fourier transform $\widehat{\psi}(\omega) = \mathscr{F}\{\psi(x)\}$ is

$$\widehat{\psi}(\omega) = \frac{1}{\sqrt{(2\pi)}} \frac{(\sin\frac{\omega}{4})^2}{\frac{\omega}{4}} e^{-i(\omega-\pi)/2} = \frac{1}{\sqrt{(2\pi)}} \left(\frac{4i}{\omega}\right) e^{-\frac{i\omega}{2}} \sin^2\left(\frac{\omega}{4}\right).$$

and

$$\int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} d\omega = \frac{8}{\pi} \int_{-\infty}^{\infty} \frac{|\sin\frac{\omega}{4}|^4}{|\omega|^3} d\omega < \infty.$$

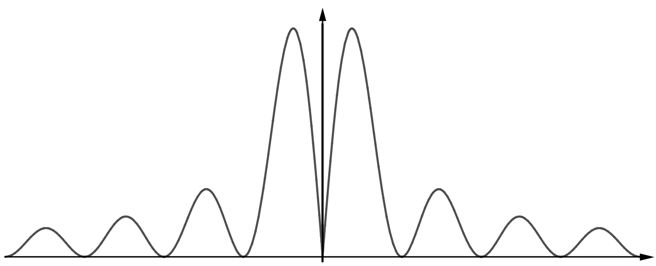

**Figure 19.2** The absolute value of the Fourier transform of the Haar wavelet.

The Haar wavelet is one of the classic examples. It is well-localized in the time domain, but it is not continuous. The absolute value of the Fourier transform of the Haar wavelet, $|\hat{\psi}(\omega)|$, is plotted in Figure 19.2. This figure clearly indicates that the Haar wavelet has poor frequency localization, since it does not have compact support in the frequency domain. The function $|\hat{\psi}(\omega)|$ is even and attains its maximum at the frequency $\omega_0 \sim 4.662$. The rate of decay as $\omega \to \infty$ is as $\omega^{-1}$. The reason for the slow decay is discontinuity of $\psi$. Its discontinuous nature is a serious weakness in many applications. However, the Haar wavelet is one of the most fundamental examples that illustrate major features of the general wavelet theory. $\square$

***THEOREM 19.2.1***

Let $\psi$ be a wavelet and let $\varphi$ be a bounded integrable function. Then the function $\psi * \varphi$ is a wavelet.

**PROOF** Since

$$\begin{aligned}\int_{-\infty}^{\infty} |\psi * \varphi(x)|^2\, dx &= \int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} \psi(x-u)\varphi(u)du\right|^2 dx \\ &\le \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\psi(x-u)||\varphi(u)|du\right)^2 dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\psi(x-u)||\varphi(u)|^{1/2}|\varphi(u)|^{1/2}du\right)^2 dx \\ &\le \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\psi(x-u)|^2|\varphi(u)|du \int_{-\infty}^{\infty} |\varphi(u)|du\right) dx \\ &\le \int_{-\infty}^{\infty} |\varphi(u)|du \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x-u)|^2|\varphi(u)|dxdu \\ &= \left(\int_{-\infty}^{\infty} |\varphi(u)|du\right)^2 \int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty,\end{aligned}$$

we have $\psi * \varphi \in L^2(\mathbb{R})$. Moreover,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{|\widehat{\psi * \varphi}(\omega)|^2}{|\omega|} d\omega = \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega)\widehat{\varphi}(\omega)|^2}{|\omega|} d\omega &= \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} |\widehat{\varphi}(\omega)|^2 d\omega \\ &\le \sup |\widehat{\varphi}(\omega)|^2 \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty\end{aligned}$$

Thus, the function $\psi * \varphi$ is a wavelet. ∎

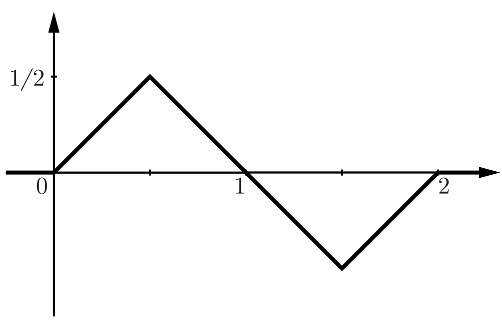


**Figure 19.3** A continuous wavelet.

***Example 19.2.2***

Theorem 19.2.1 can be used to generate examples of wavelets. For example, if we take the Haar wavelet and convolve it with the following function

$$\varphi(x)=\begin{cases}0 & x<0\\ 1 & 0\le x\le 1,\\ 0 & x\ge 1,\end{cases}$$

then we obtain a simple continuous function (see Figure 19.3). If we convolve the Haar wavelet with $\varphi(x)=e^{-x^2}$, then the obtained wavelet is smooth (see Figure 19.4). ▯

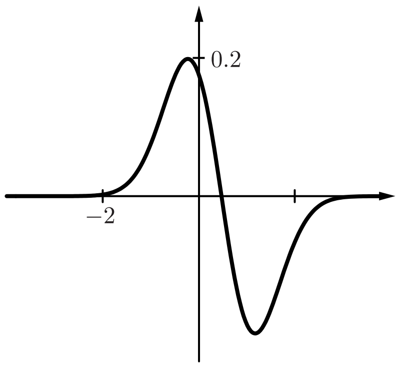


**Figure 19.4** A smooth wavelet.

***DEFINITION 19.2.2*** *(Continuous Wavelet Transform). Let $\psi\in L^2(\mathbb{R})$ and let, for $a,b\in\mathbb{R}$, $a\neq 0$,*

$$\psi_{a,b}(x)=|a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right).$$

*The integral transform $W_\psi$ defined on $L^2(\mathbb{R})$ by*

$$(W_\psi f)(a,b)=\int_{-\infty}^{\infty} f(t)\overline{\psi_{a,b}(t)}\,dt=\langle f,\psi_{a,b}\rangle \tag{19.2.4}$$

*is called a continuous wavelet transform.*

The function $\psi$ is often called the *mother wavelet* or, the *analyzing wavelet.* The parameter $b$ can be interpreted as the time translation and $a$ is a scaling parameter which measures the degree of compression.

***LEMMA 19.2.1***
*For any $f \in L^2(\mathbb{R})$, we have*

$$\mathscr{F}\{(W_\psi f)(a,b)\} = \sqrt{2\pi|a|}\,\hat{f}(\omega)\overline{\hat{\psi}(a\omega)}. \tag{19.2.5}$$

**PROOF** Using the Parseval formula for the Fourier transform, it follows from (19.2.4) that

$$\begin{aligned}(W_\psi f)(a,b) &= \langle f, \psi_{a,b}\rangle = \left\langle \hat{f}, \hat{\psi}_{a,b}\right\rangle \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\sqrt{2\pi|a|}\,\hat{f}(\omega)\,\overline{\hat{\psi}(a\omega)}\right\}e^{ib\omega}d\omega. \end{aligned}\tag{19.2.6}$$

This means that

$$\begin{aligned}\mathscr{F}\{(W_\psi f)(a,b)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ib\omega}(W_\psi f)(a,b)db \\ &= \sqrt{2\pi|a|}\,\hat{f}(\omega)\,\overline{\hat{\psi}(a\omega)}. \end{aligned}\tag{19.2.7}$$

∎

***THEOREM 19.2.2***
(*Parseval's Relation for Wavelet Transforms*). Suppose that $\psi \in L^2(\mathbb{R})$ which satisfy the admissibility condition

$$C_\psi = 2\pi\int_{-\infty}^{\infty}\frac{|\widehat{\psi}(\omega)|^2}{|\omega|}d\omega < \infty. \tag{19.2.8}$$

Then, for any $f, g \in L^2(\mathbb{R})$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(W_\psi f)(a,b)\overline{(W_\psi g)(a,b)}\,\frac{dbda}{a^2} = C_\psi\langle f, g\rangle. \tag{19.2.9}$$

**PROOF** From (19.2.6) we get

$$(W_\psi f)(a,b) = \sqrt{|a|}\int_{-\infty}^{\infty}\widehat{f}(\omega)\,e^{ib\omega}\,\overline{\widehat{\psi}(a\omega)}\,d\omega \tag{19.2.10}$$

and

$$\overline{(W_\psi g)(a,b)} = \sqrt{|a|}\int_{-\infty}^{\infty}\overline{\widehat{g}(\sigma)}\,e^{-ib\sigma}\,\widehat{\psi}(a\sigma)\,d\sigma. \tag{19.2.11}$$

Substituting (19.2.10) and (19.2.11) in the left hand-side of (19.2.9) gives

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(W_\psi f)(a,b)\overline{(W_\psi g)(a,b)}\,\frac{dbda}{a^2} \\ &\quad= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dbda}{a^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|a|\widehat{f}(\omega)\,\overline{\widehat{g}(\sigma)}\,\overline{\widehat{\psi}(a\omega)}\,\widehat{\psi}(a\sigma)e^{ib(\omega-\sigma)}d\omega d\sigma\end{aligned}$$

which is, by interchanging the order of integration,

$$= 2\pi \int_{-\infty}^{\infty} \frac{da}{|a|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f}(\omega)\, \overline{\widehat{g}(\sigma)}\, \overline{\widehat{\psi}(a\omega)}\, \widehat{\psi}(a\sigma)\, d\omega d\sigma \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ib(\omega-\sigma)} db$$
$$= 2\pi \int_{-\infty}^{\infty} \frac{da}{|a|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f}(\omega)\, \overline{\widehat{g}(\sigma)}\, \overline{\widehat{\psi}(a\omega)}\, \widehat{\psi}(a\sigma)\delta(\sigma-\omega)\, d\omega d\sigma$$
$$= 2\pi \int_{-\infty}^{\infty} \frac{da}{|a|} \int_{-\infty}^{\infty} \widehat{f}(\omega)\, \overline{\widehat{g}(\omega)} \left|\widehat{\psi}(a\omega)\right|^2 d\omega$$

and finally, again interchanging the order of integration and putting $a\omega = x$,

$$= 2\pi \int_{-\infty}^{\infty} \widehat{f}(\omega)\, \overline{\widehat{g}(\omega)}\, d\omega \int_{-\infty}^{\infty} \frac{\left|\widehat{\psi}(x)\right|^2}{|x|} dx = C_\psi \left\langle \widehat{f}, \widehat{g} \right\rangle = C_\psi \langle f, g \rangle . \quad (19.2.12)$$

∎

If $f = g$, then (19.2.9) assumes the form

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(W_\psi f)(a,b)|^2 \frac{dbda}{a^2} = C_\psi \|f\|^2. \quad (19.2.13)$$

***THEOREM 19.2.3***
(*Inversion formula*). If $f \in L^2(\mathbb{R})$, then

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)(a,b)\psi_{a,b}(x) \frac{dbda}{a^2}, \quad (19.2.14)$$

where the equality holds almost everywhere.

**PROOF** For any $g \in L^2(\mathbb{R})$, we have

$$C_\psi \langle f, g \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)(a,b) \overline{(W_\psi g)(a,b)} \frac{dbda}{a^2}$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)(a,b) \overline{\int_{-\infty}^{\infty} g(t) \overline{\psi_{a,b}(t)} dt} \frac{dbda}{a^2}$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)(a,b)\psi_{a,b}(t) \frac{dbda}{a^2} \overline{g(t)} dt$$
$$= \left\langle \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)(a,b)\psi_{a,b} \frac{dbda}{a^2},\ g \right\rangle .$$

Since $g$ is an arbitrary element of $L^2(\mathbb{R})$, the inversion formula 19.2.14 follows.
∎

The following theorem summarizes some elementary properties of the continuous wavelet transform. Proofs are straightforward, hence, are left as exercises.

***THEOREM 19.2.4***

Suppose $\psi$ and $\varphi$ are wavelets and let $f, g \in L^2(\mathbb{R})$.

(i) $(W_\psi(\alpha f + \beta g))(a,b) = \alpha(W_\psi f)(a,b) + \beta(W_\psi g)(a,b)$ for any $\alpha, \beta \in \mathbb{C}$,

(ii) $(W_\psi(T_c f))(a,b) = (W_\psi f)(a, b-c)$, where $T_c$ is the translation operator defined by $T_c f(t) = f(t-c)$,

(iii) $(W_\psi(D_c f))(a,b) = \frac{1}{\sqrt{c}}(W_\psi f)\left(\frac{a}{c}, \frac{b}{c}\right)$, where $c$ is a positive number and $D_c$ is the dilation operator defined by $D_c f(t) = \frac{1}{c} f(\frac{t}{c})$,

(iv) $(W_\psi \varphi)(a,b) = \overline{(W_\varphi \psi)\left(\frac{1}{a}, -\frac{b}{a}\right)}$, $a \neq 0$,

(v) $(W_{\alpha\psi+\beta\varphi} f)(a,b) = \overline{\alpha}\,(W_\psi f)(a,b) + \overline{\beta}\,(W_\varphi f)(a,b)$ for any $\alpha, \beta \in \mathbb{C}$,

(vi) $(W_{P\psi} P f)(a,b) = (W_\psi f)(a, -b)$, where $P$ is the parity operator defined by $Pf(t) = f(-t)$,

(vii) $(W_{T_c\psi} f)(a,b) = (W_\psi f)(a, b + ca)$,

(viii) $(W_{D_c\psi} f)(a,b) = \frac{1}{\sqrt{c}}(W_\psi f)(ac, b)$, $c > 0$.

For the wavelets to be useful analyzing functions, the mother wavelet must have certain properties. One such property is defined by the admissibility condition (19.2.1) which guarantees existence of the inversion formula for the continuous wavelet transform. If $\psi \in L^1(\mathbb{R})$, then its Fourier transform $\widehat{\psi}$ is continuous. If $\widehat{\psi}$ is continuous, $C_\psi$ can be finite only if $\widehat{\psi}(0) = 0$, or, equivalently, $\int_{-\infty}^{\infty} \psi(t)dt = 0$. This means that $\psi$ must be an oscillatory function with zero mean. Condition (19.2.1) also imposes a restriction on the rate of decay of $|\widehat{\psi}(\omega)|^2$.

In addition to the admissibility condition (19.2.1), there are other properties that may be useful in particular applications. For example, it may be necessary to require that $\psi$ be $n$ times continuously differentiable or infinitely differentiable. If the Haar wavelet is convolved $(n+1)$ times with the function $\varphi$ given in Example 19.2.2, then the resulting function $\psi * \varphi * \cdots * \varphi$ is an $n$ times differentiable wavelet. The function in Figure 19.4 is an infinitely differentiable wavelet. The so-called "Mexican hat wavelet" is another example of an infinitely differentiable wavelet.

***Example 19.2.3***

(*Mexican Hat Wavelet*). This wavelet is defined by

$$\psi(t) = (1 - t^2)e^{-at^2/2}$$

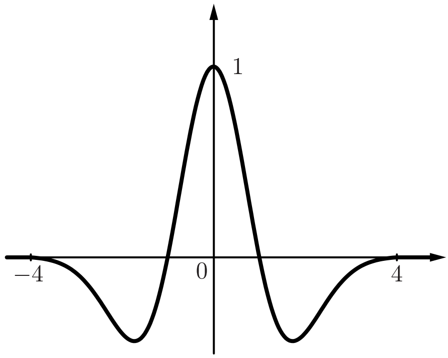


**Figure 19.5** Mexican hat wavelet.

and shown in Figure 19.5 with $a = 1$. ▯

Another desirable property of wavelets is the so called "localization property." We want $\psi$ to be well localized in both time and frequency domains. In other words, $\psi$ and its derivatives must decay very rapidly. For frequency localization, $\widehat{\psi}(\omega)$ must decay sufficiently rapidly as $\omega \to \infty$ and $\widehat{\psi}(\omega)$ should be flat in the neighborhood of $\omega = 0$. The flatness at $\omega = 0$ is associated with the number of vanishing moments of $\psi$. The $k$-th moment of $\psi$ is defined by

$$m_k = \int_{-\infty}^{\infty} t^k \psi(t) dt.$$

A wavelet is said to have $n$ vanishing moments if

$$\int_{-\infty}^{\infty} t^k \psi(t) dt = 0 \qquad \text{for } k = 0, 1, \ldots, n.$$

Or, equivalently,

$$\left[\frac{d^k \widehat{\psi}(\omega)}{d\omega^k}\right]_{\omega=0} = 0 \qquad \text{for } k = 0, 1, \ldots, n.$$

Wavelets with a larger number of vanishing moments result in more flatness when frequency $\omega$ is small.

## 19.3 The Discrete Wavelet Transform

While the continuous wavelet transform is compared to the Fourier transform, which requires calculating the integral $\int_{-\infty}^{\infty} e^{-i\omega x} f(x) dx$ for all (or, almost all)

$\omega \in \mathbb{R}$, the discrete wavelet transform can be compared to the Fourier series, which requires calculating the integral $\int_0^{2\pi} e^{-inx} f(x)dx$ for integer values of $n$. Since the continuous wavelet transform is a two parameter representation of a function

$$(W_\psi f)(a,b) = |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t)\, \overline{\psi\left(\frac{t-b}{a}\right)}\, dt,$$

we can discretize it by assuming that $a$ and $b$ take only integer values. It turns out that it is better to discretize it in a different way. First we fix two positive constants $a_0$ and $b_0$ and then define

$$\psi_{m,n}(x) = a_0^{-m/2} \psi(a_0^{-m} x - nb_0), \tag{19.3.1}$$

where $m$ and $n$ range over $\mathbb{Z}$. By the *discrete wavelet coefficients* of $f \in L^2(\mathbb{R})$ we mean the numbers $\langle f, \psi_{m,n} \rangle$, where $m, n \in \mathbb{Z}$. The fundamental question here is whether it is possible to reconstruct $f$ from those coefficients. The weakest interpretation of this problem is whether $\langle f, \psi_{m,n} \rangle = \langle g, \psi_{m,n} \rangle$ for all $m, n \in \mathbb{Z}$ implies $f = g$. In practice we expect much more than that: we want $\langle f, \psi_{m,n} \rangle$ and $\langle g, \psi_{m,n} \rangle$ to be "close" if $f$ and $g$ are "close." This will be guaranteed if there exists a $B > 0$, such that

$$\sum_{m,n=-\infty}^{\infty} |\langle f, \psi_{m,n} \rangle|^2 \leq B \|f\|^2$$

for all $f \in L^2(\mathbb{R})$. Similarly, we want $f$ and $g$ to be "close" if $\langle f, \psi_{m,n} \rangle$ and $\langle g, \psi_{m,n} \rangle$ are "close." This is important because we want to be sure that when we neglect some small terms in the representation of $f$ in terms of $\langle f, \psi_{m,n} \rangle$, then the reconstructed function will not differ much from $f$. The representation will have this property if there exists an $A > 0$, such that

$$A \|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, \psi_{m,n} \rangle|^2$$

for all $f \in L^2(\mathbb{R})$. These two requirements are best investigated in terms of the so-called frames.

***DEFINITION 19.3.1*** *(Frame). A sequence $(\varphi_1, \varphi_2, \cdots)$ in a Hilbert space $H$ is called a frame if there exist $A, B > 0$ such that*

$$A\,||f||^2 \leq \sum_{n=1}^{\infty} |\langle f,\, \varphi_n \rangle|^2 \leq B\,||f||^2 \tag{19.3.2}$$

*for all $f \in H$. The constants $A$ and $B$ are called frame bounds. If $A = B$, then the frame is called tight.*

If $(\varphi_n)$ is an orthonormal basis, then it is a tight frame since $\sum_{n=1}^{\infty} |\langle f,\, \varphi_n\rangle|^2 = ||f||^2$ for all $f \in H$. The vectors $(1, 0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ form a tight frame in $\mathbb{C}^2$ which is not a basis.

## 19.4 Examples of Orthonormal Wavelets

Since the discovery of wavelets, orthonormal wavelets play an important role in the wavelet theory and have a variety of applications. In this section we discuss several examples of orthonormal wavelets.

***DEFINITION 19.4.1*** *(Orthonormal Wavelet). A wavelet $\psi \in L^2(\mathbb{R})$ is called orthonormal if the family of functions $\psi_{m,n}$ generated from $\psi$ by*

$$\psi_{m,n}(x) = 2^{m/2}\psi\left(2^m\left(x - \frac{n}{2^m}\right)\right) = 2^{m/2}\psi(2^m x - n), \quad m, n \in \mathbb{Z}, \tag{19.4.1}$$

*is orthonormal, that is,*

$$\langle \psi_{m,n}, \psi_{k,\ell}\rangle = \int_{-\infty}^{\infty} \psi_{m,n}(x)\, \psi_{k,\ell}(x)\, dx = \delta_{m,k}\, \delta_{n,\ell}, \tag{19.4.2}$$

*for all $m, n, k, \ell \in \mathbb{Z}$.*

The following lemma is often useful when dealing with orthogonality of wavelets.

***LEMMA 19.4.1***
*If $\psi, \varphi \in L^2(\mathbb{R})$, then*

$$\langle \psi_{m,k},\, \varphi_{m,\ell}\rangle = \langle \psi_{n,k},\, \varphi_{n,\ell}\rangle, \tag{19.4.3}$$

*for all $m, n, k, \ell \in \mathbb{Z}$.*

**PROOF** We have

$$\langle \psi_{m,k}, \varphi_{m,\ell}\rangle = \int_{-\infty}^{\infty} 2^m\, \psi(2^m x - k)\, \varphi(2^m x - \ell)\, dx,$$

which is, by assuming $2^m x = 2^n t$,

$$\langle \psi_{m,k}, \varphi_{m,\ell}\rangle = \int_{-\infty}^{\infty} 2^n\, \psi(2^n t - k)\, \varphi(2^n t - \ell)\, dx = \langle \psi_{n,k}, \varphi_{n,\ell}\rangle.$$

∎

***Example 19.4.1***
(*The Haar Wavelet*). The simplest example of an orthonormal wavelet is the classic Haar wavelet. We consider the scaling function $\varphi = \chi_{[0,1)}$. The function $\varphi$ satisfies the dilation equation

$$\varphi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} c_n \varphi(2x - n), \tag{19.4.4}$$

where the coefficients $c_n$ are given by

$$c_n = \sqrt{2} \int_{-\infty}^{\infty} \varphi(x) \varphi(2x - n)\, dx. \tag{19.4.5}$$

Evaluating this integral with $\varphi = \chi_{[0,1)}$ gives $c_n$ as follows:

$$c_0 = c_1 = \frac{1}{\sqrt{2}} \quad \text{and} \quad c_n = 0 \quad \text{for} \quad n > 1.$$

Consequently, the dilation equation becomes

$$\varphi(x) = \varphi(2x) + \varphi(2x - 1). \tag{19.4.6}$$

This means that $\varphi(x)$ is a linear combination of the even and odd translates of $\varphi(2x)$ and satisfies a very simple two-scale relation (19.4.6), as shown in Figure 19.6.

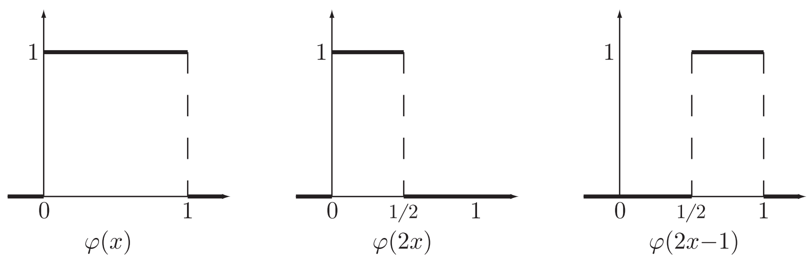


**Figure 19.6** Two-scale relation of $\varphi(x) = \varphi(2x) + \varphi(2x - 1)$.

The Haar mother wavelet is obtained as a simple two-scale relation

$$\psi(x) = \varphi(2x) - \varphi(2x - 1) \tag{19.4.7}$$

$$= \chi_{[0,\frac{1}{2}]}(x) - \chi_{[\frac{1}{2},1]}(x)$$

$$= \left\{ \begin{array}{cc} 1 & \text{if } 0 \le \mathrm{x} < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \le \mathrm{x} < 1, \\ 0 & \text{otherwise.} \end{array} \right\}. \tag{19.4.8}$$

For any $m, n \in \mathbb{Z}$, we have

$$\psi_{m,n}(t) = 2^{-m/2}\psi\left(2^{-m}t - n\right) = \begin{cases} 2^{-m/2} & 2^m n \le t < 2^m n + 2^{m-1}, \\ -2^{-m/2} & 2^m n + 2^{m-1} \le t < 2^m n + 2^m, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\|\psi_{m,n}\|_2 = \|\psi\|_2 = 1$, for all $m, n \in \mathbb{Z}$. To verify that $\{\psi_{m,n}\}$ is an orthonormal system, we observe that

$$\langle \psi_{m,n}, \psi_{k,\ell} \rangle = \int_{-\infty}^{\infty} 2^{m/2}\psi\left(2^m x - n\right) 2^{k/2}\psi\left(2^k x - \ell\right) dx,$$

which gives, by the change of variables $2^m x - n = t$,

$$\langle \psi_{m,n}, \psi_{k,\ell} \rangle = 2^{k/2} 2^{-m/2} \int_{-\infty}^{\infty} \psi(t)\, \psi\left(2^{k-m}(t+n) - \ell\right) dt. \tag{19.4.9}$$

For $m = k$, we obtain

$$\langle \psi_{m,n}, \psi_{m,\ell} \rangle = \int_{-\infty}^{\infty} \psi(t)\, \psi(t + n - \ell)\, dt = \delta_{0,n-\ell} = \delta_{n,\ell}, \tag{19.4.10}$$

where $\psi(t) \ne 0$ in $0 \le t < 1$ and $\psi\left(t - \overline{\ell - n}\right) \ne 0$ in $\ell - n \le t < 1 + \ell - n$, and these intervals are disjoint from each other unless $n = \ell$.

We now consider the case $m \ne k$. In view of symmetry, it suffices to consider the case $m > k$. Putting $r = m - k > 0$ in (19.4.9), we obtain, for $m > k$,

$$\langle \psi_{m,n}, \psi_{m,\ell} \rangle = 2^{r/2} \int_{-\infty}^{\infty} \psi(t)\, \psi\left(2^r t + s\right) dt, \tag{19.4.11}$$

where $s = 2^r n - \ell$. Thus, it suffices to show that

$$\int_0^{\frac{1}{2}} \psi\left(2^r t + s\right) dt - \int_{\frac{1}{2}}^{1} \psi\left(2^r t + s\right) dt = 0.$$

Using a simple change of variables $2^r t + s = x$, we find

$$\int_0^{\frac{1}{2}} \psi\left(2^r t + s\right) dt - \int_{\frac{1}{2}}^{1} \psi\left(2^r t + s\right) dt = \int_s^a \psi(x)\, dx - \int_a^b \psi(x)\, dx, \tag{19.4.12}$$

where $a = s + 2^{r-1}$ and $b = s + 2^r$. Since the interval $[s, a]$ contains the support $[0, 1]$ of $\psi$, the first integral in (19.4.12) is zero. Similarly, the second integral is also zero. $\square$

***Example 19.4.2***
(*The Shannon Wavelet*). The function $\psi$ whose Fourier transform satisfies

$$\widehat{\psi}(\omega) = \chi_I(\omega), \tag{19.4.13}$$

where $I = [-2\pi, -\pi] \cup [\pi, 2\pi]$, is called the *Shannon wavelet*. The function $\psi$ can directly be obtained from the inverse Fourier transform of $\widehat{\psi}$ so that

$$\begin{aligned}\psi(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\widehat{\psi}(\omega)\,d\omega \\ &= \frac{1}{2\pi}\left[\int_{-2\pi}^{-\pi} e^{i\omega t}d\omega + \int_{\pi}^{2\pi} e^{i\omega t}d\omega\right] \\ &= \frac{1}{\pi t}(\sin 2\pi t - \sin \pi t) = \frac{\sin\left(\frac{\pi t}{2}\right)}{\left(\frac{\pi t}{2}\right)}\cos\left(\frac{3\pi t}{2}\right). \qquad (19.4.14)\end{aligned}$$

This function is orthonormal to its translates by integers. Indeed, by Parseval's relation,

$$\begin{aligned}\langle \psi(t), \psi(t-n)\rangle &= \frac{1}{2\pi}\left\langle \widehat{\psi}, e^{in\omega}\widehat{\psi}\right\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\psi}(\omega)\, e^{in\omega}\overline{\widehat{\psi}}(\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} e^{in\omega}d\omega = \delta_{0,n}.\end{aligned}$$

The wavelet basis is now given by

$$\psi_{m,n}(t) = 2^{-m/2}\,\psi\left(2^{-m}t - n - \frac{1}{2}\right), \qquad m, n \in \mathbb{Z}$$

or,

$$\psi_{m,n}(t) = 2^{-\frac{m}{2}}\,\frac{\sin\left\{\frac{\pi}{2}\left(2^{-m}t - n\right)\right\}}{\frac{\pi}{2}\left(2^{-m}t-n\right)}\cos\left\{\frac{3\pi}{2}\left(2^{-m}t - n\right)\right\}. \qquad (19.4.15)$$

For any fixed $n \in \mathbb{Z}$, the functions $\psi_{m,n}(t)$ form a basis for the space of functions supported on the interval

$$\left[-2^{-m+1}\pi,\ -2^{-m}\pi\right] \cup \left[2^{-m}\pi,\ 2^{-m+1}\pi\right].$$

The system $\{\psi_{m,n}(t)\}$, $m, n \in \mathbb{Z}$, is an orthonormal basis for $L^2(\mathbb{R})$. Both $\psi(t)$ and $\widehat{\psi}(\omega)$ are shown in Figure 19.7.

The Fourier transform of $\psi_{m,n}$ is

$$\widehat{\psi}_{m,n}(\omega) = \begin{cases} 2^{m/2}\exp\left(-i\,\omega\, n\, 2^m\right) & \text{if } 2^{-m}\pi < |\omega| < 2^{-m+1}\pi, \\ 0 & \text{otherwise.}\end{cases} \qquad (19.4.16)$$

Evidently, $\widehat{\psi}_{m,n}$ and $\widehat{\psi}_{k,\ell}$ do not overlap for $m \neq k$. Hence, by the Parseval relation [(equation (3.4.37), Debnath, 2002)], it turns out that, for $m \neq k$,

$$\langle \psi_{m,n}, \psi_{k,\ell}\rangle = \frac{1}{2\pi}\left\langle \widehat{\psi}_{m,n}, \widehat{\psi}_{k,\ell}\right\rangle = 0. \qquad (19.4.17)$$

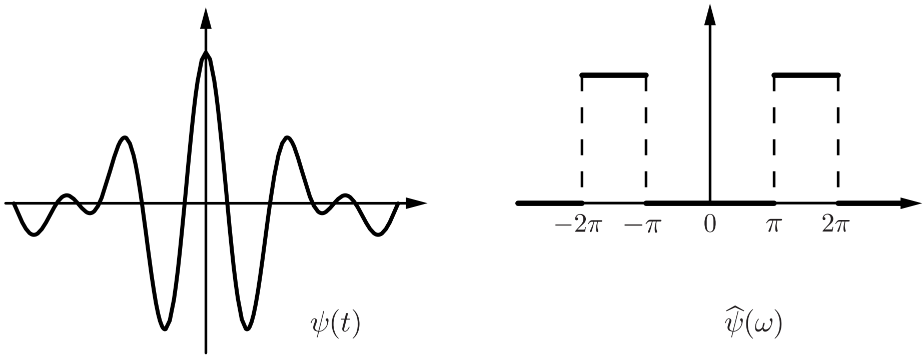


**Figure 19.7** The Shannon wavelet and its Fourier transform.

For $m = k$, we have

$$\begin{aligned}\langle \psi_{m,n}, \psi_{k,\ell} \rangle &= \frac{1}{2\pi} \left\langle \widehat{\psi}_{m,n}, \widehat{\psi}_{m,\ell} \right\rangle \\ &= \frac{1}{2\pi} 2^{-m} \int_{-\infty}^{\infty} \exp\left\{-i\,\omega\, 2^{-m}\,(n-\ell)\right\} \left|\widehat{\psi}\left(2^{-m}\omega\right)\right|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{-i\sigma\,(n-\ell)\right\} d\sigma = \delta_{n,\ell}. \end{aligned} \tag{19.4.18}$$

This shows that $\{\psi_{m,n}\,(t)\}$ is an orthonormal system. ▯

***Example 19.4.3***
(*The Daubechies Wavelets and Algorithms*). Daubechies (1988, 1992) first developed the theory and construction of continuous orthonormal wavelets with compact support. Wavelets with compact support have many interesting properties. They can be constructed to have a given number of derivatives and to have a given number of vanishing moments.

We assume that the scaling function $\varphi$ satisfies the dilation equation

$$\varphi\,(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} c_n\, \varphi\,(2x - n)\,, \tag{19.4.19}$$

where $c_n = \langle \varphi, \varphi_{1,n} \rangle$ and $\sum_{n=-\infty}^{\infty} |c_n|^2 \le \infty$.

If the scaling function $\varphi$ has compact support, then only a finite number of $c_n$ have nonzero values. The associated generating function $\widehat{m}$,

$$\widehat{m}\,(\omega) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} c_n\, e^{-i\omega n} \tag{19.4.20}$$

is a trigonometric polynomial and it satisfies the identity [(see equation (7.3.4), Debnath, 2002)] with special values $\widehat{m}(0) = 1$ and $\widehat{m}(\pi) = 0$. If coefficients $c_n$ are real, then the corresponding scaling function, as well as the mother wavelet $\psi$, will also be real-valued. The mother wavelet $\psi$ corresponding to $\varphi$ is given by the formula [(see equation (7.3.24), Debnath, 2002)] with $|\widehat{\varphi}(0)| = 1$. The Fourier transform $\widehat{\psi}(\omega)$ is $m$-times continuously differentiable and it satisfies the moment condition

$$\widehat{\psi}^{(k)}(0) = 0 \quad \text{for} \quad k = 0, 1, \ldots, m. \tag{19.4.21}$$

It follows that $\psi \in \mathscr{C}^m(\mathbb{R})$ implies that $\widehat{m}_0$ has a zero at $\omega = \pi$ of order $(m+1)$. In other words,

$$\widehat{m}_0(\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right)^{m+1} \widehat{L}(\omega), \tag{19.4.22}$$

where $\widehat{L}$ is a trigonometric polynomial.

In addition to the orthogonality condition [(see equation (7.3.4), Debnath, 2002)], we assume

$$\widehat{m}_0(\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right)^{N} \widehat{L}(\omega), \tag{19.4.23}$$

where $\widehat{L}(\omega)$ is $2\pi$-periodic and $\widehat{L} \in \mathscr{C}^{N-1}(\mathbb{R})$. Evidently,

$$|\widehat{m}_0(\omega)|^2 = \widehat{m}_0(\omega)\, \widehat{m}_0(-\omega) \tag{19.4.24}$$

$$= \left(\frac{1 + e^{-i\omega}}{2}\right)^{N} \left(\frac{1 + e^{i\omega}}{2}\right)^{N} \widehat{L}(\omega)\, \widehat{L}(-\omega)$$

$$= \left(\cos^2 \frac{\omega}{2}\right)^{N} \left|\widehat{L}(\omega)\right|^2, \tag{19.4.25}$$

where $\left|\widehat{L}(\omega)\right|^2$ is a polynomial in $\cos\omega$, that is,

$$\left|\widehat{L}(\omega)\right|^2 = Q(\cos\omega).$$

Since $\cos\omega = 1 - 2\sin^2\left(\frac{\omega}{2}\right)$, it is convenient to introduce $x = \sin^2\left(\frac{\omega}{2}\right)$ so that (19.4.25) reduces to the form

$$|\widehat{m}_0(\omega)|^2 = \left(\cos^2 \frac{\omega}{2}\right)^{N} Q(1 - 2x) = (1 - x)^N P(x), \tag{19.4.26}$$

where $P(x)$ is a polynomial in $x$.

We next use the fact that

$$\cos^2\left(\frac{\omega + \pi}{2}\right) = \sin^2\left(\frac{\omega}{2}\right) = x$$

and

$$\left|\widehat{L}(\omega+\pi)\right|^2 = Q(-\cos\omega) = Q(2x-1),$$
$$= Q(1-2(1-x)) = P(1-x) \tag{19.4.27}$$

to express the identity [(see equation (7.3.4), Debnath, 2002)] in terms of $x$ so that it becomes

$$(1-x)^N P(x) + x^N P(1-x) = 1. \tag{19.4.28}$$

Since $(1-x)^N$ and $x^N$ are two polynomials of degree $N$ which are relatively prime, then, by Bezout's theorem (Daubechies, 1992), there exists a unique polynomial $P_N$ of degree $\leq N-1$ such that (19.4.28) holds. An explicit solution for $P_N(x)$ is given by

$$P_N(x) = \sum_{k=0}^{N-1} \binom{N+k-1}{k} x^k, \tag{19.4.29}$$

which is positive for $0 < x < 1$ so that $P_N(x)$ is at least a possible candidate for $\left|\widehat{L}(\omega)\right|^2$. There also exist higher degree polynomial solutions $P_N(x)$ of (19.4.28) which can be written as

$$P_N(x) = \sum_{k=0}^{N-1} \binom{N+k-1}{k} x^k + x^N R\left(x - \frac{1}{2}\right), \tag{19.4.30}$$

where $R$ is an odd polynomial.

Since $P_N(x)$ is a possible candidate for $\left|\widehat{L}(\omega)\right|^2$ and

$$\widehat{L}(\omega)\,\widehat{L}(-\omega) = \left|\widehat{L}(\omega)\right|^2 = Q(\cos\omega) = Q(1-2x) = P_N(x), \tag{19.4.31}$$

the next problem is how to find out $\widehat{L}(\omega)$. This can be done by the following lemma.

***LEMMA 19.4.2***

*(Riesz's Lemma for Spectral Factorization). If*

$$\widehat{A}(\omega) = \sum_{k=0}^{n} a_k \cos^k \omega, \tag{19.4.32}$$

*where $a_k \in \mathbb{R}$ and $a_n \neq 0$, and if $\widehat{A}(\omega) \geq 0$ for all $\omega \in \mathbb{R}$ with $\widehat{A}(0) = 1$, then there exists a trigonometric polynomial*

$$\widehat{L}(\omega) = \sum_{k=0}^{n} b_k\, e^{-ik\omega} \tag{19.4.33}$$

*with real coefficients* $b_k$ *such that* $\widehat{L}(0)=1$ *and*

$$\widehat{A}(\omega)=\widehat{L}(\omega)\,\widehat{L}(-\omega)=\left|\widehat{L}(\omega)\right|^2 \tag{19.4.34}$$

*for all* $\omega\in\mathbb{R}$.

We refer to Daubechies (1992) for a proof of the Riesz Lemma. We also point out that the factorization of $\widehat{A}(\omega)$ given in (19.4.34) is not unique.

For a given $N$, if we select $P=P_N$, then $\widehat{A}(\omega)$ becomes a polynomial of degree $N-1$ in $\cos\omega$ and $\widehat{L}(\omega)$ is a polynomial of degree $(N-1)$ in $\exp(-i\omega)$. Therefore, the generating function $\widehat{m}_0$ given by (19.4.23) is of degree $(2N-1)$ in $\exp(-i\omega)$. The interval $[0,2N-1]$ becomes the support of the corresponding scaling function ${}_N\varphi$. The mother wavelet ${}_N\psi$ obtained from ${}_N\psi$ is called the *Daubechies wavelet.*

For $N=2$, it follows from (19.4.29) that

$$P_2(x)=\sum_{k=0}^{1}\binom{k+1}{k}x^k=1+2x$$

and hence, (19.4.31) gives

$$\left|\widehat{L}^2(\omega)\right|^2=P_2(x)=P_2\left(\sin^2\frac{\omega}{2}\right)=1+2\sin^2\frac{\omega}{2}=2-\cos\omega.$$

Using (19.4.33), we obtain that $\widehat{L}(\omega)$ is a polynomial of degree $N-1=1$ and

$$\widehat{L}(\omega)\,\widehat{L}(-\omega)=2-\frac{1}{2}\left(e^{i\omega}+e^{-i\omega}\right).$$

It follows from (19.4.33) that

$$\left(b_0+b_1e^{-i\omega}\right)\left(b_0+b_1e^{i\omega}\right)=2-\frac{1}{2}\left(e^{i\omega}+e^{-i\omega}\right). \tag{19.4.35}$$

Equating the coefficients in this identity gives

$$b_0^2+b_1^2=1 \quad \text{and} \quad 2b_0b_1=-1. \tag{19.4.36}$$

These equations admit solutions

$$b_0=\frac{1}{2}\left(1+\sqrt{3}\right) \quad \text{and} \quad b_1=\frac{1}{2}\left(1-\sqrt{3}\right). \tag{19.4.37}$$

Thus, the generating function (19.4.21) takes the form

$$\begin{aligned}\widehat{m}_0(\omega)&=\left(\frac{1+e^{-i\omega}}{2}\right)^2\left(b_0+b_1e^{-i\omega}\right)\\&=\frac{1}{8}\left[\left(1+\sqrt{3}\right)+\left(3+\sqrt{3}\right)e^{-i\omega}+\left(3-\sqrt{3}\right)e^{-2i\omega}+\left(1-\sqrt{3}\right)e^{-3i\omega}\right]\end{aligned} \tag{19.4.38}$$

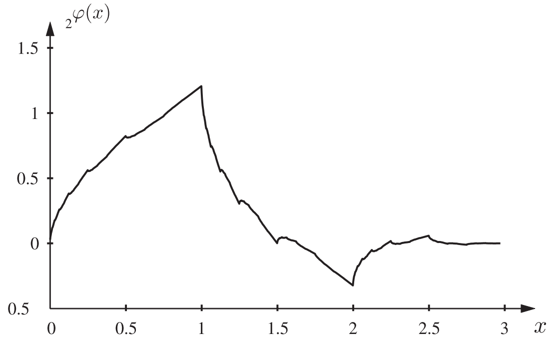


**Figure 19.8** The Daubechies scaling function $_2\varphi(x)$.

with $\widehat{m}_0(0) = 1$. Comparing coefficients of (19.4.38) with the Equation (7.3.3) (Debnath, 2002) gives $h_n = c_n$ as

$$\begin{aligned} c_0 &= \tfrac{1}{4\sqrt{2}}\left(1+\sqrt{3}\right), \quad c_1 = \tfrac{1}{4\sqrt{2}}\left(3+\sqrt{3}\right) \\ c_2 &= \tfrac{1}{4\sqrt{2}}\left(3-\sqrt{3}\right), \quad c_3 = \tfrac{1}{4\sqrt{2}}\left(1-\sqrt{3}\right). \end{aligned} \tag{19.4.39}$$

Consequently, the Daubechies scaling function $_2\varphi(x)$ takes the form, dropping the subscript,

$$\varphi(x) = \sqrt{2}\left[c_0\,\varphi(2x) + c_1\,\varphi(2x-1) + c_2\,\varphi(2x-2) + c_3\,\varphi(2x-3)\right]. \tag{19.4.40}$$

Using the equation (7.3.31) (Debnath, 2002) with $N = 2$, we obtain the Daubechies wavelet $_2\psi(x)$, dropping the subscript,

$$\begin{aligned} \psi(x) &= \sqrt{2}\left[d_0\,\varphi(2x) + d_1\,\varphi(2x-1) + d_2\,\varphi(2x-2) + d_3\,\varphi(2x-3)\right] \\ &= \sqrt{2}\left[-c_3\,\varphi(2x) + c_2\,\varphi(2x-1) - c_1\,\varphi(2x-2) + c_0\,\varphi(2x-3)\right], \end{aligned} \tag{19.4.41}$$

where the coefficients in (19.4.41) are the same as for the scaling function $\varphi(x)$, but in reverse order and with alternate terms having their signs changed from plus to minus.

On the other hand, the use of the equation (7.3.29) (see Debnath, 2002) with the equation (7.3.34) also gives the Daubechies wavelet $_2\psi(x)$ in the form

$$_2\psi(x) = \sqrt{2}\left[-c_0\,\varphi(2x-1) + c_1\,\varphi(2x) - c_2\,\varphi(2x+1) + c_3\,\varphi(2x+2)\right].$$

□

The wavelet has the same coefficients as $\psi$ given in (19.4.41) except that the wavelet is reversed in sign and runs from $x = -1$ to 2 instead of starting from

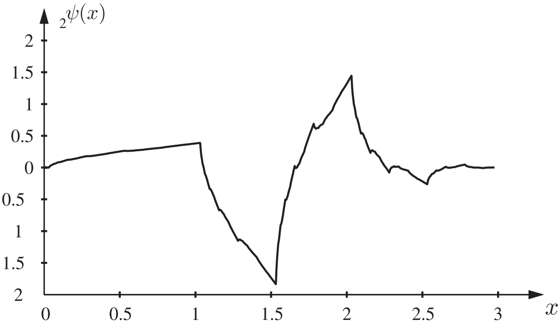


**Figure 19.9** The Daubechies wavelet $_2\psi\,(x)$.

$x = 0$. It is often referred to as the *Daubechies D4 wavelet* since it is generated by four coefficients.

Both Daubechies' scaling function $_2\varphi$ and Daubechies' wavelet $_2\psi$ are shown in Figures 19.8 and 19.9, respectively.

For $N = 1$, it follows from (19.4.29) that $P_1\,(x) \equiv 1$, and this in turn leads to the fact that $Q\,(\cos\omega) = 1$, $\widehat{L}\,(\omega) = 1$ so that the generating function is

$$\widehat{m}_0\,(\omega) = \frac{1}{2}\left(1 + e^{-i\omega}\right). \qquad (19.4.42)$$

This corresponds to the generating function for the Haar wavelet.

## 19.5 Exercises

1. For the Haar wavelet defined in Example 19.2.1, show that

(a) $\mathscr{F}\,\{\psi\,(2^m x - n)\} = \left(\dfrac{4i}{\omega}\right)\, e^{-i\omega n} \exp\left(-\dfrac{i\omega}{2.2^m}\right) \sin^2\left(\dfrac{\omega}{4.2^m}\right)$

(b) $|\mathscr{F}\,\{\psi\,(2^m x - n)\}\,| = \dfrac{4}{\omega} \sin^2\left(\dfrac{\omega}{4.2^m}\right)$ .

Explain the significance of this result.

2. Find the Fourier transforms of the following wavelets:

(a) $\psi(t) = \begin{cases} 1 - |t|, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$ (piecewise linear spline wavelet)

(b) $\psi(t) = (1 - t^2)\exp\left(-\dfrac{t^2}{2}\right) = -\dfrac{d^2}{dt^2}\exp\left(-\dfrac{t^2}{2}\right)$ (Mexcican hat wavelet)

(c) $\psi(t) = \exp\left(i\omega_0 t - \frac{t^2}{2}\right)$ (Morlet wavelet).

3. If $f$ is a homogeneous function of degree $n$, show that

$$(W_\psi f)(\lambda a, \lambda b) = \lambda^{n+\frac{1}{2}} (W_\psi f)(a, b).$$

4. In the proof of Theorem 19.2.2 we do not address the difficulty that arises from the fact that $a$ takes both positive and negative values. Provide a more detailed proof that removes the difficulty.

5. Prove Theorem 19.2.4.

6. Show that

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} \frac{\sin \pi(2x-n)}{\pi(2x-n)}\, dx = \frac{1}{2\pi n} \sin\left(\frac{\pi n}{2}\right).$$

7. Let $\psi_{m,n}(t) = 2^{-m/2}\, \psi\left(2^{-m}t - n\right)$, where $\psi(t)$ is the Haar wavelet, that is,

$$\psi_{m,n}(t) = \begin{cases} 2^{-m/2} & \text{if } 2^m n < t < 2^m n + 2^{m-1}, \\ -2^{-m/2} & \text{if } 2^m n + 2^{m-1} < t < 2^m n + 2^m, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f(t) = \begin{cases} a & \text{if } 0 < t < \frac{1}{2}, \\ b & \text{if } \frac{1}{2} < t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\langle f, \psi_{m,0} \rangle$.

(b) Show that $\sum_{m=0}^{\infty} \langle f,\, \psi_{m,0} \rangle\, \psi_{m,0}(t) = \begin{cases} a & \text{if } 0 < t < \frac{1}{2}, \\ b & \text{if } \frac{1}{2} < t < 1. \end{cases}$

8. Consider the cardinal $B$-splines $B_n(x)$ of order $n$ are defined by

$$\begin{aligned} B_1(x) &= \chi_{[0,1]}(x), \\ B_n(x) &= B_1(x) * B_1(x) * \ldots * B_1(x) = B_1(x) * B_{n-1}(x),\ n \geq 2, \end{aligned}$$

where $n$ factors are involved in the convolution product.

(a) Show that

$$\begin{aligned} B_n(x) &= \int_{-\infty}^{\infty} B_{n-1}(x-t) B_1(t)\, dt \\ &= \int_0^1 B_{n-1}(x-t)\, dt = \int_{x-1}^{x} B_{n-1}(t)\, dt. \end{aligned}$$

(b) Find $B_2(x)$, $B_3(x)$, $B_4(x)$ explicitly.

(c) Show that

$$\widehat{B}_1(\omega) = \left(\frac{2}{\omega}\right) \exp\left(-\frac{i\omega}{2}\right) \sin\left(\frac{\omega}{2}\right) = \int_0^1 e^{-i\omega t}\,dt.$$

9. Use the Fourier transform $\widehat{B}_1(\omega)$ of $B_1(x)$ to prove

$$\sum_{k=-\infty}^{\infty} \left|\widehat{B}_n(2\omega + 2\pi k)\right|^2 = -\frac{\sin^{2n}(\omega)}{(2n-1)!} \frac{d^{2n-1}}{d\omega^{2n-1}} (\cot \omega).$$

10. The *Franklin wavelet* is generated by the second-order ($n = 2$) splines. Show that the Fourier transform $\widehat{\varphi}(\omega)$ of this wavelet is

$$\widehat{\varphi}(\omega) = \frac{\sin^2 \frac{\omega}{2}}{\left(\frac{\omega}{2}\right)^2} \left(1 - \frac{2}{3} \sin^2 \frac{\omega}{2}\right)^{-\frac{1}{2}}.$$

11. The Gabor window (or the Gaussian window) function $f \in L^2(\mathbb{R})$ is defined by

$$f(t) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \exp\left(-at^2\right).$$

Show that $||f|| = 1$ and $\hat{f}(\omega) = \frac{1}{\sqrt{2a}} f\left(\frac{\omega}{2a}\right)$.

Discuss the time and frequency characteristics of the Gabor windows.

Draw graphs of $f(t)$ and $\hat{f}(\omega)$.

12. For a triangular window $f \in L^2(\mathbb{R})$ is defined by

$$f(t) = \sqrt{\frac{3}{2a^3}} \left[\chi_{[-\frac{a}{2},\frac{a}{2}]}(t) * \chi_{[-\frac{a}{2},\frac{a}{2}]}(t)\right] = \sqrt{\frac{3}{2a^3}} (a - |t|)\, \chi_{[-a,a]}(t).$$

Show that

(a) $||f|| = 1$,

(b) $\hat{f}(\omega) = \left(\frac{2}{\pi}\right) \sqrt{\frac{3a}{2}} \left(\frac{\sin \frac{a\omega}{2}}{a\omega}\right)^2.$

Examine its time and frequency characteristics.

# Appendix A

## *Some Special Functions and Their Properties*

The main purpose of this appendix is to introduce several special functions and to state their basic properties that are most frequently used in the theory and applications of integral transforms. The subject is, of course, too vast to be treated adequately in so short a space, so that only the more important results will be stated. For a fuller discussion of these topics and of further properties of these functions the reader is referred to the standard treatises on the subject.

### A-1 Gamma, Beta, and Error Functions

The *gamma function* (also called the *factorial function*) is defined by a definite integral in which a variable appears as a parameter

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0. \tag{A-1.1}$$

The integral (A-1.1) is uniformly convergent for all $x$ in [a, b] where $0 < a \le b < \infty$, and hence, $\Gamma(x)$ is a continuous function for all $x > 0$.

Integrating (A-1.1) by parts, we obtain the fundamental property of $\Gamma(x)$

$$\begin{aligned}\Gamma(x) &= [-e^{-t} t^{x-1}]_0^\infty + (x-1)\int_0^\infty e^{-t} t^{x-2} dt \\ &= (x-1)\Gamma(x-1), \quad \text{for} \quad x-1 > 0.\end{aligned}$$

Then we replace $x$ by $x+1$ to obtain the fundamental result

$$\Gamma(x+1) = x\,\Gamma(x). \tag{A-1.2}$$

In particular, when $x = n$ is a positive integer, we make repeated use of

(A-1.2) to obtain

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots$$
$$= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\ \Gamma(1) = n!, \tag{A-1.3}$$

where $\Gamma(1)=1$.

We put $t=u^2$ in (A-1.1) to obtain

$$\Gamma(x) = 2\int_0^\infty \exp(-u^2)u^{2x-1}du, \quad x>0. \tag{A-1.4}$$

Letting $x=\frac{1}{2}$, we find

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty \exp(-u^2)\,du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}. \tag{A-1.5}$$

Using (A-1.2), we deduce

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}. \tag{A-1.6}$$

Similarly, we can obtain the values of $\Gamma\left(\frac{5}{2}\right),\ \Gamma\left(\frac{7}{2}\right),\ldots,\Gamma\left(\frac{2n+1}{2}\right)$.

The gamma function can also be defined for negative values of $x$ by the rewritten form of (A-1.2) as

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \quad x\neq 0,-1,-2,\ldots. \tag{A-1.7}$$

For example

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\ \Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}, \tag{A-1.8}$$

$$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{4}{3}\sqrt{\pi}. \tag{A-1.9}$$

We differentiate (A-1.1) with respect to $x$ to obtain

$$\frac{d}{dx}\Gamma(x) = \Gamma'(x) = \int_0^\infty \frac{d}{dx}(t^x)\frac{e^{-t}}{t}dt$$
$$= \int_0^\infty \frac{d}{dx}[\exp(x\log t)]\frac{e^{-t}}{t}dt = \int_0^\infty t^{x-1}(\log t)e^{-t}dt. \tag{A-1.10}$$

At $x = 1$, this gives

$$\Gamma'(1) = \int_0^\infty e^{-t} \log t \; dt = -\gamma, \tag{A-1.11}$$

where $\gamma$ is called the *Euler constant* and has the value 0.5772.

The graph of the gamma function is shown in Figure A.1.

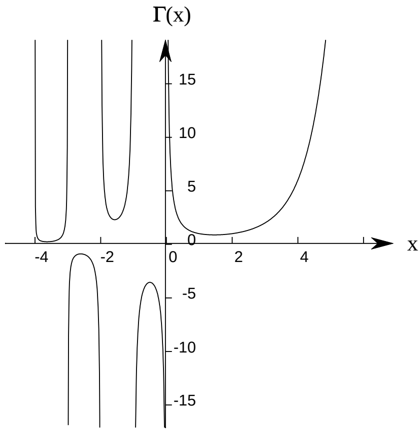


**Figure A.1** The gamma function.

The volume, $V_n$ and the surface area, $S_n$ of a sphere of radius $r$ in $\mathbb{R}^n$ are given by

$$V_n = \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^n r^n}{\Gamma\left(\frac{n}{2}+1\right)}, \qquad S_n = \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^n r^{n-1}}{\Gamma\left(\frac{n}{2}\right)}$$

Thus, $dV_n = S_n$.

In particular, when $n = 2, 3, \ldots,$, $V_2 = \pi r^2$, $S_2 = 2\pi r$; $V_3 = \frac{4}{3}\pi r^3$, $S_3 = 4\pi r^2$; ....

## Legendre Duplication Formula

Several useful properties of the gamma function are recorded below for reference without proof.

$$2^{2x-1}\,\Gamma(x)\Gamma\left(x + \frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2x). \tag{A-1.12}$$

In particular, when $x = n (n = 0, 1, 2, \ldots)$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}\,(2n)!}{2^{2n}\,n!}. \tag{A-1.13}$$

The following properties also hold for $\Gamma(x)$:

$$\Gamma(x)\Gamma(1-x) = \pi \text{ cosec } \pi x, \quad x \text{ is a noninteger,} \tag{A-1.14}$$

$$\Gamma(x) = p^x \int_0^\infty \exp(-pt)\, t^{x-1} dt, \tag{A-1.15}$$

$$\Gamma(x) = \int_{-\infty}^\infty \exp(xt - e^t) dt. \tag{A-1.16}$$

$$\Gamma(x+1) - \sqrt{2\pi}\exp(-x) x^{x+\frac{1}{2}} \qquad \text{for large } x, \tag{A-1.17}$$

$$n! \sim \sqrt{2\pi}\exp(-n) x^{n+\frac{1}{2}} \qquad \text{for large } n. \tag{A-1.18}$$

The *incomplete gamma function*, $\gamma(x,a)$, is defined by the integral

$$\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt, \quad a > 0. \tag{A-1.19}$$

The *complementary incomplete gamma* function, $\Gamma(a,x)$, is defined by the integral

$$\Gamma(a,x) = \int_x^\infty e^{-t}\ t^{a-1} dt, \quad a > 0. \tag{A-1.20}$$

Thus, it follows that

$$\gamma(a,x) + \Gamma(a,x) = \Gamma(a). \tag{A-1.21}$$

The *beta function*, denoted by $B(x,y)$ is defined by the integral

$$B(x,y) = \int_0^t t^{x-1}(1-t)^{y-1} dt, \quad x > 0, \quad y > 0. \tag{A-1.22}$$

The beta function $B(x,y)$ is *symmetric* with respect to its arguments $x$ and $y$, that is,

$$B(x,y) = B(y,x). \tag{A-1.23}$$

This follows from (A-1.22) by the change of variable $1 - t = u$, that is,

$$B(x,y) = \int_0^1 u^{y-1}(1-u)^{x-1} du = B(y,x).$$

If we make the change of variable $t = u/(1+u)$ in (A-1.22), we obtain another integral representation of the beta function

$$B(x,y) = \int_0^\infty u^{x-1}(1+u)^{-(x+y)} du = \int_0^\infty u^{y-1}(1+u)^{-(x+y)} du, \tag{A-1.24}$$

Putting $t = \cos^2\theta$ in (A-1.22), we derive

$$B(x,y) = 2\int_0^{\pi/2} \cos^{2x-1}\theta \, \sin^{2y-1}\theta \, d\theta. \tag{A-1.25}$$

Several important results are recorded below for ready reference without proof.

$$B(1,1) = 1, \qquad B\left(\frac{1}{2},\frac{1}{2}\right) = \pi, \tag{A-1.26}$$

$$B(x,y) = \left(\frac{x-1}{x+y-1}\right) B(x-1,y), \tag{A-1.27}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{A-1.28}$$

$$B\left(\frac{1+x}{2},\frac{1-x}{2}\right) = \pi \sec\left(\frac{\pi x}{2}\right), \qquad 0 < x < 1. \tag{A-1.29}$$

The *error function*, $\mathrm{erf}(x)$ is defined by the integral

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)dt, \quad -\infty < x < \infty. \tag{A-1.30}$$

Clearly it follows from (A-1.30) that

$$\mathrm{erf}(-x) = -\mathrm{erf}(x), \tag{A-1.31}$$

$$\frac{d}{dx}[\mathrm{erf}(x)] = \frac{2}{\sqrt{\pi}}\exp(-x^2), \tag{A-1.32}$$

$$\mathrm{erf}(0) = 0, \quad \mathrm{erf}(\infty) = 1. \tag{A-1.33}$$

The *complementary error function*, $\mathrm{erfc}(x)$ is defined by the integral

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)dt. \tag{A-1.34}$$

Clearly it follows that

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x), \tag{A-1.35}$$

$$\mathrm{erfc}(0)=1, \ \mathrm{erfc}(\infty) = 0. \tag{A-1.36}$$

$$\mathrm{erfc}(x) \sim \frac{1}{x\sqrt{\pi}}\exp(-x^2) \qquad \text{for large } x. \tag{A-1.37}$$

The graphs of $\mathrm{erf}(x)$ and $\mathrm{erfc}(x)$ are shown in Figure A.2.

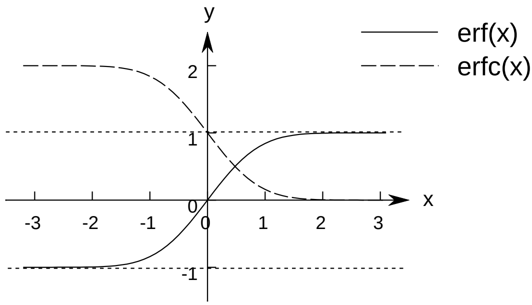


**Figure A.2** The error function and the complementary error function.

Closely associated with the error function are the Fresnel integrals, which are defined by

$$C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt \quad \text{and} \quad S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt. \tag{A-1.38}$$

These integrals arises in diffraction problems in optics, in water waves and in elasticity and elsewhere.

Clearly it follows from (A-1.38) that

$$C(0) = 0 = S(0) \tag{A-1.39}$$

$$C(\infty) = S(\infty) = \frac{\pi}{2}, \tag{A-1.40}$$

$$\frac{d}{dx}C(x) = \cos\left(\frac{\pi x^2}{2}\right), \quad \frac{d}{dx}S(x) = \sin\left(\frac{\pi x^2}{2}\right). \tag{A-1.41}$$

It also follows from (A-1.38) that $C(x)$ has extrema at the points where $x^2 = (2n+1),\; n = 0, 1, 2, 3, \ldots,$ and $S(x)$ has extrema at the points where $x^2 = 2n, n = 1, 2, 3, \ldots$. The largest maxima occur first and are found to be $C(1) = 0.7799$ and $S(\sqrt{2}) = 0.7139$. We also infer that both $C(x)$ and $S(x)$ are oscillatory about the line $y = 0.5$. The graphs of $C(x)$ and $S(x)$ for non-negative real $x$ are shown in Figure A.3.

## A-2 Bessel and Airy Functions

*The Bessel function of the first kind of order*or $v$ (non-negative real number)

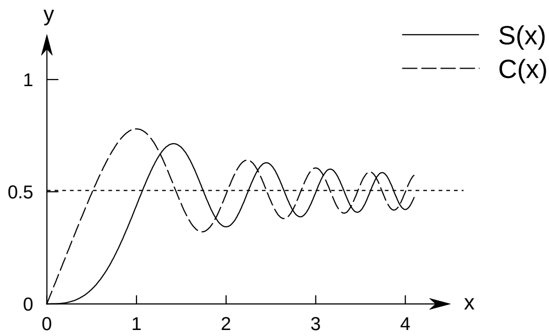


**Figure A.3** The Fresnel integrals $C(x)$ and $S(x)$.

is denoted by $J_v(x)$, and defined by

$$J_v(x) = x^v \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{2r+v}\, r!\; \Gamma(r+v+1)}. \tag{A-2.1}$$

This series is convergent for all $x$.

The Bessel function $y = J_v(x)$ satisfies the *Bessel equation*

$$x^2 y'' + xy' + (x^2 - v^2)y = 0. \tag{A-2.2}$$

When $v$ is *not* a positive integer or zero, $J_v(x)$ and $J_{-v}(x)$ are two linearly independent solutions so that

$$y = A\, J_v(x) + B\, J_{-v}(x) \tag{A-2.3}$$

is the general solution of (A-2.2), where $A$ and $B$ are arbitrary constants.

However, when $v = n$, where $n$ is a *positive integer* or *zero*, $J_n(x)$ and $J_{-n}(x)$ are no longer independent, but are related by the equation

$$J_{-n}(x) = (-1)^n J_n(x). \tag{A-2.4}$$

Thus, when $n$ is a positive integer or zero, equation (A-2.2) has only *one* solution given by

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!} \left(\frac{x}{2}\right)^{n+2r}. \tag{A-2.5}$$

A second solution, known as *Neumann's or Webber's solution*, $Y_n(x)$ is given by

$$Y_n(x) = \lim_{v \to n} Y_v(x), \tag{A-2.6}$$

where

$$Y_v(x) = \frac{(\cos v\pi) J_v(x) - J_{-v}(x)}{\sin v\pi}. \tag{A-2.7}$$

Thus, the general solution of (A-2.2) is

$$y(x) = A\ J_n(x) + B\,Y_n(x), \tag{A-2.8}$$

where A and B are arbitrary constants.

In particular, from (A-2.5),

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{x}{2}\right)^{2r}, \tag{A-2.9}$$

$$J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(r+1)!} \left(\frac{x}{2}\right)^{2r+1}. \tag{A-2.10}$$

Clearly, it follows from (A-2.9) and (A-2.10) that

$$J_0'(x) = -J_1(x). \tag{A-2.11}$$

Bessel's equation may not always arise in the standard form given in (A-2.2), but more frequently as

$$x^2 y'' + xy' + (k^2x^2 - v^2)y = 0 \tag{A-2.12}$$

with the general solution

$$y(x) = A\,J_v(kx) + B\,Y_v(kx). \tag{A-2.13}$$

The *recurrence relations* are recorded below for easy reference without proof.

$$J_{v+1}(x) = \left(\frac{v}{x}\right) J_v(x) - J_v'(x), \tag{A-2.14}$$

$$J_{v-1}(x) = \left(\frac{v}{x}\right) J_v(x) + J_v'(x), \tag{A-2.15}$$

$$J_{v-1}(x) + J_{v+1}(x) = \left(\frac{2v}{x}\right) J_v(x), \tag{A-2.16}$$

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x). \tag{A-2.17}$$

We have, from (A-2.5),

$$x^n J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r 2^{-(n+2r)}}{r!(n+r)!} x^{2n+2r}.$$

Differentiating both sides of this result with respect to $x$ and using the fact that $2(n+r)/(n+r)! = 2/(n+r-1)!$, it turns out that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} \frac{(-1)^r 2^{-(n+2r+1)}}{r!(n+r-1)!} x^{2n+2r-1} = x^n J_{n-1}(x). \tag{A-2.18}$$

Similarly, we can show

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x). \tag{A-2.19}$$

The generating function for the Bessel function is

$$\exp\left[\frac{1}{2}x\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} t^n J_n(x). \tag{A-2.20}$$

The integral representation of $J_n(x)$ is

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\,\theta - x\sin\theta)d\theta. \tag{A-2.21}$$

The following are known as the *Lommel integrals*:

$$\int_0^a x\,J_n(px)J_n(qx)\,dx$$
$$= \frac{a}{(q^2-p^2)}[p\,J_n(qa)J_n'(pa) - q\,J_n(pa)\,J_n'(qa)],\ p \neq q, \tag{A-2.22}$$

and

$$\int_0^a x\,J_n^2(px)dx = \frac{a^2}{2}\left[J_n'^2(pa) + \left(1-\frac{n^2}{p^2a^2}\right)J_n^2(pa)\right]. \tag{A-2.23}$$

When $n = \pm\frac{1}{2}$,

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x. \tag{A-2.24}$$

A rough idea of the shape of the Bessel functions when $x$ is large may be obtained from equation (A-2.2). Substitution of $y = x^{-\frac{1}{2}}u(x)$ eliminates the first derivative, and hence, gives the equation

$$u'' + \left(1 - \frac{4n^2-1}{4x^2}\right)u = 0. \tag{A-2.25}$$

For large $x$, this equation approximately becomes

$$u'' + u = 0. \tag{A-2.26}$$

This equation admits the solution $u(x) = A\cos(x+\varepsilon)$ that is,

$$y = \frac{A}{\sqrt{x}}\cos(n+\varepsilon). \tag{A-2.27}$$

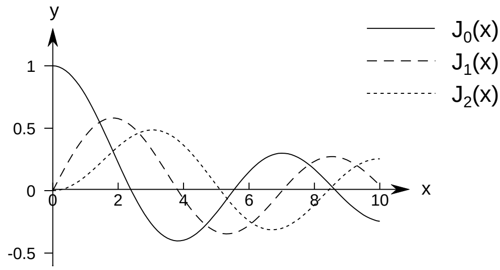


**Figure A.4** Graphs of $y = J_0(x)$, $J_1(x)$ and $J_2(x)$.

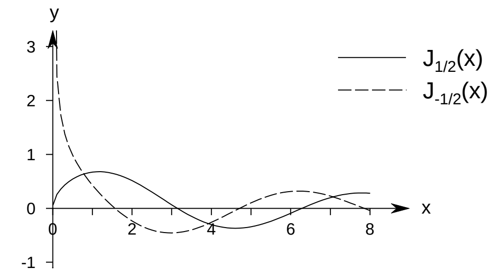


**Figure A.5** Graphs of $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$.

This suggests that $J_n(x)$ is oscillatory and has an infinite number of zeros. It also tends to zero as $x \to \infty$. The graphs of $J_n(x)$ for $n = 0, 1, 2$ and for $n = \pm\frac{1}{2}$ are shown in Figure A.4 and Figure A.5, respectively.

An important special case arises in particular physical problems when $k^2 = -1$ in equation (A-2.12). we then have the *modified Bessel equation*

$$x^2 y'' + xy' - (x^2 + v^2)y = 0, \tag{A-2.28}$$

with the general solution

$$y = A\, J_v(ix) + B\, Y_v(ix). \tag{A-2.29}$$

We now define a new function

$$I_v(x) = i^{-v} J_v(ix), \tag{A-2.30}$$

and then use the series (A-2.1) for $J_v(x)$ so that

$$I_v(x) = i^{-v} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(r+v+1)} \left(\frac{ix}{2}\right)^{v+2r} = \sum_{r=0}^{\infty} \frac{1}{r!\Gamma(r+v+1)} \left(\frac{x}{2}\right)^{v+2r}. \tag{A-2.31}$$

Similarly, we can find the second solution, $K_v(x)$ of the modified Bessel equation (A-2.28). Usually, $I_v(x)$ and $K_v(x)$ are called *modified Bessel functions* and their properties can be obtained in a similar way to those of $J_v(x)$ and $Y_v(x)$. The graphs of $Y_0(x)$, $Y_1(x)$ and $Y_2(x)$ are shown in Figure A.6.

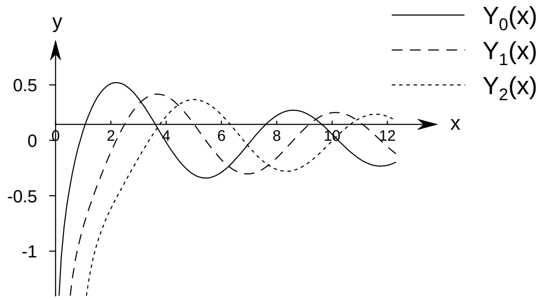


**Figure A.6** Graphs of $y = Y_0(x)$, $Y_1(x)$ and $Y_2(x)$.

We state a few important infinite integrals involving Bessel functions which arise frequently in the application of Hankel transforms.

$$\int_0^\infty \exp(-at)\, J_v(bt) t^v\, dt = \frac{(2b)^v\, \Gamma\left(v+\frac{1}{2}\right)}{\sqrt{\pi}(a^2+b^2)^{v+\frac{1}{2}}}, \quad v > -\frac{1}{2}, \quad \text{(A-2.32)}$$

$$\int_0^\infty \exp(-at)\, J_v(bt) t^{v+1}\, dt = \frac{2a(2b)^v\, \Gamma\left(v+\frac{3}{2}\right)}{\sqrt{\pi}(a^2+b^2)^{v+\frac{3}{2}}}, \quad v > -1, \quad \text{(A-2.33)}$$

$$\int_0^\infty \exp(-a^2t^2)\, J_v(bt) t^{v+1}\, dt = \frac{b^v}{(2a^2)^{v+1}} \exp\left(-\frac{b^2}{4a^2}\right), \quad v > -1, \quad \text{(A-2.34)}$$

$$\int_0^\infty \exp(-a^2t^2) J_v(bt) J_v(ct)\, t\, dt = \frac{1}{2a^2} \exp\left(-\frac{b^2+c^2}{4a^2}\right) I_v\left(\frac{bc}{2a^2}\right), \quad v > -1, \quad \text{(A-2.35)}$$

$$\int_0^\infty t^{2\mu-v-1}\, J_v(t)\, dt = \frac{2^{2\mu-v-1}\Gamma(\mu)}{\Gamma(v-\mu+1)}, \quad 0 < \mu < \frac{1}{2}, \quad v > -\frac{1}{2}. \quad \text{(A-2.36)}$$

The *Airy function*, $y = Ai(x)$ is the first solution of the differential equation

$$y'' - xy = 0. \tag{A-2.37}$$

The second solution is denoted by $Bi(x)$. Then these functions are given by

$$Ai(x) = \sqrt{\frac{x}{3}}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right)\right], \tag{A-2.38}$$

$$Bi(x) = \sqrt{\frac{x}{3}}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right) + I_{\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right)\right]. \tag{A-2.39}$$

The integral representation for $Ai(x)$ is

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + x\,t\right) dt. \tag{A-2.40}$$

The graph of $y = Ai(x)$ is shown in Figure A.7.

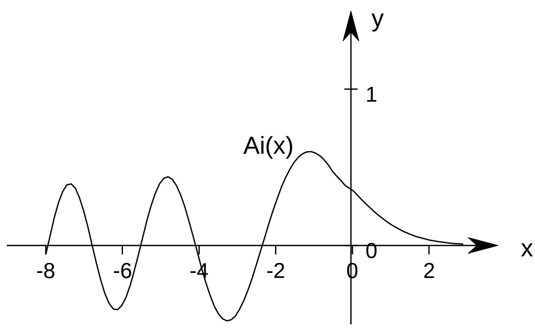


**Figure A.7** The Airy function.

## A-3 Legendre and Associated Legendre Functions

*The Legendre polynomials* $P_n(x)$ are defined by the *Rodrigues formula*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n. \tag{A-3.1}$$

The seven Legendre polynomials are

$$
\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \frac{1}{2}(3x^2-1)\\
P_3(x) &= \frac{1}{2}(5x^3-3x)\\
P_4(x) &= \frac{1}{8}(35x^4-30x^2+3)\\
P_5(x) &= \frac{1}{8}(63x^5-70x^3+15x)\\
P_6(x) &= \frac{1}{16}(231x^6-315x^4+105x^2-5).
\end{aligned}
$$

The generating function for the Legendre polynomial is

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x). \tag{A-3.2}$$

This function provides more information about the Legendre polynomials. For example

$$P_n(1)=1, \qquad P_n(-1)=(-1)^n, \tag{A-3.3}$$

$$P_{2n}(0) = (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\ n!} = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \tag{A-3.4}$$

$$P_{2n+1}(0) = 0,\ n=0,1,2,\ldots, \tag{A-3.5}$$

$$P_n(-x) = (-1)^n P_n(x), \qquad \frac{d^n}{dx^n}P_n(x) = \frac{(2n)!}{2^n\ n!}, \tag{A-3.6}$$

where the double factorial is defined by

$$(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1) \text{ and } (2n)!! = 2\cdot 4\cdot 6\cdots(2n).$$

The graphs of the first four Legendre polynomials are shown in Figure A.8.

The recurrence relations for the Legendre polynomials are

$$(n+1)\,P_{n+1}(x) = (2n+1)x\,P_n(x) - n\,P_{n-1}(x), \tag{A-3.7}$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \tag{A-3.8}$$

$$(1-x^2)\,P'_n(x) = n\,P_{n-1}(x) - n\,x\,P_n(x), \tag{A-3.9}$$

$$(1-x^2)\,P'_n(x) = (n+1)x\,P_n(x) - (n+1)\,P_{n+1}(x). \tag{A-3.10}$$

The Legendre polynomials $y = L_n(x)$ satisfy the Legendre differential equation

$$(1-x^2)y'' - 2x\,y' + n(n+1)y = 0. \tag{A-3.11}$$

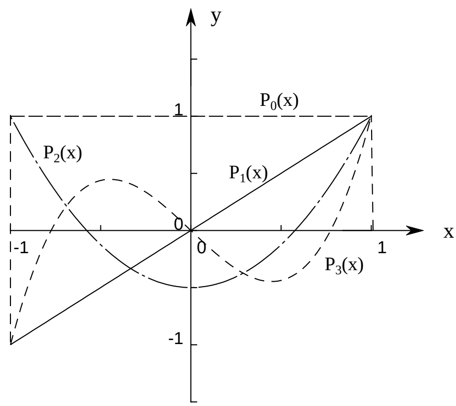


**Figure A.8** Graphs of $y = P_0(x)$, $P_1(x)$, $P_2(x)$ and $P_3(x)$.

If $n$ is *not* an integer, both solutions of (A-3.11) diverge at $x = \pm 1$.

The orthogonal relation is

$$\int_{-1}^{1} P_n(x)\, P_m(x) dx = \frac{2}{(2n+1)}\, \delta_{nm}. \tag{A-3.12}$$

The *associated Legendre functions* are defined by

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x) = \frac{1}{2^n\, n!} (1-x^2)^{\frac{m}{2}} \frac{d^{m+n}}{dx^{m+n}} (x^2-1)^n, \tag{A-3.13}$$

where $0 \le m \le n$.

Clearly, it follows that

$$P_n^0(x) = P_n(x), \tag{A-3.14}$$

$$P_n^m(-x) = (-1)^{n+m} P_n^m(x), \quad P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x). \tag{A-3.15}$$

The generating function for $P_n^m(x)$ is

$$\frac{(2m)!(1-x^2)^{\frac{m}{2}}}{2^m\, m!(1-2\ t\ x+t^2)^{m+\frac{1}{2}}} = \sum_{r=0}^{\infty} P_{r+m}^m(x)\, t^r. \tag{A-3.16}$$

The recurrence relations are

$$(2n+1)\, x\, P_n^m(x) = (n+m) P_{n-1}^m(x) + (n-m+1) P_{n+1}^m(x), \tag{A-3.17}$$

$$2(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_n^m(x) = P_n^{m+1}(x) - (n+m)(n-m+1) P_n^{m-1}(x). \tag{A-3.18}$$

The associated Legendre functions $P_n^m(x)$ are solutions of the differential equation

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{(1-x^2)}\right] y = 0. \qquad \text{(A-3.19)}$$

This reduces to the Legendre equation when $m=0$.

Listed below are few associated Legendre functions with $x = \cos\theta$:

$$\begin{aligned}
P_1^1(x) &= (1-x^2)^{\frac{1}{2}} = \sin\theta, \\
P_2^1(x) &= 3x(1-x^2)^{\frac{1}{2}} = 3\cos\theta\sin\theta \\
P_2^2(x) &= 3(1-x^2) = 3\sin^2\theta \\
P_3^1(x) &= \frac{3}{2}(5x^2-1)(1-x^2)^{\frac{1}{2}} = \frac{3}{2}(5\cos^2\theta - 1)\sin\theta \\
P_3^2(x) &= 15x(1-x^2) = 15\cos\theta\sin^2\theta \\
P_3^3(x) &= 15(1-x^2)^{3/2} = 15\sin^3\theta.
\end{aligned}$$

The orthogonal relations are

$$\int_{-1}^{1} P_n^m(x) P_t^m(x)\, dx = \frac{2}{(2\ell+1)} \cdot \frac{(\ell+m)!}{(\ell-m)!}\, \delta_{n\ell}, \qquad \text{(A-3.20)}$$

$$\int_{-1}^{1} (1-x^2)^{-1} P_n^m(x) P_n^\ell(x)\, dx = \frac{(n+m)!}{m(n-m)!}\, \delta_{m\ell}. \qquad \text{(A-3.21)}$$

## A-4 Jacobi and Gegenbauer Polynomials

The *Jacobi polynomials* $P_n^{(\alpha,\beta)}(x)$ of degree $n$ are defined by the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right], \quad \text{(A-4.1)}$$

where $\alpha > -1$ and $\beta > -1$.

When $\alpha = \beta = 0$, the Jacobi polynomials become Legendre polynomials, that is,

$$P_n(x) = P_n^{(0,0)}(x), \qquad n = 0, 1, 2, \ldots. \qquad \text{(A-4.2)}$$

On the other hand, the associated Laguerre functions arise as the limit

$$L_n^\alpha(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right). \qquad \text{(A-4.3)}$$

The recurrence relations for $P_n^{(\alpha,\beta)}(x)$ are

$$\begin{aligned}&2(n+1)(\alpha+\beta+n+1)(\alpha+\beta+2n)P_{n+1}^{(\alpha,\beta)}(x)\\&=(\alpha+\beta+2n+1)[(\alpha^2-\beta^2)+x(\alpha+\beta+2n+2)(\alpha+\beta+2n)]P_n^{(\alpha,\beta)}(x)\\&\qquad -2(\alpha+n)(\beta+n)(\alpha+\beta+2n+2)P_{n-1}^{(\alpha,\beta)}(x),\end{aligned}\quad \text{(A-4.4)}$$

where $n=1,2,3,\ldots$, and

$$P_n^{(\alpha,\beta-1)}(x)-P_n^{(\alpha-1,\beta)}(x)=P_{n-1}^{(\alpha,\beta)}(x). \quad \text{(A-4.5)}$$

The generating function for Jacobi polynomials is

$$2^{(\alpha+\beta)}R^{-1}(1-t+R)^{-\alpha}(1+t+R)^{-\beta}=\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)\ t^n, \quad \text{(A-4.6)}$$

where $R=(1-2x\,t+t^2)^{\frac{1}{2}}$.

The Jacobi polynomials, $y=P_n^{(\alpha,\beta)}(x)$, satisfy the differential equation

$$(1-x^2)y''+[(\beta-\alpha)-(\alpha+\beta+2)x]\,y'+n(n+\alpha+\beta+1)y=0. \quad \text{(A-4.7)}$$

The orthogonal relation is

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx=\begin{Bmatrix}0, & n\neq m\\ \delta_n, & n=m\end{Bmatrix}, \quad \text{(A-4.8)}$$

where

$$\delta_n=\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}. \quad \text{(A-4.9)}$$

When $\alpha=\beta=v-\dfrac{1}{2}$, the Jacobi polynomials reduce to the *Gegenbauer polynomials* $C_n^v(x)$, which are defined by the Rodrigues formula

$$C_n^v(x)=\frac{(-1)^n}{2^n n!}(1-x^2)^{v-\frac{1}{2}}\frac{d^n}{dx^n}\left[(1-x^2)^{v+n-\frac{1}{2}}\right]. \quad \text{(A-4.10)}$$

The generating function for $C_n^v(x)$ of degree $n$ is

$$(1-2x\,t+t^2)^{-v}=\sum_{n=0}^{\infty}C_n^v(x)\,t^n, \quad |t|<1, \quad |x|\leq 1, \quad v>-\frac{1}{2}. \quad \text{(A-4.11)}$$

The recurrence relations are

$$(n+1)C_{n+1}^v(x)-2(v+n)x\,C_n^v(x)+(2v+n-1)C_{n-1}^v(x)=0, \quad \text{(A-4.12)}$$

$$(n+1)C_{n+1}^v(x)-2v\,C_n^{v+1}(x)+2v\ C_{n-1}^{v+1}(x)=0, \quad \text{(A-4.13)}$$

$$\frac{d}{dx}\left[C_n^v(x)\right] = 2v\, C_{n+1}^{v+1}(x). \tag{A-4.14}$$

The differential equation satisfied by $y = C_n^v\,(x)$ is

$$(1-x^2)y'' - (2v+1)\,xy' + n(n+2v)\,y = 0. \tag{A-4.15}$$

The orthogonal property is

$$\int_{-1}^{1} (1-x^2)^{v-\frac{1}{2}} C_n^v(x)\, C_m^v\,(x)\,dx = \delta_n \delta_{nm}, \tag{A-4.16}$$

where

$$\delta_n = \frac{2^{1-2v} n\Gamma(n+2\,v)}{n!(n+v)[\Gamma(v)]^2}. \tag{A-4.17}$$

When $v = \frac{1}{2}$, the Gegenbauer polynomials reduce to Legendre polynomials, that is,

$$C_n^{\frac{1}{2}}(x) = P_n(x). \tag{A-4.18}$$

The Hermite polynomials can also be obtained from the Gegenbauer polynomials as the limit

$$H_n(x) = n! \lim_{v\to\infty} v^{-n/2}\; C_n^v\left(\frac{x}{\sqrt{v}}\right). \tag{A-4.19}$$

Finally, when $\alpha = \beta = \frac{1}{2}$, the Gegenbauer polynomials reduce to the well-known *Chebyshev polynomials*, $T_n(x)$, which are defined by a solution of the second order difference equation (see Example 12.6.7)

$$u_{n+2} - 2x\; u_{n+1} + u_n = 0, \quad |x| \leq 1 \tag{A-4.20}$$

$$u(0) = u_0 \text{ and } u(1) = u_1. \tag{A-4.21}$$

The generating function for $T_n(x)$ is

$$\frac{(1-t^2)}{(1-2x\,t+t^2)} = T_0(x) + 2\sum_{n=1}^{\infty} T_n(x)t^n, \quad |x| \leq 1 \quad t < 1. \tag{A-4.22}$$

The first seven Chebyshev polynomials of degree $n$ of the first kind are

$$\begin{aligned}
T_0(x) &= 1\\
T_1(x) &= x\\
T_2(x) &= 2x^2 - 1\\
T_3(x) &= 4x^3 - 3x\\
T_4(x) &= 8x^4 - 8x^2 + 1\\
T_5(x) &= 16x^5 - 20x^3 + 5x\\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1.
\end{aligned}$$

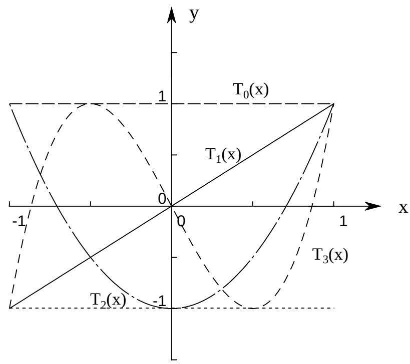


**Figure A.9** Chebyshev polynomials $y = T_n(x)$.

The graphs of the first four Chebyshev polynomials are shown in Figure A.9.

The Chebyshev polynomials $y = T_n(x)$ satisfy the differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0. \tag{A-4.23}$$

It follows from (A-4.22) that $T_n(x)$ satisfies the recurrence relations

$$T_{n+1}(x) - 2x\,T_n(x) + T_{n-1}(x) = 0, \tag{A-4.24}$$

$$T_{n+m}(x) - 2\,T_n(x)T_m(x) + T_{n-m}(x) = 0, \tag{A-4.25}$$

$$(1-x^2)T_n'(x) + nx\,T_n(x) - n\,T_{n-1}(x) = 0. \tag{A-4.26}$$

The parity relation for $T_n(x)$ is

$$T_n(-x) = (-1)^n\,T_n(x). \tag{A-4.27}$$

The Rodrigues formula is

$$T_n(x) = \frac{\sqrt{\pi}(-1)^n(1-x^2)^{\frac{1}{2}}}{2^n\left(n-\frac{1}{2}\right)!}\cdot\frac{d^n}{dx^n}\left[(1-x^2)^{n-\frac{1}{2}}\right]. \tag{A-4.28}$$

The orthogonal relation for $T_n(x)$ is

$$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}T_m(x)\,T_n(x)\,dx = \left\{\begin{matrix} 0, & m\neq n \\ \frac{\pi}{2}, & m=n \\ \pi, & m=n=0 \end{matrix}\right\}. \tag{A-4.29}$$

The *Chebyshev polynomials of the second kind*, $U_n(x)$, are defined by

$$U_n(x) = (1-x^2)^{-\frac{1}{2}}\sin[(n+1)\cos^{-1}x], \quad -1\leq x\leq 1. \tag{A-4.30}$$

The generating function for $U_n(x)$ is

$$(1-2x\,t+t^2)^{-1}=\sum_{n=0}^{\infty}U_n(x)t^n, \quad |x|<1, \quad |t|<1. \tag{A-4.31}$$

The first seven Chebyshev polynomials $U_n(x)$ are given by

$$\begin{aligned}
U_0(x) &= 1\\
U_1(x) &= 2x\\
U_2(x) &= 4x^2-1\\
U_3(x) &= 8x^3-4x\\
U_4(x) &= 16x^4-12x^2+1\\
U_5(x) &= 32x^5-32x^3+6x\\
U_6(x) &= 64x^6-80x^4+24x^2-1.
\end{aligned}$$

The differential equation for $y=U_n(x)$ is

$$(1-x^2)y''-3xy'+n(n+2)y=0. \tag{A-4.32}$$

The recurrence relations are

$$U_{n+1}(x)-2x\,U_n(x)+U_{n-1}(x)=0. \tag{A-4.33}$$

$$(1-x^2)U_n'(x)+nx\,U_n(x)-(n+1)U_{n-1}(x)=0. \tag{A-4.34}$$

The parity relation is

$$U_n(-x)=(-1)^nU_n(x). \tag{A-4.35}$$

The Rodrigues formula is

$$U_n(x)=\frac{\sqrt{\pi}(-1)^n(n+1)}{2^{n+1}\left(n+\frac{1}{2}\right)!(1-x^2)^{\frac{1}{2}}}\frac{d^n}{dx^n}\left[(1-x^2)^{n+\frac{1}{2}}\right]. \tag{A-4.36}$$

The orthogonal relation for $U_n(x)$ is

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}\,U_m(x)\,U_n(x)\,dx=\frac{\pi}{2}\,\delta_{mn}. \tag{A-4.37}$$

## A-5 Laguerre and Associated Laguerre Functions

The Laguerre polynomials $L_n(x)$ are defined by the *Rodrigues formula*

$$L_n(x)=e^x\frac{d^n}{dx^n}(x^n\;e^{-x}), \tag{A-5.1}$$

where $n = 0, 1, 2, 3, \ldots$.

The first seven Laguerre polynomials are

$$\begin{aligned}
L_0(x) &= 1\\
L_1(x) &= 1 - x\\
L_2(x) &= 2 - 4x + x^2\\
L_3(x) &= 6 - 18x + 9x^2 - x^3\\
L_4(x) &= 24 - 96x + 72x^2 - 16x^3 + x^4\\
L_5(x) &= 120 - 600x + 600x^2 - 200x^3 + 25x^4 - x^5\\
L_6(x) &= 720 - 4320x + 5400x^2 - 2400x^3 + 450x^4 - 36x^5 + x^6.
\end{aligned}$$

The generating function is

$$(1-t)^{-1} \exp\left(\frac{x\,t}{1-t}\right) = \sum_{n=0}^{\infty} t^n\, L_n(x). \tag{A-5.2}$$

In particular

$$L_n(0) = 1. \tag{A-5.3}$$

The orthogonal relation for the Laguerre polynomial is

$$\int_0^{\infty} e^{-x} L_m(x)\, L_n(x) dx = (n!)^2 \delta_{nm}. \tag{A-5.4}$$

The recurrence relations are

$$(n+1) L_{n+1}(x) = (2n + 1 - x)\, L_n(x) - n\, L_{n-1}(x), \tag{A-5.5}$$

$$x\, L_n'(x) = n\, L_n(x) - n\, L_{n-1}(x), \tag{A-5.6}$$

$$L_n'(x) = L_{n-1}'(x) - L_{n-1}(x). \tag{A-5.7}$$

The Laguerre polynomials $y = L_n(x)$ satisfy the *Laguerre differential equation*

$$xy'' + (1-x)y' + ny = 0. \tag{A-5.8}$$

The *associated Laguerre polynomials* are defined by

$$L_n^m(x) = \frac{d^m}{dx^m} L_n(x) \qquad \text{for } n \geq m. \tag{A-5.9}$$

The generating function for $L_n^m(x)$ is

$$(1-z)^{-(m+1)} \exp\left(-\frac{x\,z}{1-z}\right) = \sum_{n=0}^{\infty} L_n^m(x) z^n, \quad |z| < 1. \tag{A-5.10}$$

It follows from this that

$$L_n^m(0) = \frac{(n+m)!}{n!m!}. \tag{A-5.11}$$

The associated Laguerre function satisfies the *recurrence relation*

$$(n+1)L_{n+1}^m(x) = (2n+m+1-x)L_n^m(x) - (n+m)L_{n-1}^m(x), \tag{A-5.12}$$

$$x\frac{d}{dx}L_n^m(x) = n\,L_n^m(x) - (n+m)L_{n-1}^m(x). \tag{A-5.13}$$

The associated Laguerre function $y = L_n^m(x)$ satisfies the associated Laguerre differential equation

$$x\,y'' + (m+1-x)y' + n\,y = 0. \tag{A-5.14}$$

The Rodrigues formula for $L_n^m(x)$ is

$$L_n^m(x) = \frac{e^x x^{-m}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+m}). \tag{A-5.15}$$

The *orthogonal* relation for $L_n^m(x)$ is

$$\int_0^\infty e^{-x}\ x^m\ L_n^m(x)L_l^m(x)\,dx = \frac{(n+m)!}{n!}\delta_{n\,l}. \tag{A-5.16}$$

## A-6 Hermite Polynomials and Weber-Hermite Functions

The Hermite polynomials $H_n(x)$ are defined by the Rodrigues formula

$$H_n(x) = (-1)^n \exp(x^2)\frac{d^n}{dx^n}[\exp(-x^2)], \tag{A-6.1}$$

where $n = 0, 1, 2, 3, \ldots$.

The first seven Hermite polynomials are

$$\begin{aligned}
H_0(x) &= 1\\
H_1(x) &= 2x\\
H_2(x) &= 4x^2 - 2\\
H_3(x) &= 8x^3 - 12x\\
H_4(x) &= 16x^4 - 48x^2 + 12\\
H_5(x) &= 32x^5 - 16x^3 + 120x\\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120.
\end{aligned}$$

The *generating function* is

$$\exp(2x\,t - t^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x). \tag{A-6.2}$$

It follows from (A-6.2) that $H_n(x)$ satisfies the *parity relation*

$$H_n(-x) = (-1)^n H_n(x). \tag{A-6.3}$$

Also, it follows from (A-6.2) that

$$H_{2n+1}(0) = 0, \quad H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}. \tag{A-6.4}$$

The *recurrence relations* for Hermite polynomials are

$$H_{n+1}(x) - 2x\,H_n(x) + 2n\,H_{n-1}(x) = 0, \tag{A-6.5}$$

$$H_n'(x) = 2x\,H_{n-1}(x). \tag{A-6.6}$$

The Hermite polynomials, $y = H_n(x)$, are solutions of the *Hermite differential equation*

$$y'' - 2xy' + 2ny = 0. \tag{A-6.7}$$

The orthogonal property of Hermite polynomials is

$$\int_{-\infty}^{\infty} \exp(-x^2) H_n(x) H_m(x)\,dx = 2^n n! \sqrt{\pi}\,\delta_{mn}. \tag{A-6.8}$$

With repeated use of integration by parts, it follows from (A-6.1) that

$$\int_{-\infty}^{\infty} \exp(-x^2)\,H_n(x)\,x^m\,dx = 0, \quad m = 0, 1, \ldots, (n-1), \tag{A-6.9}$$

$$\int_{-\infty}^{\infty} \exp(-x^2)\,H_n(x)\,x^n\,dx = \sqrt{\pi}\ n!. \tag{A-6.10}$$

The *Weber-Hermite function* or, simply, *Hermite functions*

$$y = h_n(x) = \exp\left(-\frac{x^2}{2}\right) H_n(x) \tag{A-6.11}$$

satisfies the Hermite differential equation

$$y'' + (\lambda - x^2)y = 0, \quad x \in \mathbb{R} \tag{A-6.12}$$

where $\lambda = 2n + 1$. If $\lambda \neq 2n + 1$, then $y$ is not finite as $|x| \to \infty$.

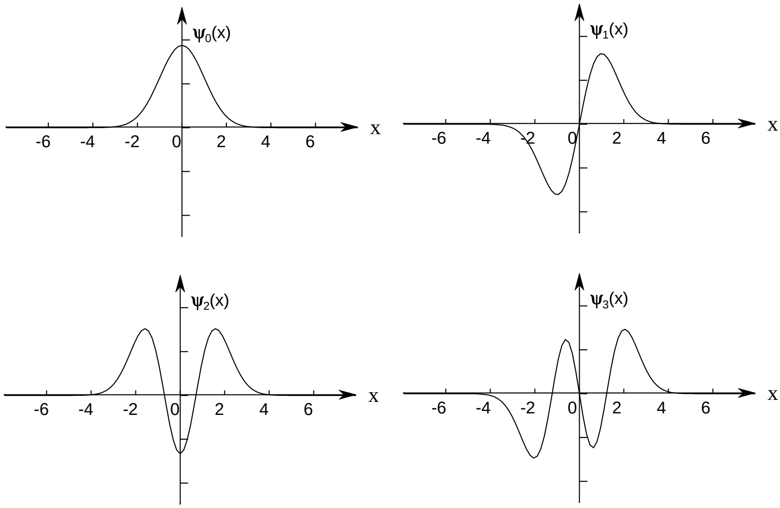


**Figure A.10** The normalized Weber-Hermite functions.

The Hermite functions $\{h_n(x)\}_0^\infty$ form an orthogonal basis for the Hilbert space $L^2(\mathbb{R})$ with weight function 1. They satisfy the following fundamental properties:

$$h_n'(x) + x\,h_n(x) - 2_n\,h_{n-1}(x) = 0,$$
$$h_n'(x) - x\,h_n(x) + h_{n+1}(x) = 0,$$
$$h_n''(x) - x^2\,h_n(x) + (2n+1)h_x = 0,$$
$$\mathscr{F}\,\{h_n(x)\} = \tilde{h_n}(k) = (-i)^n\,h_n(k).$$

The normalized Weber-Hermite functions are given by

$$\psi_n(x) = 2^{-n/2}\pi^{-\frac{1}{4}}(n!)^{-\frac{1}{2}}\exp\left(-\frac{x^2}{2}\right)H_n(x). \tag{A-6.13}$$

Physically, they represent quantum mechanical oscillator wave functions. The graphs of these functions are shown in Figure A.10.

## A-7 Mittag Leffler Function

Another important function that has widespred use in fractional calculus and fractional differential equation is the *Mittag-Leffler function.* The Mittag-

Leffler function is an entire function defined by the series

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad \alpha > 0. \tag{A-7.1}$$

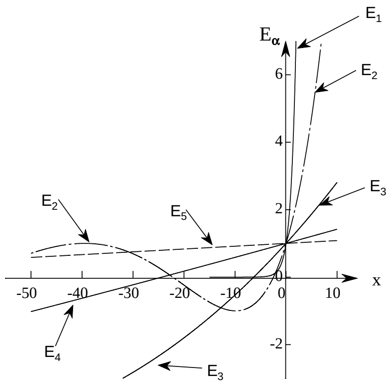


**Figure A.11** Graph of the Mittag-Leffler function $E_\alpha(x)$.

The graph of the Mittag-Leffler function is shown in Figure A.11.

*The generalized Mittag-Leffler function*, $E_{\alpha,\beta}(z)$, is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \alpha, \beta > 0. \tag{A-7.2}$$

Also the inverse Laplace transform yields

$$\mathscr{L}^{-1}\left\{\frac{m!\, s^{\alpha-\beta}}{(s^\alpha \mp a)^{m+1}}\right\} = t^{\alpha m+\beta-1} E_{\alpha,\beta}^{(m)}\left(\pm a t^\alpha\right), \tag{A-7.3}$$

where

$$E_{\alpha,\beta}^{(m)}(z) = \frac{d^m}{dz^m} E_{\alpha,\beta}(z). \tag{A-7.4}$$

Obviously,

$$E_{\alpha,1}(z) = E_\alpha(z), \qquad E_{1,1}(z) = E_1(z) = e^z. \tag{A-7.5}$$

# Appendix B

## *Tables of Integral Transforms*

In this appendix we provide a set of *short* tables of integral transforms of the functions that are either cited in the text or in most common use in mathematical, physical, and engineering applications. In these tables no attempt is made to give complete lists of transforms. For exhaustive lists of integral transforms, the reader is referred to Erdélyi et al. (1954), Campbell and Foster (1948), Ditkin and Prudnikov (1965), Doetsch (1950–1956, 1970), Marichev (1983), and Oberhettinger (1972, 1974).

**TABLE B-1 Fourier Transforms**

| | $f(x)$ | $F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ikx) f(x) dx$ |
|---|---|---|
| 1 | $\exp(-a\|x\|), \quad a > 0$ | $\left(\sqrt{\frac{2}{\pi}}\right) a(a^2 + k^2)^{-1}$ |
| 2 | $x \exp(-a\|x\|), \quad a > 0$ | $\left(\sqrt{\frac{2}{\pi}}\right) (-2aik)(a^2 + k^2)^{-2}$ |
| 3 | $\exp(-ax^2), \quad a > 0$ | $\frac{1}{\sqrt{2a}} \exp\left(-\frac{k^2}{4a}\right)$ |
| 4 | $(x^2 + a^2)^{-1}, \quad a > 0$ | $\sqrt{\frac{\pi}{2}} \frac{\exp(-a\|k\|)}{a}$ |
| 5 | $x(x^2 + a^2)^{-1}, \quad a > 0$ | $\sqrt{\frac{\pi}{2}} \left(\frac{ik}{2a}\right) \exp(-a\|k\|)$ |
| 6 | $\begin{Bmatrix} c, & a \le x \le b \\ 0, & \text{outside} \end{Bmatrix}$ | $\frac{ic}{\sqrt{2\pi}} \frac{1}{k}(e^{-ibk} - e^{-iak})$ |
| 7 | $\|x\| \exp(-a\|x\|), \quad a > 0$ | $\sqrt{\frac{2}{\pi}} (a^2 - k^2)(a^2 + k^2)^{-2}$ |

| | $f(x)$ | $F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ikx) f(x) dx$ |
|---|---|---|
| 8 | $\frac{\sin ax}{x}$ | $\sqrt{\frac{\pi}{2}}\, H(a - \|k\|)$ |
| 9 | $\exp\{-x(a - i\omega)\}\, H(x)$ | $\frac{1}{\sqrt{2\pi}} \frac{i}{(\omega - k + ia)}$ |
| 10 | $(a^2 - x^2)^{-\frac{1}{2}} H(a - \|x\|)$ | $\sqrt{\frac{\pi}{2}}\, J_0(ak)$ |
| 11 | $\frac{\sin\left[b(x^2 + a^2)^{\frac{1}{2}}\right]}{(x^2 + a^2)^{\frac{1}{2}}}$ | $\sqrt{\frac{\pi}{2}}\, J_0\left(a\sqrt{b^2 - k^2}\right) H(b - \|k\|)$ |
| 12 | $\frac{\cos\left(b\sqrt{a^2 - x^2}\right)}{(a^2 - x^2)^{\frac{1}{2}}} H(a - \|x\|)$ | $\sqrt{\frac{\pi}{2}}\, J_0\left(a\sqrt{b^2 + k^2}\right)$ |
| 13 | $e^{-ax} H(x), \quad a > 0$ | $\frac{1}{\sqrt{2\pi}}\, (a - ik)(a^2 + k^2)^{-1}$ |
| 14 | $\frac{1}{\sqrt{\|x\|}} \exp(-a\|x\|)$ | $(a^2 + k^2)^{-\frac{1}{2}} \left[a + (a^2 + k^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}$ |
| 15 | $\delta(x)$ | $\frac{1}{\sqrt{2\pi}}$ |
| 16 | $\delta^{(n)}(x)$ | $\frac{1}{\sqrt{2\pi}}\, (ik)^n$ |
| 17 | $\delta(x - a)$ | $\frac{1}{\sqrt{2\pi}}\, \exp(-iak)$ |
| 18 | $\delta^{(n)}(x - a)$ | $\frac{1}{\sqrt{2\pi}}\, (ik)^n \exp(-iak)$ |
| 19 | $\exp(iax)$ | $\sqrt{2\pi}\, \delta(k - a)$ |
| 20 | $1$ | $\sqrt{2\pi}\, \delta(k)$ |
| 21 | $x$ | $\sqrt{2\pi}\, i\, \delta'(k)$ |
| 22 | $x^n$ | $\sqrt{2\pi}\, i^n\, \delta^{(n)}(k)$ |

| | $f(x)$ | $F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ikx)f(x)dx$ |
|---|---|---|
| 23 | $H(x)$ | $\sqrt{\frac{\pi}{2}}\left[\frac{1}{i\pi k}+\delta(k)\right]$ |
| 24 | $H(x-a)$ | $\sqrt{\frac{\pi}{2}}\left[\frac{\exp(-ika)}{\pi ik}+\delta(k)\right]$ |
| 25 | $H(x)-H(-x)$ | $\sqrt{\frac{2}{\pi}}\left(-\frac{i}{k}\right)$ |
| 26 | $x^n\exp(iax)$ | $\sqrt{2\pi}\,i^n\,\delta^{(n)}(k-a)$ |
| 27 | $\lvert x\rvert^{-1}$ | $\frac{1}{\sqrt{2\pi}}(A-2\log\lvert k\rvert),\ A$ is a constant |
| 28 | $\log(\lvert x\rvert)$ | $-\sqrt{\frac{\pi}{2}}\frac{1}{\lvert k\rvert}$ |
| 29 | $H(a-\lvert x\rvert)$ | $\sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right)$ |
| 30 | $\lvert x\rvert^{\alpha}\quad(\alpha<1,$ not a negative integer$)$ | $\sqrt{\frac{2}{\pi}}\Gamma(\alpha+1)\lvert k\rvert^{-(1+\alpha)}\times\cos\left[\frac{\pi}{2}(\alpha+1)\right]$ |
| 31 | $sgn\,x$ | $\sqrt{\frac{2}{\pi}}\frac{1}{(ik)}$ |
| 32 | $x^{-n-1}\,sgn\ x$ | $\frac{1}{\sqrt{2\pi}}\frac{(-ik)^n}{n!}(A-2\ \log\lvert k\rvert)$ |
| 33 | $\frac{1}{x}$ | $-i\sqrt{\frac{\pi}{2}}\,sgn\,k$ |
| 34 | $\frac{1}{x^n}$ | $-i\sqrt{\frac{\pi}{2}}\left[\frac{(-ik)^{n-1}}{(n-1)!}\,sgn\,k\right]$ |
| 35 | $x^n\exp(iax)$ | $\sqrt{2\pi}\ i^n\delta^{(n)}(k-a)$ |

| | $f(x)$ | $F(k)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}\exp(-ikx)f(x)dx$ |
|---|---|---|
| 36 | $x^{\alpha}H(x)$, ($\alpha$ not an integer) | $\dfrac{\Gamma(\alpha+1)}{\sqrt{2\pi}}\,|k|^{-(\alpha+1)} \times \exp\left[-\left(\dfrac{\pi i}{2}\right)(\alpha+1)\,sgn\,k\right]$ |
| 37 | $x^n\,\exp(iax)\,H(x)$ | $\sqrt{\frac{\pi}{2}}\left[\frac{n!}{i\pi(k-a)^{n+1}}+i^n\,\delta^{(n)}(k-a)\right]$ |
| 38 | $\exp(iax)\,H(x-b)$ | $\sqrt{\dfrac{\pi}{2}}\left[\dfrac{\exp[-ib(k-a)]}{i\pi(k-a)}+\delta(k-a)\right]$ |
| 39 | $\dfrac{1}{x-a}$ | $-i\sqrt{\dfrac{\pi}{2}}\exp(-iak)sgn\ k$ |
| 40 | $\dfrac{1}{(x-a)^n}$ | $-i\sqrt{\dfrac{\pi}{2}}\ \exp(-iak)\dfrac{(-ik)^{n-1}}{(n-1)!}sgn\ k$ |
| 41 | $\dfrac{e^{iax}}{(x-b)}$ | $i\sqrt{\dfrac{\pi}{2}}\ \exp[ib(a-k)][1-2H(k-a)]$ |
| 42 | $\dfrac{e^{iax}}{(x-b)^n}$ | $i\sqrt{\dfrac{\pi}{2}}\,[1-2\,H(k-a)] \times\dfrac{\exp\{ib(a-k)\}}{(n-1)!}\,[-i(k-a)]^{n-1}$ |
| 43 | $|x|^{\alpha}\ sgn\ x$ ($\alpha$ not integer) | $\sqrt{\dfrac{2}{\pi}}\,\dfrac{(-i)\Gamma(\alpha+1)}{|k|^{\alpha+1}}\cos\left(\dfrac{\pi\alpha}{2}\right)\ sgn\ k$ |
| 44 | $x^n\,f(x)$ | $(-i)^n\,\dfrac{d^n}{dk^n}F(k)$ |
| 45 | $\dfrac{d^n}{dx^n}f(x)$ | $(ik)^n\,F(k)$ |
| 46 | $e^{iax}\,f(bx)$ | $\dfrac{1}{b}\,F\left(\dfrac{k-a}{b}\right)$ |
| 47 | $\begin{matrix}\sin\\ \cos\end{matrix}\,(ax^2)$ | $\dfrac{1}{\sqrt{2a}}\,\begin{matrix}\sin\\ \cos\end{matrix}\,\left(\frac{k^2}{4a}-\frac{\pi}{4}\right)$ |

**TABLE B-2 Fourier Cosine Transforms**

| | $f(x)$ | $F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(kx) f(x) dx$ |
|---|---|---|
| 1 | $\exp(-ax), \quad a > 0$ | $\left(\sqrt{\frac{2}{\pi}}\right) a(a^2+k^2)^{-1}$ |
| 2 | $x\,\exp(-ax), \quad a > 0$ | $\left(\sqrt{\frac{2}{\pi}}\right) (a^2-k^2)(a^2+k^2)^{-2}$ |
| 3 | $\exp(-a^2x^2)$ | $\frac{1}{\lvert a\rvert\sqrt{2}} \exp\left(-\frac{k^2}{4a^2}\right)$ |
| 4 | $H(a-x)$ | $\sqrt{\frac{2}{\pi}} \left(\frac{\sin ak}{k}\right)$ |
| 5 | $x^{a-1}, \quad 0 < a < 1$ | $\sqrt{\frac{2}{\pi}}\,\Gamma(a)\,k^{-a} \cos\left(\frac{a\pi}{2}\right)$ |
| 6 | $\cos(ax^2)$ | $\frac{1}{2\sqrt{a}} \left[\cos\left(\frac{k^2}{4a}\right) + \sin\left(\frac{k^2}{4a}\right)\right]$ |
| 7 | $\sin(ax^2), \quad a > 0$ | $\frac{1}{2\sqrt{a}} \left[\cos\left(\frac{k^2}{4a}\right) - \sin\left(\frac{k^2}{4a}\right)\right]$ |
| 8 | $(a^2-x^2)^{v-\frac{1}{2}} H(a-x),\ v > -\frac{1}{2}$ | $2^{v-\frac{1}{2}}\Gamma\left(v+\frac{1}{2}\right) \left(\frac{a}{k}\right)^v J_v(ak)$ |
| 9 | $(a^2+x^2)^{-1}\ J_0(bx),\ a, b > 0$ | $\sqrt{\frac{\pi}{2}}\,a^{-1}\,e^{-ak} I_0(ab),\ b < k < \infty$ |
| 10 | $x^{-v} J_v(ax), \qquad v > -\frac{1}{2}$ | $\frac{(a^2-k^2)^{v-\frac{1}{2}} H(a-k)}{2^{v-\frac{1}{2}} a^v\,\Gamma\left(v+\frac{1}{2}\right)}$ |
| 11 | $(x^2+a^2)^{-\frac{1}{2}} e^{-b(x^2+a^2)^{\frac{1}{2}}}$ | $K_0\left[a(k^2+b^2)^{\frac{1}{2}}\right],\ a > 0,\ b > 0$ |
| 12 | $(2ax-x^2)^{v-\frac{1}{2}} H(2a-x),\ v > -\frac{1}{2}$ | $\sqrt{2}\,\Gamma\left(v+\frac{1}{2}\right)\left(\frac{2a}{k}\right)^v \times \cos(ak)\,J_v(ak)$ |

| | $f(x)$ | $F_c(k)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty \cos(kx)f(x)dx$ |
|---|---|---|
| 13 | $x^{\nu-1}e^{-ax}, \qquad \nu>0,\ a>0$ | $\sqrt{\frac{2}{\pi}}\,\Gamma(\nu)r^{-\nu}\cos\nu\theta$, where $r=(a^2+k^2)^{\frac{1}{2}},\ \theta=\tan^{-1}\left(\frac{k}{a}\right)$ |
| 14 | $\dfrac{2}{x}\,e^{-x}\sin x$ | $\sqrt{\dfrac{2}{\pi}}\ \tan^{-1}\left(\dfrac{2}{k^2}\right)$ |
| 15 | $\sin\left[a(b^2-x^2)^{\frac{1}{2}}\right]\ H(b-x)$ | $\sqrt{\dfrac{\pi}{2}}\,(ab)(a^2+k^2)^{-\frac{1}{2}} \times J_1\left[b(a^2+k^2)^{\frac{1}{2}}\right]$ |
| 16 | $\dfrac{(1-x^2)}{(1+x^2)^2}$ | $\sqrt{\dfrac{\pi}{2}}\,k\exp(-k)$ |
| 17 | $x^{-\alpha}, \qquad 0<\alpha<1$ | $\sqrt{\dfrac{\pi}{2}}\dfrac{k^{\alpha-1}}{\Gamma(\alpha)}\sec\left(\dfrac{\pi\alpha}{2}\right)$ |
| 18 | $\left(\dfrac{1}{a}+x\right)e^{-ax}$ | $\sqrt{\dfrac{2}{\pi}}\dfrac{2a^2}{(a^2+k^2)^2}$ |
| 19 | $\log\left(1+\dfrac{a^2}{x^2}\right), \quad a>0$ | $\sqrt{2\pi}\,\dfrac{(1-e^{-ak})}{k}$ |
| 20 | $\log\left(\dfrac{a^2+x^2}{b^2+x^2}\right), \qquad a,b>0$ | $\sqrt{2\pi}\,\dfrac{(e^{-bk}-e^{-ak})}{k}$ |
| 21 | $a(x^2+a^2)^{-1}, \qquad a>0$ | $\sqrt{\dfrac{\pi}{2}}\ \exp(-ak), \qquad k>0$ |
| 22 | $(a^2-x^2)^{-1}$ | $\sqrt{\dfrac{\pi}{2}}\,\dfrac{\sin(ak)}{k}$ |
| 23 | $e^{-bx}\sin(ax)$ | $\dfrac{1}{\sqrt{2\pi}}\left[\dfrac{a+k}{b^2+(a+k)^2}+\dfrac{a-k}{b^2+(a-k)^2}\right]$ |
| 24 | $e^{-bx}\cos(ax)$ | $\dfrac{b}{\sqrt{2\pi}}\left[\dfrac{1}{b^2+(a-k)^2}+\dfrac{1}{b^2+(a+k)^2}\right]$ |

**TABLE B-3 Fourier Sine Transforms**

| | $f(x)$ | $F_s(k)=\sqrt{\frac{2}{\pi}}\int_0^\infty \sin(kx)\ f(x)dx$ |
|---|---|---|
| 1 | $\exp(-ax), \quad a>0$ | $\sqrt{\frac{2}{\pi}}\,k(a^2+k^2)^{-1}$ |
| 2 | $x\exp(-ax), \quad a>0$ | $\sqrt{\frac{2}{\pi}}(2ak)(a^2+k^2)^{-2}$ |
| 3 | $x^{\alpha-1}, \quad 0<\alpha<1$ | $\sqrt{\frac{2}{\pi}}\,k^{-\alpha}\Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right)$ |
| 4 | $\frac{1}{\sqrt{x}}$ | $\frac{1}{\sqrt{k}}, \qquad k>0$ |
| 5 | $x^{\alpha-1}e^{-ax}, \quad \alpha>-1,\ a>0$ | $\sqrt{\frac{2}{\pi}}\,\Gamma(\alpha)\ r^{-\alpha}\sin(\alpha\theta)$, where |
| | | $r=(a^2+k^2)^{\frac{1}{2}},\ \theta=\tan^{-1}\left(\frac{k}{a}\right)$ |
| 6 | $x^{-1}e^{-ax}, \qquad a>0$ | $\sqrt{\frac{2}{\pi}}\ \tan^{-1}\left(\frac{k}{a}\right), \qquad k>0$ |
| 7 | $x\ \exp(-a^2x^2)$ | $2^{-3/2}\left(\frac{k}{a^3}\right)\exp\left(-\frac{k^2}{4a^2}\right)$ |
| 8 | $erfc(ax)$ | $\sqrt{\frac{2}{\pi}}\frac{1}{k}\left[1-\exp\left(-\frac{k^2}{4a^2}\right)\right]$ |
| 9 | $x(a^2+x^2)^{-1}$ | $\sqrt{\frac{\pi}{2}}\ \exp(-ak), \quad a>0$ |
| 10 | $x(a^2+x^2)^{-2}$ | $\frac{1}{\sqrt{2\pi}}\left(\frac{k}{a}\right)\exp(-ak), \quad (a>0)$ |
| 11 | $x(a^2-x^2)^{v-\frac{1}{2}}H(a-x)$, $v>-\frac{1}{2}$ | $2^{v-\frac{1}{2}}a^{v+1}k^{-v}\Gamma\left(v+\frac{1}{2}\right)$ $\times J_{v+1}(ak)$ |
| 12 | $\tan^{-1}\left(\frac{x}{a}\right)$ | $\sqrt{\frac{\pi}{2}}\,k^{-1}\exp(-ak)$ |

| | $f(x)$ | $F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(kx)\ f(x)dx$ |
|---|---|---|
| 13 | $x^{-v} J_{v+1}(ax), \quad v > -\frac{1}{2}$ | $\frac{k(a^2-k^2)^{v-\frac{1}{2}}}{2^{v-\frac{1}{2}} a^{v+1} \Gamma\left(v+\frac{1}{2}\right)} H(a-k)$ |
| 14 | $x^{-1} J_0(ax)$ | $\begin{cases} \sqrt{\frac{2}{\pi}} \sin^{-1}\left(\frac{k}{a}\right), 0<k<a \\ \sqrt{\frac{\pi}{2}}, \quad a<k<\infty \end{cases}$ |
| 15 | $x(a^2+x^2)^{-1} J_0(bx),\ a>0,\ b>0$ | $\sqrt{\frac{\pi}{2}}\, e^{-ak} I_0(ab), \quad a<k<\infty$ |
| 16 | $J_0(a\sqrt{x}), \quad a>0$ | $\sqrt{\frac{2}{\pi}}\,\frac{1}{k}\cos\left(\frac{a^2}{4k}\right)$ |
| 17 | $(x^2-a^2)^{v-\frac{1}{2}} H(x-a),\ \lvert v\rvert < \frac{1}{2}$ | $2^{v-\frac{1}{2}} \left(\frac{a}{k}\right)^v \Gamma\left(v+\frac{1}{2}\right) J_{-v}(ak)$ |
| 18 | $x^{1-v}(x^2+a^2)^{-1} J_v(ax),$ $v>-\frac{3}{2}, \quad a,b>0$ | $\sqrt{\frac{\pi}{2}}\, a^{-v} \exp(-ak)\, I_v(ab),$ $a<k<\infty$ |
| 19 | $H(a-x), \quad a>0$ | $\sqrt{\frac{2}{\pi}}\,\frac{1}{k}(1-\cos ak)$ |
| 20 | $erfc(ax)$ | $\sqrt{\frac{2}{\pi}}\,\frac{1}{k}\left[1-\exp\left(-\frac{k^2}{4a^2}\right)\right]$ |
| 21 | $x^{-\alpha}, \quad 0<\alpha<2$ | $\Gamma(1-\alpha)\, k^{\alpha-1}\cos\left(\frac{\alpha\pi}{2}\right)$ |
| 22 | $(ax-x^2)^{\alpha-\frac{1}{2}} H(a-x),\ \alpha>-\frac{1}{2}$ | $\sqrt{2}\,\Gamma\left(\alpha+\frac{1}{2}\right)\left(\frac{a}{k}\right)^\alpha$ $\times \sin\left(\frac{ak}{2}\right) J_\alpha\left(\frac{ak}{2}\right)$ |
| 23 | $e^{-bx}\sin(ax)$ | $\frac{b}{\sqrt{2\pi}}\left[\frac{1}{b^2+(a-k)^2} - \frac{1}{b^2+(a+k)^2}\right]$ |
| 24 | $\ln\left\lvert\frac{a+x}{b-x}\right\rvert$ | $\sqrt{2\pi}\,\frac{\sin(ak)}{k}$ |

**TABLE B-4 Laplace Transforms**

| | $f(t)$ | $\bar{f}(s)=\int_0^\infty \exp(-st)\ f(t)dt$ |
|---|---|---|
| 1 | $t^n \quad (n=0,1,2,3,\ldots)$ | $\dfrac{n!}{s^{n+1}}$ |
| 2 | $e^{at}$ | $\dfrac{1}{s-a}$ |
| 3 | $\cos at$ | $\dfrac{s}{s^2+a^2}$ |
| 4 | $\sin at$ | $\dfrac{a}{s^2+a^2}$ |
| 5 | $\cosh at$ | $\dfrac{s}{s^2-a^2}$ |
| 6 | $\sinh at$ | $\dfrac{a}{s^2-a^2}$ |
| 7 | $t^n e^{-at}$ | $\dfrac{\Gamma(n+1)}{(s+a)^{n+1}}$ |
| 8 | $t^a \quad (a>-1)$ | $\dfrac{\Gamma(a+1)}{s^{a+1}}$ |
| 9 | $e^{at}\cos bt$ | $\dfrac{s-a}{(s-a)^2+b^2}$ |
| 10 | $e^{at}\sin bt$ | $\dfrac{b}{(s-a)^2+b^2}$ |
| 11 | $(e^{at}-e^{bt})$ | $\dfrac{a-b}{(s-a)(s-b)}$ |
| 12 | $\dfrac{1}{(a-b)}(a\,e^{at}-be^{bt})$ | $\dfrac{s}{(s-a)(s-b)}$ |
| 13 | $t\sin at$ | $\dfrac{2as}{(s^2+a^2)^2}$ |
| 14 | $t\cos at$ | $\dfrac{s^2-a^2}{(s^2+a^2)^2}$ |
| 15 | $\sin at \sinh at$ | $\dfrac{2sa^2}{(s^4+4a^4)}$ |

| | $f(t)$ | $\bar{f}(s)=\int_0^\infty \exp(-st)\ f(t)dt$ |
|---|---|---|
| 16 | $(\sinh at - \sin at)$ | $\frac{2a^3}{(s^4-a^4)}$ |
| 17 | $(\cosh at - \cos at)$ | $\frac{2a^2 s}{(s^4-a^4)}$ |
| 18 | $\frac{\cos at - \cos bt}{(b^2-a^2)}\ (a^2 \neq b^2)$ | $\frac{s}{(s^2+a^2)(s^2+b^2)}$ |
| 19 | $\frac{1}{\sqrt{t}}$ | $\sqrt{\frac{\pi}{s}}$ |
| 20 | $2\sqrt{t}$ | $\frac{1}{s}\sqrt{\frac{\pi}{s}}$ |
| 21 | $t\cosh at$ | $(s^2+a^2)(s^2-a^2)^{-2}$ |
| 22 | $t\sinh at$ | $2as(s^2-a^2)^{-2}$ |
| 23 | $\frac{\sin(at)}{t}$ | $\tan^{-1}\left(\frac{a}{s}\right)$ |
| 24 | $t^{-1/2}\exp\left(-\frac{a}{t}\right)$ | $\sqrt{\frac{\pi}{s}}\exp(-2\sqrt{as})$ |
| 25 | $t^{-3/2}\exp\left(-\frac{a}{t}\right)$ | $\sqrt{\frac{\pi}{a}}\exp(-2\sqrt{as})$ |
| 26 | $\frac{1}{\sqrt{\pi t}}(1+2at)e^{at}$ | $\frac{s}{(s-a)\sqrt{s-a}}$ |
| 27 | $(1+at)e^{at}$ | $\frac{s}{(s-a)^2}$ |
| 28 | $\frac{1}{2\sqrt{\pi t^3}}(e^{bt}-e^{at})$ | $\sqrt{s-a}-\sqrt{s-b}$ |
| 29 | $\exp(a^2 t)\,erf(a\sqrt{t})$ | $\frac{a}{\sqrt{s}(s-a^2)}$ |
| 30 | $\exp(a^2 t)\,erfc\,(a\sqrt{t})$ | $\frac{1}{\sqrt{s}\,(\sqrt{s}+a)}$ |
| 31 | $\frac{1}{\sqrt{\pi t}} + a\exp(a^2 t)\,erf(a\sqrt{t})$ | $\frac{\sqrt{s}}{(s-a^2)}$ |

| | $f(t)$ | $\bar{f}(s)=\int_0^\infty \exp(-st)\ f(t)dt$ |
|---|---|---|
| 32 | $\frac{1}{\sqrt{\pi t}}-a\ \exp(a^2t)\ erfc(a\sqrt{t})$ | $\frac{1}{\sqrt{s}+a}$ |
| 33 | $\frac{\exp(-at)}{\sqrt{b-a}}\ erf\left(\sqrt{(b-a)t}\right)$ | $\frac{1}{(s+a)\sqrt{s+b}}$ |
| 34 | $\frac{1}{2}e^{i\omega t}\left[e^{-\lambda z}\ erfc(\zeta-\sqrt{i\omega t})\right.$ $\left.+\exp(\lambda z)\ erfc(\zeta+\sqrt{i\omega t})\right]$, where $\zeta=z/2\sqrt{vt},\ \lambda=\sqrt{\frac{i\omega}{v}}$. | $(s-i\omega)^{-1}\,e^{-z\sqrt{\frac{s}{v}}}$ |
| 35 | $\frac{1}{2}\left[e^{-ab}\ erfc\left(\frac{b-2at}{2\sqrt{t}}\right)\right.$ $\left.+\exp(ab)\ erfc\left(\frac{b+2at}{2\sqrt{t}}\right)\right]$ | $e^{-b(s+a^2)^{\frac{1}{2}}}$ |
| 36 | $Si(t)=\int_0^t \frac{\sin x}{x}dx$ | $\frac{1}{s}\cot^{-1}(s)$ |
| 37 | $Ci(t)=-\int_t^\infty \frac{\cos x}{x}dx$ | $-\frac{1}{2s}\ \log(1+s^2)$ |
| 38 | $-Ei(-t)=\int_t^\infty \frac{e^{-x}}{x}dx$ | $\frac{1}{s}\ \log(1+s)$ |
| 39 | $J_0(at)$ | $(s^2+a^2)^{-\frac{1}{2}}$ |
| 40 | $I_0(at)$ | $(s^2-a^2)^{-\frac{1}{2}}$ |
| 41 | $t^{\alpha-1}\exp(-at),\quad a>0$ | $\Gamma(\alpha)(s+a)^{-\alpha}$ |
| 42 | $\frac{\sqrt{\pi}}{\Gamma\left(v+\frac{1}{2}\right)}\left(\frac{t}{2a}\right)^v J_v(at)$ | $(s^2+a^2)^{-\left(v+\frac{1}{2}\right)}$, $\mathrm{Re}\ v>-\frac{1}{2}$ |
| 43 | $t^{-1}\ J_v(at)$ | $\frac{a^v}{v\left(\sqrt{s^2+a^2}+s\right)^v}$, $\mathrm{Re}\,v>-\frac{1}{2}$ |
| 44 | $J_0(a\sqrt{t})$ | $\frac{1}{s}\ \exp\left(-\frac{a^2}{4s}\right)$ |

| | $f(t)$ | $\bar{f}(s)=\int_0^\infty \exp(-st)\ f(t)dt$ |
|---|---|---|
| 45 | $\left(\frac{2}{a}\right)^v t^{v/2} J_v(a\sqrt{t})$ | $s^{-(v+1)}\exp\left(-\frac{a^2}{4s}\right)$, $\mathrm{Re}\, v>-\frac{1}{2}$ |
| 46 | $\frac{a}{2t\sqrt{\pi t}}\exp\left(-\frac{a^2}{4t}\right)$ | $\exp(-a\sqrt{s}), \quad a>0$ |
| 47 | $\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{a^2}{4t}\right)$ | $\frac{1}{\sqrt{s}}\exp(-a\sqrt{s}), \quad a\geq 0$ |
| 48 | $\exp\left(-\frac{a^2t^2}{4}\right)$ | $\frac{\sqrt{\pi}}{a}\exp\left(\frac{s^2}{a^2}\right)\ erfc\left(\frac{s}{a}\right), \ a>0$ |
| 49 | $(t^2-a^2)^{-\frac{1}{2}}H(t-a)$ | $K_0(as), \quad a>0$ |
| 50 | $\delta(t-a)$ | $\exp(-as), \quad a\geq 0$ |
| 51 | $H(t-a)$ | $\frac{1}{s}\exp(-as), \quad a\geq 0$ |
| 52 | $\delta'(t-a)$ | $s\,e^{-as}, \quad a\geq 0$ |
| 53 | $\delta^{(n)}(t-a)$ | $s^n\exp(-as)$ |
| 54 | $\lvert\sin at\rvert, \quad (a>0)$ | $\frac{a}{(s^2+a^2)}\coth\left(\frac{\pi s}{2a}\right)$ |
| 55 | $\frac{1}{\sqrt{\pi t}}\cos(2\sqrt{at})$ | $\frac{1}{\sqrt{s}}\exp\left(-\frac{a}{s}\right)$ |
| 56 | $\frac{1}{\sqrt{\pi t}}\sin(2\sqrt{at})$ | $\frac{1}{s\sqrt{s}}\exp\left(-\frac{a}{s}\right)$ |
| 57 | $\frac{1}{\sqrt{\pi a}}\cosh(2\sqrt{at})$ | $\frac{1}{\sqrt{s}}\exp\left(\frac{a}{s}\right)$ |
| 58 | $\frac{1}{\sqrt{\pi a}}\sinh(2\sqrt{at})$ | $\frac{1}{s\sqrt{s}}\exp\left(\frac{a}{s}\right)$ |
| 59 | $erf\left(\frac{t}{2a}\right)$ | $\frac{1}{s}\exp(a^2s^2)\ erfc(as), \quad a>0$ |

| | $f(t)$ | $\bar{f}(s)=\int\limits_0^\infty \exp(-st)\ f(t)dt$ |
|---|---|---|
| 60 | $erfc\left(\frac{a}{2\sqrt{t}}\right)$ | $\frac{1}{s}\exp(-a\sqrt{s}),\quad a\geq 0$ |
| 61 | $\sqrt{\frac{4t}{\pi}}e^{-\frac{a^2}{4t}}-a\ erfc\left(\frac{a}{2\sqrt{t}}\right)$ | $\frac{1}{s\sqrt{s}}\exp(-a\sqrt{s}),\quad a\geq 0$ |
| 62 | $e^{a(b+at)}\,erfc\left(a\sqrt{t}+\frac{b}{2\sqrt{t}}\right)$ | $\frac{\exp(-b\sqrt{s})}{\sqrt{s}(\sqrt{s}+a)},\quad a\geq 0$ |
| 63 | $J_0\left(a\sqrt{t^2-\omega^2}\right)\ H(t-\omega)$ | $(s^2+a^2)^{-\frac{1}{2}}\exp\left\{-\omega\sqrt{s^2+a^2}\right\}$ |
| 64 | $\frac{1}{t}(e^{bt}-e^{at})$ | $\log\left(\frac{s-a}{s-b}\right)$ |
| 65 | $\{\pi(t+a)\}^{-\frac{1}{2}}$ | $\frac{1}{\sqrt{s}}\exp(as)\ erfc(\sqrt{as}),\ a>0$ |
| 66 | $\frac{1}{\pi t}\sin(2a\sqrt{t})$ | $erf\left(\frac{a}{\sqrt{s}}\right)$ |
| 67 | $\frac{1}{\sqrt{\pi t}}\ \exp(-2a\sqrt{t}),\quad a\geq 0$ | $\frac{1}{\sqrt{s}}\ \exp\left(\frac{a^2}{s}\right)\ erfc\left(\frac{a}{\sqrt{s}}\right)$ |
| 68 | $C(t)=\frac{1}{\sqrt{2\pi}}\int\limits_0^t \frac{\cos u}{\sqrt{u}}\,du$ | $\frac{1}{2s}\left[\frac{1}{\sqrt{1+s^2}}+\frac{s}{1+s^2}\right]^{\frac{1}{2}}$ |
| 69 | $S(t)=\frac{1}{\sqrt{2\pi}}\int\limits_0^t \frac{\sin u}{\sqrt{u}}\,du$ | $\frac{1}{2s}\left[\frac{1}{\sqrt{1+s^2}}-\frac{s}{1+s^2}\right]^{\frac{1}{2}}$ |
| 70 | $\mathscr{I}(t)=1+2\sum\limits_{n=1}^{\infty}\exp(-n^2\pi t)$ | $(\sqrt{s}\ \tanh\sqrt{s})^{-1}$ |
| 71 | $t^{m\alpha+\beta-1}E_{\alpha,\beta}^{(m)}(\pm at)$ | $\frac{m!s^{\alpha-\beta}}{(s^\alpha\mp a)^{m+1}}$ |
| 72 | $\frac{1+2at}{\sqrt{\pi t}}$ | $\frac{s+a}{s\sqrt{s}}$ |

**TABLE B-5 Hankel Transforms**

| | $f(r)$ | order | $\tilde{f}_n(k) = \int_0^\infty r J_n(kr)\, f(r)\, dr$ |
|---|---|---|---|
| 1 | $H(a-r)$ | 0 | $\frac{a}{k} J_1(ak)$ |
| 2 | $\exp(-ar)$ | 0 | $a(a^2+k^2)^{-\frac{3}{2}}$ |
| 3 | $\frac{1}{r}\exp(-ar)$ | 0 | $(a^2+k^2)^{-\frac{1}{2}}$ |
| 4 | $(a^2-r^2)\, H(a-r)$ | 0 | $\frac{4a}{k^3} J_1(ak) - \frac{2a^2}{k^2} J_0(ak)$ |
| 5 | $a(a^2+r^2)^{-\frac{3}{2}}$ | 0 | $\exp(-ak)$ |
| 6 | $\frac{1}{r}\cos(ar)$ | 0 | $(k^2-a^2)^{-\frac{1}{2}} H(k-a)$ |
| 7 | $\frac{1}{r}\sin(ar)$ | 0 | $(a^2-k^2)^{-\frac{1}{2}} H(a-k)$ |
| 8 | $\frac{1}{r^2}(1-\cos ar)$ | 0 | $\cosh^{-1}\left(\frac{a}{k}\right) H(a-k)$ |
| 9 | $\frac{1}{r} J_1(ar)$ | 0 | $\frac{1}{a} H(a-k), \quad a > 0$ |
| 10 | $Y_0(ar)$ | 0 | $\left(\frac{2}{\pi}\right)(a^2-k^2)^{-1}$ |
| 11 | $K_0(ar)$ | 0 | $(a^2+k^2)^{-1}$ |
| 12 | $\frac{\delta(r)}{r}$ | 0 | 1 |
| 13 | $(r^2+b^2)^{-\frac{1}{2}} \times \exp\left\{-a(r^2+b^2)^{\frac{1}{2}}\right\}$ | 0 | $(k^2+a^2)^{-\frac{1}{2}} \times \exp\left\{-b(k^2+a^2)^{\frac{1}{2}}\right\}$ |
| 14 | $\frac{\sin r}{r^2}$ | 0 | $\begin{cases} \frac{\pi}{2}, & k < 1 \\ \sin^{-1}\left(\frac{1}{k}\right), & k > 1 \end{cases}$ |

| | $f(r)$ | order | $\tilde{f}_n(k) = \int_0^\infty r J_n(kr)\, f(r)\, dr$ |
|---|---|---|---|
| 15 | $(r^2+a^2)^{-\frac{1}{2}}$ | 0 | $\frac{1}{k}\exp(-ak)$ |
| 16 | $\exp(-ar)$ | 1 | $k(a^2+k^2)^{-3/2}$ |
| 17 | $\frac{\sin ar}{r}$ | 1 | $\frac{a\,H(k-a)}{k(k^2-a^2)^{\frac{1}{2}}}$ |
| 18 | $\frac{1}{r}\exp(-ar)$ | 1 | $\frac{1}{k}\left[1-\frac{a}{(k^2+a^2)^{\frac{1}{2}}}\right]$ |
| 19 | $\frac{1}{r^2}\exp(-ar)$ | 1 | $\frac{1}{k}\left[(k^2+a^2)^{\frac{1}{2}}-a\right]$ |
| 20 | $r^n\,H(a-r)$ | $>-1$ | $\frac{1}{k}a^{n+1}\,J_{n+1}(ak)$ |
| 21 | $r^n\exp(-ar),\ \ \mathrm{Re}\,a>0$ | $>-1$ | $\frac{1}{\sqrt{\pi}}\frac{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)ak^n}{(a^2+k^2)^{n+\frac{3}{2}}}$ |
| 22 | $r^n\exp(-ar^2)$ | $>-1$ | $\frac{k^n}{(2a)^{n+1}}\exp\left(-\frac{k^2}{4a}\right)$ |
| 23 | $r^{a-1}$ | $>-1$ | $\frac{2^a\Gamma\left[\frac{1}{2}(a+n+1)\right]}{k^{a+1}\Gamma\left[\frac{1}{2}(1-a+n)\right]}$ |
| 24 | $r^n(a^2-r^2)^{m-n-1}$ $\times H(a-r)$ | $>-1$ | $2^{m-n-1}\Gamma(m-n)a^m$ $\times k^{n-m}J_m(ak)$ |
| 25 | $r^m\exp(-r^2/a^2)$ | $>-1$ | ${}_1F_1\left(1+\frac{m}{2}+\frac{n}{2};n+1;-\frac{1}{4}a^2k^2\right)$ $\times\frac{k^n a^{m+n+2}}{2^{n+1}\Gamma(n+1)}\,\Gamma\left(1+\frac{m}{2}+\frac{n}{2}\right)$ |
| 26 | $\frac{1}{r}J_{n+1}(ar)$ | $>-1$ | $k^n a^{-(n+1)}H(a-k),\ \ a>0$ |
| 27 | $r^n(a^2-r^2)^m H(a-r),$ $m>-1$ | $>-1$ | $2^m a^n\Gamma(m+1)\left(\frac{a}{k}\right)^{m+1}$ $\times J_{n+m+1}(ak)$ |

| | $f(r)$ | order | $\tilde{f}_n(k) = \int_0^\infty r J_n(kr)\, f(r)\, dr$ |
|---|---|---|---|
| 28 | $\frac{1}{r^2} J_n(ar)$ | $> \frac{1}{2}$ | $\begin{Bmatrix} \frac{1}{2n}\left(\frac{k}{a}\right)^n, & 0 < k \le a \\ \frac{1}{2n}\left(\frac{a}{k}\right)^n, & a < k < \infty \end{Bmatrix}$ |
| 29 | $\frac{r^n}{(a^2+r^2)^{m+1}}, \quad a > 0$ | $> -1$ | $\left(\frac{k}{2}\right)^m \frac{a^{n-m}}{\Gamma(m+1)} K_{n-m}(ak)$ |
| 30 | $\exp(-p^2r^2)\, J_n(ar),$ | $> -1$ | $(2p^2)^{-1} \exp\left(-\frac{a^2+k^2}{4p^2}\right) \times I_n\left(\frac{ak}{2p^2}\right)$ |
| 31 | $\frac{1}{r}\exp(-ar)$ | $> -1$ | $\frac{\left\{(k^2+a^2)^{\frac{1}{2}} - a\right\}^n}{k^n(k^2+a^2)^{\frac{1}{2}}}$ |
| 32 | $\frac{r^n}{(r^2+a^2)^{n+1}}$ | $> -1$ | $\left(\frac{k}{2}\right)^n \frac{K_0(ak)}{\Gamma(n+1)}$ |
| 33 | $\frac{r^n}{(a^2-r^2)^{n+\frac{1}{2}}} H(a-r)$ | $< 1$ | $\frac{1}{\sqrt{\pi}}\left(\frac{k}{2}\right)^n \Gamma\left(\frac{1}{2}-n\right)\left(\frac{\sin ak}{k}\right)$ |
| 34 | $\frac{1}{\sqrt{r}} J_{n-1}(ar)$ | $> -1$ | $\begin{Bmatrix} 0, & 0 < k \le a \\ \frac{1}{\sqrt{a}}\left(\frac{a}{k}\right)^{n-\frac{1}{2}}, & a < k < \infty \end{Bmatrix}$ |
| 35 | $\frac{1}{r\sqrt{r}} J_n(ar)$ | $> 0$ | $\begin{Bmatrix} \frac{\sqrt{a}}{2n}\left(\frac{k}{a}\right)^{n+\frac{1}{2}}, & 0 < k \le a \\ \frac{\sqrt{a}}{2n}\left(\frac{a}{k}\right)^{n+\frac{1}{2}}, & a < k < \infty \end{Bmatrix}$ |
| 36 | $\frac{1}{\sqrt{r}} J_{n+1}(ar)$ | $> -\frac{3}{2}$ | $\begin{Bmatrix} \frac{1}{\sqrt{a}}\left(\frac{k}{a}\right)^{n+\frac{1}{2}}, & 0 < k \le a \\ 0, & a < k < \infty \end{Bmatrix}$ |
| 37 | $r^{n-1}e^{-ar}$ | $> -1$ | $\frac{(2k)^n(n-\frac{1}{2})!}{\sqrt{\pi}(k^2+a^2)^{n+\frac{1}{2}}}$ |
| 38 | $e^{-ar^2} J_0(br)$ | $0$ | $\frac{a}{2}\exp\left(\frac{k^2-b^2}{4a}\right) I_0\left(\frac{bk}{2a}\right)$ |

**TABLE B-6 Mellin Transforms**

| | $f(x)$ | $\tilde{f}(p)=\int_0^\infty x^{p-1}f(x)dx$ |
|---|---|---|
| 1 | $\exp(-nx)$ | $n^{-p}\,\Gamma(p),\quad \operatorname{Re} p>0$ |
| 2 | $\exp(-ax^2),\quad a>0$ | $\frac{1}{2}a^{-(p/2)}\,\Gamma\left(\frac{p}{2}\right),\quad \operatorname{Re} p>0$ |
| 3 | $\cos(ax)$ | $a^{-p}\,\Gamma(p)\cos\left(\frac{\pi p}{2}\right),\quad 0<\operatorname{Re} p<1$ |
| 4 | $\sin(ax)$ | $a^{-p}\,\Gamma(p)\sin\left(\frac{\pi p}{2}\right),\quad 0<\operatorname{Re} p<1$ |
| 5 | $(a+x)^{-1},\qquad \lvert\arg a\rvert<\pi$ | $\pi a^{p-1}\operatorname{cosec}(\pi p),\quad 0<\operatorname{Re} p<1$ |
| 6 | $(a-x)^{-1}$ | $\pi a^{p-1}\cot(\pi p),\quad 0<\operatorname{Re} p<1$ |
| 7 | $(1+x)^{-a},\quad \operatorname{Re} a>0$ | $\frac{\Gamma(p)\Gamma(a-p)}{\Gamma(a)}$ |
| 8 | $(1+x^a)^{-s}$ | $\frac{\Gamma(p/a)\,\Gamma(s-p/a)}{a\,\Gamma(s)}$ |
| 9 | $(a^2+x^2)^{-1}$ | $\frac{\pi}{2}\,a^{(p-2)}\operatorname{cosec}\left(\frac{\pi p}{2}\right)$ |
| 10 | $\begin{Bmatrix}1, & 0\le x\le a\\ 0, & x>a\end{Bmatrix}$ | $p^{-1}a^p$ |
| 11 | $Ci\,(x)=-\int_x^\infty \frac{\cos t}{t}dt$ | $-p^{-1}\Gamma(p)\cos\left(\frac{p\pi}{2}\right),\ 0<\operatorname{Re} p<1$ |
| 12 | $Si(x)=\int_0^x \frac{\sin t}{t}\,dt$ | $-p^{-1}\Gamma(p)\sin\left(\frac{\pi p}{2}\right),\ -1<\operatorname{Re} p<0$ |
| 13 | $\begin{Bmatrix}(1-x)^{a-1}, & 0<x<1\\ 0, & x\ge 1\end{Bmatrix}$ | $\frac{\Gamma(a)\,\Gamma(p)}{\Gamma(a+p)}$ |
| 14 | $\begin{Bmatrix}0, & 0<x\le 1\\ (x-1)^{-a}, & x>1\end{Bmatrix}$ | $\frac{\Gamma(a-p)\,\Gamma(1-a)}{\Gamma(1-p)}$ |

| | $f(x)$ | $\tilde{f}(p) = \int_0^\infty x^{p-1} f(x) dx$ |
|---|---|---|
| 15 | $\exp(-ax)\, H(x-b)$ | $a^{-p}\, \Gamma(p, ab)$ |
| 16 | $\exp(-ax)\, H(b-x)$ | $a^{-p}\, \gamma(p, ab)$ |
| 17 | ${}_2F_1(a, b, c; -x)$ | $\dfrac{\Gamma(p)\, \Gamma(a-p)\, \Gamma(b-p)\, \Gamma(c)}{\Gamma(c-p)\, \Gamma(a)\, \Gamma(b)}$ |
| 18 | $x^{\frac{1}{2}} J_v(x)$ | $\dfrac{2^{p-\frac{1}{2}} \Gamma\left[\frac{1}{2}\left(p+v+\frac{1}{2}\right)\right]}{\Gamma\left[\frac{1}{2}\left(v-p+\frac{3}{2}\right)\right]}$ |
| 19 | $x^{-v} J_v(ax)$ | $\dfrac{2^{p-v-1} a^{v-p} \Gamma\left(\frac{p}{2}\right)}{\Gamma\left(v-\frac{1}{2}p+1\right)}$ |
| 20 | $P_n(x)\, H(1-x)$ | $\dfrac{\Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{p}{2}+\frac{1}{2}\right)}{2\Gamma\left(\frac{p}{2}-\frac{n}{2}+\frac{1}{2}\right) \Gamma\left(\frac{p}{2}+\frac{n}{2}+1\right)}$ |
| 21 | $\begin{cases} \log\left(\frac{a}{x}\right), & x < a \\ 0, & x \geq a \end{cases}$ | $\dfrac{a^p}{p^2}$ |
| 22 | $x^{-1} \log(1+x)$ | $\pi(1-p)^{-1}\ \text{cosec}\,(\pi p)$ |
| 23 | $(e^x - 1)^{-1}$ | $\Gamma(p)\, \zeta(p)$ |
| 24 | $(e^x + e^{-x})^{-1}$ | $L(p)\, \Gamma(p)$ |
| 25 | $\log \left\lvert \dfrac{1+x}{1-x} \right\rvert$ | $\left(\dfrac{\pi}{p}\right)\ \tan\left(\dfrac{p\pi}{2}\right)$ |
| 26 | $(1+x)^{-m} P_{m-1}\left(\frac{1-x}{1+x}\right)$ | $\dfrac{\Gamma(p)\{\Gamma(m-p)\}^2}{\Gamma(1-p)\, \{\Gamma(m)\}^2}$ |

| | $f(x)$ | $\tilde{f}(p) = \int_0^\infty x^{p-1} f(x) dx$ |
|---|---|---|
| 27 | $x^a (1+x)^{-b}$ | $\dfrac{\Gamma(a+p)\,\Gamma(b-a-p)}{\Gamma(b)}$ |
| 28 | $x^{-2v} J_v(x) K_v(x)$ | $\dfrac{2^{p-2v-2}\,\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{p}{2}-v\right)}{\Gamma\left(1+v-\frac{p}{4}\right)}$ |
| 29 | $\log(1+ax),\ \lvert\arg a\rvert < \pi$ | $\dfrac{\pi}{p} a^{-p} \operatorname{cosec}(\pi p),\ -1 < \operatorname{Re} p < 0$ |
| 30 | $x^{v+1} J_v(ax)$ | $2^{p+v} a^{-(p+v+1)} \dfrac{\Gamma\left(\frac{p}{2}+v+\frac{1}{2}\right)}{\Gamma\left(\frac{1-p}{2}\right)}$ |
| 31 | $(1+x^2)^{-(1+\alpha)}\, H(x-1)$ | $\dfrac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\alpha+1-\frac{p}{2}\right)}{2\,\Gamma(\alpha+1)}$ |
| 32 | $\cos(x^\alpha)$ | $\dfrac{1}{\alpha}\,\Gamma\left(\dfrac{p}{\alpha}\right)\cos\left(\dfrac{p\pi}{2\alpha}\right)$ |
| 33 | $\sin(x^\alpha)$ | $\dfrac{1}{\alpha}\,\Gamma\left(\dfrac{p}{\alpha}\right)\sin\left(\dfrac{p\pi}{2\alpha}\right)$ |
| 34 | $(1+ax)^{-n}$ | $\dfrac{\Gamma(p)\,\Gamma(n-p)}{a^p\,\Gamma(n)},\ 0 < \operatorname{Re} p < n$ |
| 35 | $e^{-x} (\log\, x)^n$ | $\dfrac{d^n}{dp^n}\Gamma(p) \quad Re\, p > 0$ |
| 36 | $e^{-ax} I_v(ax),\ Re; a > 0$ | $\dfrac{\Gamma\left(\frac{1}{2}-p\right)\Gamma(v+p)}{2^p a^p \sqrt{\pi}\,\Gamma(1+v-p)}$ |
| 37 | $e^{-ax} K_v(ax),\ Re\, a > 0$ | $\dfrac{\sqrt{\pi}\,\Gamma(p+v)\,\Gamma(p-v)}{2^p a^p \Gamma\left(p+\frac{1}{2}\right)}$ |
| 38 | erfc(x) | $\dfrac{\Gamma\left(\frac{p+1}{2}\right)}{p\sqrt{\pi}}$ |

**TABLE B-7 Hilbert Transforms**

| | $f(t)$ | $\hat{f}_H(x) = \dfrac{1}{\pi} \displaystyle\oint_{-\infty}^{\infty} \dfrac{f(t)}{(t-x)}\,dt$ |
|---|---|---|
| 1 | $1$ | $0$ |
| 2 | $\begin{Bmatrix} 0, & -\infty < t < a \\ 1, & a < t < b \\ 0, & b < t < \infty \end{Bmatrix}$ | $\dfrac{1}{\pi} \log \left\| \dfrac{b-x}{a-x} \right\|$ |
| 3 | $(t+a)^{-1}, \quad \mathrm{Im}\, a > 0$ | $i(x+a)^{-1}$ |
| 4 | $(t+a)^{-1}, \quad \mathrm{Im}\, a < 0$ | $-i(x+a)^{-1}$ |
| 5 | $\begin{Bmatrix} 0, & -\infty < t < 0 \\ (at+b)^{-1}, & 0 < t < \infty \\ & a, b > 0 \end{Bmatrix}$ | $\frac{1}{\pi}(ax+b)^{-1} \log \left\| \frac{b}{ax} \right\|, \; ax \neq -b$ |
| 6 | $\dfrac{t}{(t^2+a^2)}, \quad \mathrm{Re}\, a > 0$ | $\dfrac{a}{(x^2+a^2)}$ |
| 7 | $\dfrac{1}{(t^2+a^2)}, \quad \mathrm{Re}\, a > 0$ | $-\dfrac{x}{a(x^2+a^2)}$ |
| 8 | $\dfrac{\alpha t + \beta a}{(t^2+a^2)}, \quad \mathrm{Re}\ a > 0$ | $\dfrac{\alpha a - \beta x}{(x^2+a^2)}$ |
| 9 | $\exp(iat), \quad a > 0$ | $i \exp(iax)$ |
| 10 | $\cos(at), \quad a > 0$ | $-\sin(ax)$ |
| 11 | $\sin(at), \quad a > 0$ | $\cos(ax)$ |
| 12 | $\dfrac{a}{a^2+(t+b)^2}, \quad a > 0$ | $-\dfrac{(b+x)}{a^2+(b+x)^2}$ |
| 13 | $\begin{Bmatrix} 0, & -\infty < t < -a \\ (a^2-t^2)^{-\frac{1}{2}}, & -a < t < a \\ 0, & a < t < \infty \end{Bmatrix}$ | $\begin{Bmatrix} (x^2-a^2)^{-\frac{1}{2}}, & -\infty < x < -a \\ 0, & -a < x < a \\ -(x^2-a^2)^{-\frac{1}{2}}, & a < x < \infty \end{Bmatrix}$ |
| 14 | $H(t-a) - H(t-b), \; b > a > 0$ | $\dfrac{1}{\pi} \log \left\| \dfrac{x-b}{x-a} \right\|$ |

| | $f(t)$ | $\hat{f}_H(x) = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(t)}{(t-x)}\,dt$ |
|---|---|---|
| 15 | $\frac{1}{t}\,H(t-a), \quad a>0$ | $\frac{1}{\pi x}\log\left\|\frac{a}{x-a}\right\|, \quad x \neq 0,\, x \neq a$ |
| 16 | $\left\{\begin{matrix} -(t^2-a^2)^{-\frac{1}{2}}, & -\infty<t<-a \\ 0, & -a<t<a \\ (t^2-a^2)^{-\frac{1}{2}}, & a<t<\infty \end{matrix}\right\}$ | $\left\{\begin{matrix} 0, & -\infty<x<-a \\ (a^2-x^2)^{-\frac{1}{2}}, & -a<x<a \\ 0, & a<x<\infty \end{matrix}\right\}$ |
| 17 | $\frac{\sin at}{t}, \quad a>0$ | $\frac{1}{x}(\cos ax - 1)$ |
| 18 | $\left\{\begin{matrix} 0, & -\infty<t<0 \\ \sin(a\sqrt{t}), & 0<t<\infty,\ a>0 \end{matrix}\right\}$ | $\left\{\begin{matrix} \exp\left(-a\sqrt{\|x\|}\right), & -\infty<x<0 \\ \cos\left(a\sqrt{x}\right), & 0<x<\infty \end{matrix}\right\}$ |
| 19 | $sgn\,t\sin\left(a\sqrt{\|t\|}\right), \quad a>0$ | $\cos\left(a\sqrt{\|x\|}\right) + \exp\left(-a\sqrt{\|x\|}\right)$ |
| 20 | $\frac{1}{t}(1-\cos\, at), \quad a>0$ | $\frac{1}{x}(\sin\, ax)$ |
| 21 | $J_n(t)\,\sin(t-x), \quad n=0,1,\ldots$ | $J_n(x)$ |
| 22 | $sgn\,t\,\|t\|^v\,J_v(a\|t\|)$, where $a>0, \quad -\frac{1}{2}<\text{Re}\ v<\frac{3}{2}$ | $-\|x\|^v\ Y_v(a\|x\|)$ |
| 23 | $\sin(at)\,J_1(at), \quad a>0$ | $\cos(ax)\,J_1(ax)$ |
| 24 | $\sin(at)\,J_n(bt)$, where $0<b<a, \quad n=0,1,2,\ldots$ | $\cos(ax)\,J_n(bx)$ |
| 25 | $\cos(at)\,J_1(at), \quad a>0$ | $-\sin(ax)\,J_1(ax)$ |
| 26 | $\cos(at)\,J_n(bt), \quad 0<b<a$ where $n=0,1,2,\ldots$ | $-\sin(ax)\,J_n(bx)$ |
| 27 | $\exp(-at)\,I_0(at)\ H(t),\ a>0$ | $\frac{1}{\pi}\exp(-ax)\,K_0(a\|x\|)$ |

| | $f(t)$ | $\hat{f}_H(x) = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(t)}{(t-x)}\, dt$ |
|---|---|---|
| 28 | $\exp(-a\|t\|)\, I_0(at),\ a > 0$ | $-\frac{2}{\pi} \sinh(ax)\, K_0(a\|x\|)$ |
| 29 | $sgn\, t \exp(-a\|t\|)\, I_0(at),\ a > 0$ | $\frac{2}{\pi} \cosh(ax)\ K_0(a\|x\|)$ |
| 30 | $\exp(at)\, K_0(a\|t\|), \quad a > 0$ | $\pi \exp(ax)\, I_0(ax)\ H(-x)$ |
| 31 | $\|t\|^v\, Y_v(a\|t\|),\ a > 0,$ $-\frac{1}{2} < \mathrm{Re}\, v < \frac{3}{2}$ | $\|x\|^v J_v(a\|x\|)\, sgn\, x$ |
| 32 | $\sinh(at)\, K_0(a\|t\|), \quad a > 0$ | $\frac{\pi}{2} \exp(-a\|x\|)\ I_0(ax)$ |
| 33 | $\cosh(at)\, K_0(a\|t\|), \quad a > 0$ | $-\frac{\pi}{2}\ \exp(-a\|x\|)\, I_0(ax)\, sgn\, x$ |
| 34 | $\log \left\| \frac{b-t}{t-a} \right\|, \quad a < b$ | $\left\{ \begin{array}{ll} 0, & -\infty < x < a \\ -\pi, & a < x < b \\ 0, & b < x < \infty \end{array} \right\}$ |
| 35 | $\log \left\| \frac{t^2-a^2}{t^2-b^2} \right\|, \quad 0 < a < b$ | $\left\{ \begin{array}{ll} -\pi, & -b < x < a \\ \pi, & a < x < b \\ 0, & \text{elsewhere} \end{array} \right\}$ |
| 36 | $\frac{1-\cos at}{t}, \quad a > 0$ | $\frac{\sin ax}{x}$ |

**TABLE B-8 Stieltjes Transforms**

| | $f(t)$ | $\tilde{f}(x)=\int_0^\infty \frac{f(t)}{(t+x)}\,dt$ |
|---|---|---|
| 1 | $(a+t)^{-1}, \quad \lvert\arg a\rvert<\pi$ | $(a-x)^{-1}\log\left(\frac{a}{x}\right)$ |
| 2 | $\frac{1}{(a^2+t^2)}, \quad \text{Re } a>0$ | $(a^2+x^2)^{-1}\left[\frac{\pi x}{2a}-\log\left(\frac{x}{a}\right)\right]$ |
| 3 | $\frac{t}{(a^2+t^2)}, \quad \text{Re } a>0$ | $(a^2+x^2)^{-1}\left[\frac{\pi a}{2}+x\log\left(\frac{x}{a}\right)\right]$ |
| 4 | $t^v, \quad -1<\text{Re } v<0$ | $-\pi x^v \text{ cosec}(\pi v)$ |
| 5 | $\left\{\begin{matrix} -1, & 2n<x<2n+1 \\ +1, & 2n+1<x<2n+2 \\ & n=0,1,2,3\ldots \end{matrix}\right\}$ | $\log\left[\frac{x}{2}\left\{\Gamma\left(\frac{x}{2}\right)\Big/\Gamma\left(\frac{x+1}{2}\right)\right\}^2\right]$ |
| 6 | $\frac{t^v}{(a+t)}, \quad \lvert\arg a\rvert<\pi,$ where $-1<\text{Re } v<1$ | $(a-x)^{-1}\,\pi(a^v-x^v)\,\text{cosec}(\pi v)$ |
| 7 | $\left(\frac{t^v-a^v}{t-a}\right), \; -1<\text{Re } v<1$ | $\left(\frac{\pi}{a+x}\right)\left[x^v \text{ cosec}(v\pi) - a^v\, ctn(v\pi)+\frac{a^v}{\pi}\log\left(\frac{a}{x}\right)\right]$ |
| 8 | $t^{v-1}(a+t)^{1-\mu}, \; \lvert\arg a\rvert<\pi,$ $0<\text{Re } v<\text{Re } \mu$ | $\frac{\Gamma(v)\Gamma(\mu-v)}{\Gamma(\mu)}\left(\frac{x^{v-1}}{a^{\mu-1}}\right) \times {}_2F_1\left(\mu-1,v,\mu;1-\frac{x}{a}\right)$ |
| 9 | $t^{-\rho}(a+t)^{-\sigma}, \; \lvert\arg a\rvert<\pi,$ $-\text{Re } \sigma<\text{Re } \rho<1$ | $\pi\,\text{cosec}(\rho\pi)x^{-\rho}(a-x)^{-\sigma} \times I_{\left(1-\frac{x}{a}\right)}(\sigma,\rho)$ |
| 10 | $\exp(-at), \quad \text{Re}\,a>0$ | $-\exp(ax)\,Ei(-ax)$ |
| 11 | $\left\{\begin{matrix} \exp(-at), & 0<t<b \\ 0, & b<t<\infty \end{matrix}\right\}$ | $e^{ax}[Ei(-ab-ax)-Ei(-ax)]$ |

| | $f(t)$ | $\tilde{f}(x)=\int_0^\infty \frac{f(t)}{(t+x)}\,dt$ |
|---|---|---|
| 12 | $\left\{\begin{matrix} 0, & 0<t<b \\ \exp(-at), & b<t<\infty \\ & \text{Re } a>0 \end{matrix}\right\}$ | $-\exp(-ax)\,Ei(-ab-ax)$ |
| 13 | $\frac{1}{\sqrt{t}}\exp(-at), \quad \text{Re } a>0$ | $\frac{\pi}{\sqrt{x}}\exp(ax)\,erfc\,(\sqrt{ax})$ |
| 14 | $\sqrt{t}\exp(-at), \quad \text{Re } a>0$ | $\sqrt{\frac{\pi}{a}}-\pi\sqrt{x}\exp(ax)\,erfc\,(\sqrt{ax})$ |
| 15 | $t^{-v}\exp(-at), \quad \text{Re } a>0, \text{Re } v<1$ | $\Gamma(1-v)\,x^{-v}\,\exp(ax)\Gamma(v,ax)$ |
| 16 | $t^{v-1}\exp\left(-\frac{a}{t}\right), \quad \text{Re } a>0, \text{Re } v<1$ | $\Gamma(1-v)\,x^{v-1}\,\exp\left(\frac{a}{x}\right)\Gamma\left(v,\frac{a}{x}\right)$ |
| 17 | $\exp(-a\sqrt{t}), \quad \text{Re } a>0$ | $2\left[\cos(a\sqrt{x})\,Ci(a\sqrt{x}) - \sin(a\sqrt{x})\,Si(a\sqrt{x})\right]$ |
| 18 | $\frac{1}{\sqrt{t}}\exp(-a\sqrt{t}), \quad \text{Re } a>0$ | $-\frac{2}{\sqrt{x}}\left[\sin(a\sqrt{x})\,Ci(a\sqrt{x}) + \cos(a\sqrt{x})\,Si(a\sqrt{x})\right]$ |
| 19 | $(a+t)^{-1}\,\log\left(\frac{t}{a}\right), \; \lvert\arg a\rvert<\pi$ | $\frac{1}{2}(x-a)^{-1}\left[\log\left(\frac{x}{a}\right)\right]^2$ |
| 20 | $(t-a)^{-1}\,\log\left(\frac{t}{a}\right), \; a>0$ | $\frac{1}{2}(x+a)^{-1}\left[\pi^2+\left\{\log\left(\frac{x}{a}\right)\right\}^2\right]$ |
| 21 | $\frac{1}{\sqrt{t}}\log(at+b), \text{Re } a>0, \text{Re } b>0$ | $\frac{2\pi}{\sqrt{x}}\,\log\left(\sqrt{ax}+\sqrt{b}\right)$ |
| 22 | $t^v\log t, \quad -1<\text{Re } v<0$ | $-\pi x^v\text{cosec}\,(v\pi)[\log x-\pi\text{ctn}(v\pi)]$ |
| 23 | $\sin at, \quad a>0$ | $-[\sin(ax)Ci(ax)+\cos(ax)Si(ax)]$ |
| 24 | $\sin(a\sqrt{t}), \quad a>0$ | $\pi\exp(-a\sqrt{x})$ |

| | $f(t)$ | $\tilde{f}(x) = \int_0^\infty \frac{f(t)}{(t+x)}\,dt$ |
|---|---|---|
| 25 | $t^{-1}\sin(a\sqrt{t}), \quad a>0$ | $\left(\frac{\pi}{x}\right)\left[1-\exp(-a\sqrt{x})\right]$ |
| 26 | $t^{-\alpha}\sin(a\sqrt{t}+\alpha\pi)$, where $a>0,\ -\frac{1}{2}<\operatorname{Re}\alpha<1$ | $\left(\frac{\pi}{x^{\alpha}}\right)\exp(-a\sqrt{x})$ |
| 27 | $\sin\left(a\sqrt{t}-\frac{b}{\sqrt{t}}\right), \quad a,b>0$ | $\pi\,\exp\left[-\left(a\sqrt{x}+\frac{b}{\sqrt{x}}\right)\right]$ |
| 28 | $\frac{1}{\sqrt{t}}\sin^2(a\sqrt{t}), \quad a>0$ | $\left(\frac{\pi}{2\sqrt{x}}\right)\left[1-\exp(-2a\sqrt{x})\right]$ |
| 29 | $\cos(at), \quad a>0$ | $\cos(ax)\,Ci(ax)-\sin(ax)\,Si(ax)$ |
| 30 | $\frac{1}{\sqrt{t}}\cos(a\sqrt{t}), \quad a>0$ | $\left(\frac{\pi}{\sqrt{x}}\right)\exp(-a\sqrt{x})$ |
| 31 | $\frac{1}{\sqrt{t}}\cos\left(a\sqrt{t}-\frac{b}{\sqrt{t}}\right),\ a,b>0$ | $\left(\frac{\pi}{\sqrt{x}}\right)\exp\left[-\left(a\sqrt{x}+\frac{b}{\sqrt{x}}\right)\right]$ |
| 32 | $\frac{1}{\sqrt{t}}\cos(a\sqrt{t})\cos(b\sqrt{t})$, $a\ge b>0$ | $\frac{\pi}{\sqrt{x}}\exp(-a\sqrt{x})\cosh(b\sqrt{x})$ |
| 33 | $t^{\left(\frac{v}{2}+k\right)}J_v(a\sqrt{t})$ | $2(-1)^k\,x^{\left(\frac{1}{2}v+k\right)}K_v(a\sqrt{x})$ |
| 34 | $\sin(a\sqrt{t})\,J_0(b\sqrt{t}),\ 0<b<a$ | $\pi\exp(-a\sqrt{x})\,I_0(b\sqrt{x})$ |
| 35 | $\frac{1}{\sqrt{t}}\sin(a\sqrt{t})\,J_0(b\sqrt{t})$, $0<a<b$ | $\frac{2}{\sqrt{x}}\sinh(a\sqrt{x})\,K_0(b\sqrt{x})$ |
| 36 | $\cos(a\sqrt{t})J_0(b\sqrt{t}),\ 0<a<b$ | $2\cosh(a\sqrt{x})\,K_0(b\sqrt{x})$ |
| 37 | $\frac{1}{\sqrt{t}}\cos(a\sqrt{t})J_0(b\sqrt{t})$, $0<b<a$ | $\frac{\pi}{\sqrt{x}}\exp(-a\sqrt{x})I_0(b\sqrt{x})$ |
| 38 | $J_v^2(at), \quad a>0$ | $2\,I_v(a\sqrt{x})\,K_v(a\sqrt{x})$ |

**TABLE B-9 Finite Fourier Cosine Transforms**

| | $f(x)$ | $\tilde{f}_c(n) = \int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)\,dx$ |
|---|---|---|
| 1 | $1$ | $\begin{Bmatrix} a, & n=0 \\ 0, & n\neq 0 \end{Bmatrix}$ |
| 2 | $x$ | $\begin{Bmatrix} \frac{a^2}{2}, & n=0 \\ \left(\frac{a}{n\pi}\right)^2 [(-1)^n - 1], & n\neq 0 \end{Bmatrix}$ |
| 3 | $x^2$ | $\begin{Bmatrix} \frac{1}{3}a^3, & n=0 \\ 2a\left(\frac{a}{n\pi}\right)^2 (-1)^n, & n=1,2,\ldots \end{Bmatrix}$ |
| 4 | $x^3$ | $\begin{Bmatrix} \frac{1}{4}a^4 & n=0 \\ \frac{3a^4(-1)^n}{(n\pi)^2} + 6\left(\frac{a}{n\pi}\right)^4 & [(-1)^n - 1], \\ & n=1,2,3,\ldots \end{Bmatrix}$ |
| 5 | $\begin{Bmatrix} 1, & 0<x<\frac{a}{2} \\ -1, & \frac{1}{2}a<x<a \end{Bmatrix}$ | $\begin{Bmatrix} 0, & n=0 \\ \left(\frac{2a}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right), & n=1,2,3,\ldots \end{Bmatrix}$ |
| 6 | $\left(1-\frac{x}{a}\right)^2$ | $\begin{Bmatrix} \frac{1}{3}a, & n=0 \\ \frac{2a}{(n\pi)^2}, & n=1,2,\ldots \end{Bmatrix}$ |
| 7 | $\sin(bx)$ | $\frac{ba^2}{(n\pi)^2-(ab)^2}[(-1)^n\cos(ab) - 1],$ $n\pi\neq ab$ |
| 8 | $\cos(bx)$ | $\frac{(-1)^n ba^2\sin(ab)}{(ab)^2-(n\pi)^2}, \quad n\pi\neq ab$ |
| 9 | $\sin\left(\frac{m\pi x}{a}\right)$, $m$ an integer | $\begin{Bmatrix} 0, & n=m \\ \frac{m\pi[(-1)^{n+m}-1]}{\pi(n^2-m^2)}, & n\neq m \end{Bmatrix}$ |

| | $f(x)$ | $\tilde{f}_c(n) = \int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right) dx$ |
|---|---|---|
| 10 | $\exp(bx)$ | $(a^2 b)\left[\frac{(-1)^n \exp(ab) - 1}{(n\pi)^2 + (ba)^2}\right]$ |
| 11 | $x^{-\frac{1}{2}}(a^2 - x^2)^{-\frac{1}{2}}$ | $\left(\frac{\pi}{2}\right)^{3/2} \left(\frac{n\pi}{a}\right)^{\frac{1}{2}} \left\{J_{-1/4}\left(\frac{n\pi}{2}\right)\right\}^2$ |
| 12 | $(a^2 - x^2)^{v-\frac{1}{2}}$ | $\sqrt{\pi}\ 2^{v-1}\Gamma\left(v + \frac{1}{2}\right) \left(\frac{a^2}{n\pi}\right)^v J_v(n\pi)$ |
| 13 | $\sin\left\{b(a^2 - x^2)^{\frac{1}{2}}\right\}$ | $\left(\frac{\pi ab}{2}\right) \left(b^2 + \frac{n^2\pi^2}{a^2}\right)^{-\frac{1}{2}}$ $\times J_1\left[\{(ab)^2 + (n\pi)^2\}^{\frac{1}{2}}\right]$ |
| 14 | $(a^2 - x^2)^{-\frac{1}{2}}$ $\times \cos\left\{b(a^2 - x^2)^{\frac{1}{2}}\right\}$ | $\left(\frac{\pi}{2}\right) J_0\left[\{(ab)^2 + (n\pi)^2\}^{\frac{1}{2}}\right]$ |
| 15 | $J_0\left\{b(a^2 - x^2)^{\frac{1}{2}}\right\}$ | $\left(b^2 + \frac{n^2\pi^2}{a^2}\right)^{-\frac{1}{2}}$ $\times \sin\left[\{(ab)^2 + (n\pi)^2\}^{\frac{1}{2}}\right]$ |

**TABLE B-10 Finite Fourier Sine Transforms**

| | $f(x)$ | $\tilde{f}_s(n) = \int_0^a \sin\left(\frac{n\pi x}{a}\right) f(x)\,ds$ |
|---|---|---|
| 1 | $1$ | $\left(\frac{a}{n\pi}\right)[1-(-1)^n]$ |
| 2 | $x$ | $(-1)^{n+1}\left(\frac{a^2}{n\pi}\right)$ |
| 3 | $x^2$ | $\frac{a^3(-1)^{n-1}}{n\pi} - \frac{2a^3\left[1+(-1)^{n+1}\right]}{(n\pi)^3}$ |
| 4 | $x^3$ | $(-1)^n\frac{a^4}{\pi^5}\left(\frac{6}{n^3}-\frac{\pi^2}{n}\right)$ |
| 5 | $\left(\frac{a-x}{a}\right)$ | $\left(\frac{a}{n\pi}\right)$ |
| 6 | $x(a-x)$ | $2\left(\frac{a}{n\pi}\right)^3\left[1+(-1)^{n+1}\right]$ |
| 7 | $x(a^2-x^2)$ | $(-1)^{n+1}\,6a\left(\frac{a}{n\pi}\right)^3$ |
| 8 | $\exp(bx)$ | $\frac{n\pi a}{(n\pi)^2+(ab^2)}\left[1+(-1)^{n+1}\exp(ab)\right]$ |
| 9 | $\cos(bx)$ | $\frac{n\pi a}{(n\pi)^2-(ab)^2}\left[1+(-1)^{n+1}\cos(ab)\right], \quad n\pi \neq ab$ |
| 10 | $\sin(bx)$ | $\frac{(-1)^n\, an\pi\sin(ab)}{(n\pi)^2-(ab)^2}, \quad n\pi \neq ab$ |
| 11 | $\cosh(bx)$ | $\frac{n\pi a}{[(n\pi)^2+(ab)^2]}[1+(-1)^{n+1}\cosh(ab)]$ |
| 12 | $\sin\left(\frac{m\pi x}{a}\right)$, $m$ integer | $\begin{Bmatrix} 0, & n \neq m \\ \frac{1}{2}a, & n = m \end{Bmatrix}$ |
| 13 | $\cos\left(\frac{m\pi x}{a}\right)$, $m$ integer | $\begin{Bmatrix} \frac{na}{\pi(n^2-m^2)}\left[1+(-1)^{n+m+1}\right], & n \neq m \\ 0, & n = m \end{Bmatrix}$ |
| 14 | $x^{-1}$ | $Si(n\pi)$ |

| | $f(x)$ | $\tilde{f}_s(n) = \int_0^a \sin\left(\frac{n\pi x}{a}\right)\ f(x)\,ds$ |
|---|---|---|
| 15 | $x^{-\frac{1}{2}}(x^2 - a^2)^{-\frac{1}{2}}$ | $\left(\frac{\pi}{2}\right)^{3/2}\left(\frac{n\pi}{a}\right)^{\frac{1}{2}}\left\{J_{1/4}\left(\frac{n\pi}{2}\right)\right\}^2$ |
| 16 | $x(a^2 - x^2)^{\alpha-\frac{1}{2}}$ | $\sqrt{\pi}\ 2^{\alpha-1}a^{\alpha+1}\Gamma\left(\alpha+\frac{1}{2}\right) \times \left(\frac{n\pi}{a}\right)^{-\alpha} J_{\alpha+1}(n\pi)$ |
| 17 | $(a^2 - x^2)^{-\frac{1}{2}}\, T_{2n+1}\left(\frac{x}{a}\right)$ | $\left(\frac{\pi}{2}\right)(-1)^n J_{2n+1}(n\pi)$ |
| 18 | $(ax - x^2)^{\alpha-\frac{1}{2}}$ | $\sqrt{\pi}\ \Gamma\left(\alpha+\frac{1}{2}\right)\left(\frac{a^2}{n\pi}\right)^{\alpha} J_{\alpha}\left(\frac{n\pi}{2}\right)$ |

**TABLE B-11 Finite Laplace Transforms**

| | $f(t)$ | $\mathscr{L}_T\{f(t)\} = \bar{f}(s,T) = \int_0^T e^{-st} f(t)dt$ |
|---|---|---|
| 1 | 1 | $\frac{1}{s}(1 - e^{-sT})$ |
| 2 | t | $\frac{1}{s^2} - \frac{1}{s}e^{-sT}\left(\frac{1}{s} + T\right)$ |
| 3 | $t^n$ | $\frac{n!}{s^{n+1}} - \frac{e^{-sT}}{s^{n+1}}\big[(sT)^n + n(sT)^{n-1}$ $+ n(n-1)(sT)^{n-2} + \cdots + n!\big]$ |
| 4 | $t^a, \quad (a > -1)$ | $\frac{1}{s^{a+1}}\gamma(a+1,\ sT)$ |
| 5 | $\exp(-at), \quad a > 0$ | $(s+a)^{-1}[1 - \exp\{-T(s+a)\}]$ |
| 6 | $t^n \exp(-at), \quad a > 0$ | $\frac{n!}{(s+a)^{n+1}} - \frac{e^{-(a+s)T}}{(s+a)^{n+1}}[\{(s+a)T\}^n$ $+ n\{T(s+a)\}^{n-1}$ $+n(n-1)\{T(s+a)\}^{n-2} + \cdots + n!]$ |
| 7 | $H(t-a), \quad a > 0$ | $\frac{1}{s}\left[e^{-sa} - e^{-sT}\right] H(T-a)$ |
| 8 | $\cos(at)$ | $\frac{s}{(s^2+a^2)} + \frac{e^{-sT}}{(s^2+a^2)}$ $\times (a \sin aT - s \cos aT)$ |
| 9 | $\sin(at)$ | $\frac{a}{(s^2+a^2)} - \frac{e^{-sT}}{(s^2+a^2)}$ $\times (s \sin aT + a \cos aT)$ |
| 10 | $e^{-at}\sin(bt)$ | $\frac{b}{(s+a)^2+b^2} - \frac{\exp(-sT)}{(s+a)^2+b^2}$ $\times(s \sin bT + a \sin bT + b \cos bT)$ |
| 11 | $e^{-at}\cos(bt)$ | $\frac{s+a}{(s+a)^2+b^2} + \frac{\exp(-sT)}{(s+a)^2+b^2}$ $\times(b \sin bT - s \cos bT - a \cos bT)$ |

| | $f(t)$ | $\mathscr{L}_T\{f(t)\} = \bar{f}(s,T) = \int_0^T e^{-st} f(t)dt$ |
|---|---|---|
| 12 | $\sinh(at)$ | $\frac{a}{(s^2-a^2)} - \frac{\exp(-sT)}{(s^2-a^2)}(a\cosh aT + s\sinh aT)$ |
| 13 | $\cosh(at)$ | $\frac{s}{(s^2-a^2)} - \frac{\exp(-sT)}{(s^2-a^2)}(s\cosh aT + a\sinh aT)$ |
| 14 | $t^{\frac{1}{2}}$ | $-\frac{\sqrt{T}\exp(-sT)}{s} + \frac{\sqrt{\pi}}{2}\frac{erf(\sqrt{sT})}{s^{3/2}}$ |
| 15 | $t^{-\frac{1}{2}}$ | $\frac{\pi}{s}\ erf(\sqrt{sT})$ |
| 16 | $erfc\left(\frac{t}{2a}\right)$ | $\frac{1}{s}\{1-\exp(a^2s^2)erfc(as)\} - \frac{e^{-sT}}{s}\ erfc\left(\frac{T}{2a}\right)$ $+\frac{\exp(a^2s^2)}{s}\ erfc\left(\frac{T}{2a}+as\right)$ |
| 17 | $erfc(bt)$ | $-\frac{1}{s}\exp\left(\frac{s^2}{4b^2}\right) erf\left(\frac{s}{2b}\right) - \frac{\exp\left(\frac{s^2}{4b^2}\right)}{s} erfc\left(\frac{s}{2b}\right)$ $+\frac{1}{s}\exp\left(\frac{s^2}{4b^2}\right)\ erfc\left(bT+\frac{s}{2b}\right) - \frac{e^{-sT}}{s} erf(bT)$ |
| 18 | $erf(t)$ | $-\frac{e^{\frac{s^2}{4}}}{s}\ erf\left(\frac{s}{2}\right) + \frac{e^{\frac{s^2}{4}}}{s}\ erf\left(T+\frac{s}{2}\right) - \frac{e^{-sT}}{s} erf(T)$ |
| 19 | $erf(\sqrt{t})$ | $\frac{erf(\sqrt{T})}{s(s+1)} - \frac{\exp(-sT)erf(\sqrt{T})}{s}$ |
| 20 | $e^{bt}\ erf(\sqrt{bT})$ | $\frac{\sqrt{b}\ erf(\sqrt{sT})}{\sqrt{s}(s-b)} + \frac{e^{-(s-b)T} erf(\sqrt{bT})}{(s-b)}$ |
| 21 | $e^{bt}\ erfc(\sqrt{bT})$ | $\frac{1}{(s-b)}\left\{1 - \frac{\sqrt{b}}{\sqrt{s}}\ erf(\sqrt{sT})\right\}$ $-\frac{e^{-(s-b)T}\ erf(\sqrt{bT})}{(s-b)}$ |

**TABLE B-12 Z Transforms**

| | $f(n)$ | $F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}$ |
|---|---|---|
| 1 | $\begin{Bmatrix} 1, & n = 0 \\ 0, & n \neq 0 \end{Bmatrix}$ | $1$ |
| 2 | $1$ | $\dfrac{z}{z-1}$ |
| 3 | $a^n$ | $\dfrac{z}{z-a}$ |
| 4 | $n$ | $\dfrac{z}{(z-1)^2}$ |
| 5 | $n^2$ | $\dfrac{z(z+1)}{(z-1)^3}$ |
| 6 | $\dfrac{1}{n!}$ | $\exp\left(\dfrac{1}{z}\right)$ |
| 7 | $\cos nx$ | $\dfrac{z(z-\cos x)}{z^2 - 2z\cos x + 1}$ |
| 8 | $\sin nx$ | $\dfrac{z \sin x}{z^2 - 2z\cos x + 1}$ |
| 9 | $\exp(\pm nx)$ | $\dfrac{z}{z - \exp(\pm x)}$ |
| 10 | $n^k\, e^{nx}$ | $\dfrac{\partial}{\partial x^k}\left(\dfrac{z}{z - e^x}\right)$ |
| 11 | $n\, e^{-nx}$ | $\dfrac{z\ \exp(-x)}{(z - e^{-x})^2}$ |
| 12 | $n^2\ e^{-nx}$ | $\dfrac{z(z + e^{-x})e^{-x}}{(z - e^{-x})^3}$ |
| 13 | $\exp(-nx)\sin(an)$ | $\dfrac{z\ \exp(-x)\ \sin a}{z^2 - 2ze^{-x}\ \cos a + e^{-2x}}$ |
| 14 | $\exp(-nx)\cos(an)$ | $\dfrac{z(z - e^{-x}\cos a)}{z^2 - 2z\, e^{-x}\cos a + e^{-2x}}$ |

| | $f(n)$ | $F(z)=\sum_{n=0}^{\infty} f(n)\, z^{-n}$ |
|---|---|---|
| 15 | $\sinh(nx)$ | $\dfrac{z \sinh x}{z^2-2z \cosh x+1}$ |
| 16 | $\cosh(nx)$ | $\dfrac{z(z-\cosh x)}{z^2-2z \cosh x+1}$ |
| 17 | $H(n-1)$ | $\dfrac{1}{z-1}$ |
| 18 | $H(n)-H(n-1)$ | $1$ |
| 19 | $H(n-m), \quad m=1,2,3$ | $\dfrac{1}{z^{m-1}(z-1)}$ |
| 20 | $H(n-1)-H(n-2)$ | $\dfrac{1}{z}$ |
| 21 | $H(n-m)-H[n-(m+1)]$ | $\dfrac{1}{z^m}$ |
| 22 | $\dfrac{m(m-1)\cdots(m-n+1)}{n!}$ | $\left(1+\dfrac{1}{z}\right)^m$ |
| 23 | $\dfrac{1}{(2n+1)!}$ | $\sqrt{z}\sinh\left(\dfrac{1}{\sqrt{z}}\right)$ |
| 24 | $\dfrac{1}{(2n)!}$ | $\cosh\left(\dfrac{1}{\sqrt{z}}\right)$ |
| 25 | $\dfrac{a^n}{(2n+1)!}$ | $\sqrt{\dfrac{z}{a}}\sinh\left(\sqrt{\dfrac{a}{z}}\right)$ |
| 26 | $\dfrac{a^n}{(2n)!}$ | $\cosh\left(\sqrt{\dfrac{a}{z}}\right)$ |
| 27 | $a^n \sinh(nx)$ | $\dfrac{za \sinh x}{z^2-2za \cosh x+a^2}$ |
| 28 | $a^n \cosh(nx)$ | $\dfrac{z(z-a\cosh x)}{z^2-2za \cosh x+a^2}$ |

**TABLE B-13 Finite Hankel Transforms**

| | $f(r)$ | order $n$ | $\tilde{f}_n(k_i) = \int_0^a r\, J_n(rk_i) f(r) dr$ |
|---|---|---|---|
| 1 | $c$, where $c$ is a constant | 0 | $\left(\frac{ac}{k_i}\right) J_1(ak_i)$ |
| 2 | $(a^2 - r^2)$ | 0 | $\frac{4a}{k_i^3} J_1(ak_i)$ |
| 3 | $(a^2 - r^2)^{-\frac{1}{2}}$ | 0 | $k_i^{-1} \sin(ak_i)$ |
| 4 | $\frac{J_0(\alpha r)}{J_0(\alpha a)}$ | 0 | $-\frac{ak_i}{(\alpha^2 - k_i^2)} J_1(ak_i)$ |
| 5 | $\frac{1}{r}$ | 1 | $k_i^{-1}\{1 - J_0(ak_i)\}$ |
| 6 | $r^{-1}(a^2 - r^2)^{-\frac{1}{2}}$ | 1 | $\frac{(1 - \cos ak_i)}{(ak_i)}$ |
| 7 | $r^n$ | $> -1$ | $\frac{a^{n+1}}{k_i} J_{n+1}(ak_i)$ |
| 8 | $\frac{J_v(\alpha r)}{J_v(\alpha a)}$ | $> -1$ | $\frac{ak_i}{(\alpha^2 - k_i^2)} J_v'(ak_i)$ |
| 9 | $r^{-n}(a^2 - r^2)^{-\frac{1}{2}}$ | $> -1$ | $\frac{\pi}{2}\left\{J_{\frac{n}{2}}\left(\frac{ak_i}{2}\right)\right\}^2$ |
| 10 | $r^n(a^2 - r^2)^{-\left(n+\frac{1}{2}\right)}$ | $< \frac{1}{2}$ | $\frac{\Gamma\left(\frac{1}{2} - n\right)}{\sqrt{\pi}\, 2^n} k_i^{n-1} \sin(ak_i)$ |
| 11 | $r^{n-1}(a^2 - r^2)^{n-\frac{1}{2}}$ | $> -\frac{1}{2}$ | $\frac{\sqrt{\pi}}{2}\Gamma\left(n + \frac{1}{2}\right)\left(\frac{2}{k_i}\right)^n \times a^{2n} J_n^2\left(\frac{ak_i}{2}\right)$ |

# *Answers and Hints to Selected Exercises*

## 2.19 Exercises

1. (a) $\sqrt{\dfrac{\pi}{2}}\exp(-|k|)$. (c) $\dfrac{1}{\sqrt{2\pi}}(ik)^n$.

   (e) Hint: Put $e^x = y, \quad F(k) = \dfrac{\Gamma(1-ik)}{\sqrt{2\pi}}$.

   (f) Hint: $f(x) = -\dfrac{1}{a}\dfrac{d}{dx}\exp\left(-\dfrac{1}{2}ax^2\right)$.

   (h) $F(k) = \dfrac{1}{\sqrt{2\pi}}\left(\dfrac{k}{2}\right)^{-2}\sin^2\left(\dfrac{k}{2}\right)$.

   (j) $\mathscr{F}\{h_n(x)\} = (-i)^n h_n(k)$,

   $h_n(k)$ is an eigenfunction for the Fourier transform.

   (k) $F(k) = \dfrac{i}{\sqrt{2\pi}}\dfrac{1}{(\alpha-k)}\left[e^{ia(\alpha-k)} - e^{ib(\alpha-k)}\right]$.

   (l) $\dfrac{1}{\sqrt{2a}}\begin{matrix}\cos\\ \sin\end{matrix}\left(\dfrac{k^2}{4a} \mp \dfrac{\pi}{4}\right)$.

5. (b) Hint: Use $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$.

6. (a) $(f*g)(x) = e^{ax}\displaystyle\int_0^\infty e^{-ay}\,dy = \dfrac{1}{a}e^{ax}$.

   (b)
$$\begin{aligned}(f*g)(x) &= \int_{-\infty}^{\infty} \sin b(x-y)e^{-a|y|}dy\\ &= \int_0^\infty [\sin b(x+y) + \sin b(x-y)]\, e^{-ay}dy\\ &= 2\sin bx\int_0^\infty e^{-ay}\cos\, by\, dy = \frac{2a\sin bx}{a^2+b^2}.\end{aligned}$$

7. (j) Verify this result using the Fourier transform of convolution.

8. (a) $y(x) = e^{-x} \int_{-a}^{x} e^{\alpha} f(\alpha)\, d\alpha + e^{x} \int_{x}^{a} e^{-\alpha} f(\alpha)\, d\alpha.$

For (b)–(c) use Exercises 3(a) and 3(b).

(b) $y(x) = A \exp\left(-\frac{1}{4}x^2\right)$, where $A$ is a constant.

(e) $y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{F(k)\exp(ikt)\, dk}{(\omega^2 - k^2 + 2i\alpha k)}.$

9. (b) $f(x) = \frac{a}{\sqrt{\pi(a-b)}} \exp\left(-\frac{abx^2}{a-b}\right).$

(c) $f(x) = \sqrt{\frac{2}{\pi}} \left\{\frac{b}{a}(a-b)\right\} \frac{1}{(a-b)^2 + x^2}.$

(d) $f(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{ac}{b(x^2+c^2)}, \quad c = b - a.$

(e) $F(k) = \frac{1}{\pi} \cdot \frac{i\pi\Phi(k)}{sgn\, k}, \quad f(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} (x-t)^{-1} \phi(t)\, dt.$

10. $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{A(k)\exp[i(kx+\omega t)] + B(k)\exp[i(kx-\omega t)]\}\, dk,$

where $A(k) = \frac{1}{2}\left[F(k) + \frac{1}{i\omega}G(k)\right], \quad B(k) = \frac{1}{2}\left[F(k) - \frac{1}{i\omega}G(k)\right]$
and $\omega^2 = \left(c^2k^2 + a^2\right)$.

11. Hint: $u(x,t) = 2\int_{0}^{\infty} A(k)\exp(-k^2bt)\cos\{(x+at)k\}\, dk$

$$\approx \sqrt{\frac{\pi}{bt}} A(0) \exp\left[-\frac{(x+at)^2}{4bt}\right] \text{ as } t \to \infty,$$

where $A(k)$ is expanded in Taylor series and only the first term is retained at $k = 0$.

12. Hint: $\mathscr{F}^{-1}\{\cos(k^2t)\} = \frac{1}{\sqrt{2t}} \cos\left(\frac{x^2}{4t} - \frac{\pi}{4}\right).$

14. (a) Hint: Differentiate both sides of the integral

$$\int_0^\infty e^{-ax}\sin kx\,dx = \frac{a}{k^2+a^2} \text{ with respect to } a \text{ to obtain}$$

$$F_s(k) = \sqrt{\frac{a}{\pi}}\frac{2ak}{(k^2+a^2)^2}.$$

(c) $\mathscr{F}_c\left\{\frac{1}{x}\right\}$ does not exist.

(d) $\sqrt{\frac{\pi}{2}}(a^2+k^2)^{-\frac{1}{2}}$. Hint: Use $K_0(ax) = \int_0^\infty \exp(-ax\cosh u)\,du$ and interchange the order of integration.

15. (a) Differentiate both sides of the integral $\int_0^\infty e^{-ax}\sin kx\,dx = \frac{k}{k^2+a^2}$ with respect to $a$ to obtain the result.

(b) Integrate the above integral with respect to $a$ from $a$ to $\infty$ to obtain the answer.

(c) $\sqrt{\frac{\pi}{2}}(-i\; sgn\, k)$.

(d) $\sqrt{\frac{\pi}{2}}\,e^{-ak}$.

16. (a) Hint: If $f(x) = \exp(-ax^2)$, then $f(x)$ satisfies the equation

$$f'(x) + 2\,a\,x\,f(x) = 0.$$

We take the Fourier transform and use 3(a) to obtain

$$2a\,F'(k) + k\,F(k) = 0, \quad F(0) = \frac{1}{\sqrt{2a}}.$$

Solving this equation yields

$$F(k) = \frac{1}{\sqrt{2a}}\exp\left(-\frac{k^2}{4a}\right).$$

(b) Use the definition of the Fourier cosine transform and integrate by parts.

17. Hint: Use the Parseval formula for the gate function.

19. Hint: $U_s(k,t) = F_s(k)\,G_c(k,t)$, where $G_c(k,t) = \exp(-\kappa\,k^2 t)$.

20. $u(x,y) = -\frac{2}{\pi}\int_0^\infty \frac{1}{k}\sin ak\,\cos kx\,e^{-ky}\,dk.$

21. (a) $f(x) = \dfrac{1}{\sqrt{x}}$, (b) $f(x) = \exp(-ax)$,

(c) $f(x) = \dfrac{H(x-a)}{\sqrt{x^2 - a^2}}$, (d) $H(a-x)$.

22. Hint: Use the Fourier sine transform.

$$u(x,t) = \frac{1}{\sqrt{4\pi\,\kappa t}} \int_0^\infty f(\xi) \left[\exp\left\{-\frac{(x-\xi)^2}{4\kappa t}\right\} - \exp\left\{-\frac{(x+\xi)^2}{4\kappa t}\right\}\right] d\xi.$$

23. (a) $\dfrac{\pi a^3}{2}$, (b) $\dfrac{\pi}{b^2}(1 - e^{-ab})$, (c) $\pi a$.

24. Hint: Use the Convolution Theorem for the Fourier cosine transform.

25. Hint: Use the Convolution Theorem for the Fourier cosine transform.

26. (a) $\dfrac{1}{2}(\pi - a)$, (d) $\pi \left(\dfrac{\exp(2\pi a) + 1}{\exp(2\pi a) - 1}\right)$.

28. Hint: $\mathscr{F}^{-1}\left\{\begin{matrix}\cos\\ \sin\end{matrix}(atk^2)\right\} = \dfrac{1}{2\sqrt{at}}\left[\cos\left(\dfrac{x^2}{4at}\right) \pm \sin\left(\dfrac{x^2}{4at}\right)\right]$.

29. $u(x,z) = \dfrac{P}{2\pi\mu} \displaystyle\int_{-\infty}^{\infty} \frac{1}{\alpha} \exp(ikx - \alpha z)\, dx, \quad \alpha = \sqrt{k^2 - \frac{\omega^2}{c_2^2}}.$

Hint: Write $(x,y) = r(\cos\theta, \sin\theta)$ along with $k = \dfrac{\omega}{c_2}\cos\phi$ and $\alpha = \dfrac{i\omega}{c_2}\sin\phi$ to obtain

$$u(x,z) = \frac{P}{2\pi i\mu} \int_{0-i\infty}^{\pi+i\infty} \exp\left[-\frac{i\omega r}{c_2}\sin(\theta+\phi)\right] d\phi.$$

30. $\phi(x,z,t) = -\dfrac{Pg}{2\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\sin\omega t}{\omega} \exp(ikx + |k|z)\, dk,$

$$\eta(x,t) = \frac{P}{2\pi} \int_{-\infty}^{\infty} \cos\omega t \exp(ikx)\, dk, \text{ where } \omega^2 = g|k|,$$

$$\eta(x,t) \approx \frac{Pt}{2\sqrt{2\pi}} \frac{\sqrt{g}}{x^{3/2}} \cos\left(\frac{gt^2}{4x}\right) \quad \text{for } gt^2 >> 4x.$$

32. $\phi(x,z,t) = \dfrac{iP\,\exp(\epsilon t)}{2\pi\rho} \displaystyle\int_{-\infty}^{\infty} \dfrac{(Uk - i\epsilon)\exp(|k|z + ikx)}{(Uk - i\epsilon)^2 - g|k|}\,dk,$

$$\eta(x,t) = \frac{P\,\exp(\epsilon t)}{2\pi\rho} \int_{-\infty}^{\infty} \frac{|k|\exp(ikx)\,dk}{(Uk - i\epsilon)^2 - g|k|}.$$

36. $u(x,t) = \dfrac{1}{2}[f(x - ct) + f(x + ct)] + \dfrac{1}{2c} \displaystyle\int_{x-ct}^{x+ct} g(\xi)d\xi \quad \text{for } x > ct.$

Similar result for $x < ct$.

37. Hint: $\mathscr{F}_s\{u_{xxxx}\} = \sqrt{\dfrac{2}{\pi}}[k^4 U_s(k,y) - k^3 u(0,y)],$

$$\mathscr{F}_s\{u_{xxyy}\} = \sqrt{\frac{2}{\pi}}\frac{\partial^2}{\partial y^2}[-k^2 U_s(k,y) + ku(0,y)].$$

43. (a) $\phi(t) = \left(1 - \dfrac{it}{a}\right)^{-p}.$

(b) $\phi(t) = \exp(i\mu t - \lambda|t|).$

(c) $\phi(t) = (1 + \lambda^2 t^2)^{-1}\exp(i\mu t).$

44. $f(x) = \dfrac{1 - \cos x}{\pi x^2}.$

45. $\phi(t) = \dfrac{1}{it}[\exp(ita) - 1].$

47. $U(k,y) = F(k)\cos(k^2 y),\;\; u(x,y) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} F(k)\cos(k^2 y)\exp(ikx)\,dk.$

48. $u(x,y,t) = \dfrac{1}{2\pi} \displaystyle\int\!\!\int_{-\infty}^{\infty} F(k,l)\cos\left[c(k^2 + l^2)^{\frac{1}{2}} t\right]\exp[i(kx + ly)]\,dk\,dl.$

52. $u(x,t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \left[F(k)\cos(x\alpha) + \dfrac{G(k)}{\alpha}\sin x\alpha\right]\exp(ikt)\,dk,$

where $-\alpha^2 = \dfrac{b + ika - k^2}{c^2}.$

53. $u(x,y) = \dfrac{2T_0}{\pi} \displaystyle\int_{0}^{\infty} \dfrac{\sin ak\,\cos xk\,\cosh y\alpha}{k\cosh\alpha}dk,\;\; \alpha = \sqrt{h + k^2}.$

54. (a) $u(x,y) = \dfrac{1}{4\pi^2} \displaystyle\int_{-\infty}^{\infty}\!\!\int \dfrac{F(k,l)\exp\{i(kx + ly)\}}{(k^4 + l^2 + 2)}dk\,dl.$

(b) $u(x,y) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int \frac{F(k,l)\exp\{i(kx+ly)\}dk\,dl}{(k^2+2l^2-3ik+4)}.$

55. Hint: Seek a solution of the form $\psi = \phi_n(x,t)\sin n\pi y$ with $\psi_0(x,y) = \psi_{0n}(x)\sin n\pi y$ so that $\phi_n$ satisfies the equation

$$\frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial x^2}\phi_n - \alpha^2\phi_n\right] + \beta\frac{\partial \phi_n}{\partial x} = 0, \ \ \alpha^2 = (n\pi)^2 + \kappa^2.$$

Apply the Fourier transform of $\phi_n(x,t)$ with respect to $x$ and use $\Psi_n(k,0) = \mathscr{F}\{\psi_{0n}(x)\}$.

$$\phi_n(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi_n(k,0)\exp[i\{kx-\omega(k)t\}]\,dk,$$

where $\omega(k) = -\beta k(k^2+\alpha^2)^{-1}$.

Examine the case for $\psi_{0n}(x) = \frac{1}{a\sqrt{2}}\exp\left\{ik_0x - \left(\frac{x}{a}\right)^2\right\}$.

56. Hint: The Fourier transform of the equation gives

$$\left(i\omega R + \frac{1}{C}\right)Q(\omega) = E(\omega),$$

so that the transfer function in the frequency domain is

$$\Phi(\omega) = \frac{Q(\omega)}{E(\omega)} = \frac{C}{(1+i\omega RC)}.$$

The inverse Fourier transform gives the impulse response function

$$\phi(t) = \mathscr{F}^{-1}\{\Phi(\omega)\} = \frac{1}{R}\exp\left(-\frac{t}{RC}\right)H(t).$$

57. (a) See Titchmarsh (1959) pages 60-61.

(b) $\sqrt{a}\left[\frac{1}{2} + \sum_{n=1}^{\infty} e^{-na}\right] = \sqrt{\frac{2b}{\pi}}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(1+n^2b^2\right)^{-1}\right].$

(c) $\sqrt{a}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\exp\left(-\frac{1}{2}a^2n^2\right)\right] = \sqrt{b}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\exp\left(-\frac{1}{2}n^2b^2\right)\right].$

(d) $$\sqrt{a}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\exp\left(-\frac{1}{2}a^2n^2\right)\cos(\alpha a n)\right]$$
$$= \sqrt{b}\exp\left(-\frac{1}{2}\alpha^2\right)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\exp\left(-\frac{1}{2}b^2n^2\right)\cosh(\alpha b n)\right].$$

(e) $\sqrt{\frac{a}{b}}\, f(0)\left[\frac{1}{2}+\sum_{na\le 1}^{\infty}\left(1-a^2n^2\right)^{\nu-\frac{1}{2}}\right]=\frac{1}{2}F_c(0)+\sum_{n=1}^{\infty}(nb)^{-\nu}J_\nu(nb),$

where, in the case $\nu=\frac{1}{2}$, the term $na=1$, if it is present, is to be halved.

## 3.9 Exercises

1. (a) $\frac{2}{s^2}+\frac{a^2}{s^2+a^2}$, (b) $s(s+2)^{-2}$,

 (c) $\frac{s^2-a^2}{(s^2+a^2)^2}$, (e) $\frac{\exp(-3s)}{(s-1)}$,

 (g) $\frac{2}{s^3}\exp(-3s)$, (h) $(1+sa)s^{-2}\exp(-as)$,

 (i) $s\sqrt{\pi}(s-a)^{-3/2}$, (j) $\frac{a\left(s^2+2\omega^2\right)}{s\left(s^2+4\omega^2\right)}$.

2. Hint: $\int_0^\infty t^{-n}\exp(-st)\,dt \ge e^{-s}\int_0^1 t^{-n}\,dt+\int_1^\infty t^{-n}\exp(-st)\,dt,$

 since $\exp(-st)\ge\exp(-s)$ for $0\le t\le 1$. But $\int_0^1 t^{-n}\,dt$ does not exist.

5. Hint: Use (3.6.7).

6. Hint: Use definition 3.2.5 and result (3.6.7).

7. (a) $\frac{1}{(a^2-b^2)}(\cos bt-\cos at)$, (b) $\left(\frac{t}{c^2}-\frac{\sin ct}{c^3}\right)$,

 (c) $(t-a)H(t-a)$, (d) $\exp(2t)-(t+1)\exp(t)$,

 (e) $\frac{1}{2}\exp(-t)\sin 2t$,

 (f) Hint: $\frac{1}{s^2(s+1)(s+2)}=\frac{1}{2s^2}-\frac{3}{4s}+\frac{1}{s+1}-\frac{1}{4(s+2)}$,

 (g) $\frac{1}{a^2}\left[1+(at-1)\,e^{at}\right]$, (h) $\frac{1}{a^3}\left[2+at\,(at-2)\,e^{at}\right]$,

 (i) $\frac{1}{a^2}\left(e^{at}-at-1\right)$.

8. (a) $\dfrac{1}{2a}(\sin at + at\cos at)$, (b) $\dfrac{1}{2}\,erf(2\sqrt{t})$,

(c) $\displaystyle\int_0^t f(\tau)\,d\tau$, (d) $\dfrac{t}{2a}\sin at$,

(e) $\displaystyle\int_0^t f(t-\tau)\sin\omega\tau\,d\tau$, (f) $\dfrac{1}{2a^3}\left(\sin at - at\cos at\right)$,

(g) $\dfrac{1}{(a^2+b^2)}(b\sin bt - a\cos bt + a\,e^{at})$, (i) $erfc\left(\dfrac{a}{2\sqrt{t}}\right)$,

(j) $\dfrac{1}{a^3}\left(at - \sin at\right)$, (k) $t\cos\,at$, (l)Use(3.6.7). $\left(\dfrac{1-\cos at}{t}\right)$.

9. (b) Hint: $\dfrac{1}{(\sqrt{s}-\sqrt{a})} = \dfrac{1}{\sqrt{s}}\left(\dfrac{a}{s-a}+1\right) + \dfrac{\sqrt{a}}{s-a}$.

(c) Hint: $\bar{f}(s)$ has simple poles at $s=0$ and at $s=\pm(2n+1)\dfrac{a\pi i}{b}=\pm s_n$. The residue at $s=0$ is $\dfrac{x}{a}$, and the residue at $s=s_n$ is

$$\frac{\sinh\left\{(2n+1)\dfrac{\pi i x}{a}\right\}\exp\left\{(2n+1)\dfrac{\pi i a x}{b}\right\}}{\left(\dfrac{b}{2a}\right)\left\{(2n+1)\dfrac{\pi a i}{b}\right\}^2\sinh\left\{(2n+1)\dfrac{\pi i}{2}\right\}}.$$

Grouping the residues at $s=\pm s_n$ together and using $\sinh\left\{(2n+1)\dfrac{\pi i}{2}\right\} = i\sin(2n+1)\dfrac{\pi}{2} = i(-1)^n$, we obtain the result.

(d) Hint: $\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s+ia}}\right\} = \frac{e^{-iat}}{\sqrt{\pi t}}$

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s^2+a^2}}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s+ia}\sqrt{s-ia}}\right\}, \\
&= \frac{1}{\pi}\int_0^t \tau^{-\frac{1}{2}}e^{-ia\tau}(t-\tau)^{-\frac{1}{2}}e^{ia(t-\tau)}d\tau \\
&= \frac{1}{\pi}\int_0^t \frac{1}{\sqrt{\tau(t-\tau)}}e^{ia(t-2\tau}d\tau, \\
&= \frac{1}{\pi}\int_0^1 \frac{e^{iat(1-2v)}}{\sqrt{v(1-v)}}\,dv \qquad (\tau = tv) \\
&= \frac{1}{\pi}\int_{-1}^1 \frac{e^{iatx}}{\sqrt{1-x^2}}\,dx \qquad (x = 1-2v).
\end{aligned}$$

10. (a) Use result (3.6.7).

(b) Use 10(a) and result (3.7.6).

11. (a) $J_0(at)$, (b) $\frac{1}{t}\sinh at$, (c) $1+\frac{1}{2}\cdot\frac{t^2}{3!}+\frac{1.3}{2.4}\cdot\frac{t^4}{5!}+\cdots$,

(d) $2\sum_{n=0}^{\infty} erfc\left[\frac{(2n+1)x}{2\sqrt{t}}\right\}$, (e) $J_0(2\sqrt{t})$,

(f) $a\left[1+\frac{(at)^2}{2^2\cdot 3}+\frac{(at)^4}{2^2\cdot 4^2\cdot 5^2}+\cdots\right]$.

12. Hint: (i) $\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\}=\int_0^t f(\tau)d\tau = g(t)$,

(ii) $$\mathscr{L}^{-1}\left\{\frac{\mathscr{L}\{g(t)\}}{s}\right\}=\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\}=\int_0^t g(t_1)dt_1$$
$$=\int_0^t\left\{\int_0^{t_1} f(\tau)d\tau\right\}dt_1=\int_0^t\int_0^{t_1} f(\tau)d\tau\, dt_1.$$

13. $\frac{1}{s}\{\exp(s)-1\}^{-1}$.

15. (b) Hint: Use Example 3.6.1(a).

17. $s(s^2+1)^{-1}\exp\left(-\frac{\pi s}{2}\right)$.

18. (f) Use result (3.6.7).

22. (a) $-\frac{1}{2s}\log(1+s^2)$, (b) $\frac{1}{s}\log(1+s)$.

23. (a) Hint: Use (3.6.2) and then the shifting property (3.4.1).

(c) $L_n(t)=\sum_{r=0}^{n}\binom{n}{r}\frac{(-t)^r}{r!}$.

24. (a) Hint: Use the definition and then interchange the order of integration.

26. $\bar{f}(s)=s^{-2}\tanh\left(\frac{as}{2}\right)$, $s>0$; Hint: $f(t+2a)=f(t)$.

27. (a) $f(0)=1,\ f'(0)=5$.

29. Hint: Use the identities

$$\begin{aligned} s\bar{f}(s)\bar{g}(s) &= f(0)\bar{g}(s) + \{s\bar{f}(s) - f(0)\}\bar{g}(s) \\ &= f(0)\bar{g}(s) + \mathscr{L}\{f'(t)\}\mathscr{L}\{g(t)\}. \\ s\bar{f}(s)\bar{g}(s) &= g(0)\bar{f}(s) + \{s\bar{g}(s) - g(0)\}\bar{f}(s). \end{aligned}$$

30. (a) $\bar{f}(s) \sim \dfrac{1}{s}\left(1 - \dfrac{2!}{s^2} + \dfrac{4!}{s^4} - \cdots\right).$

(b) Hint: Put $t = x + 1$ and then write the binomial expansion of $(x^2 + 2x)^{\frac{1}{2}}$ for $|x| < 2$.

$$K_0(s) \sim \frac{e^{-s}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\left\{\Gamma\left(n + \frac{1}{2}\right)\right\}^2}{(2s)^{n+\frac{1}{2}}} \quad \text{as } s \to \infty.$$

31. (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+1)}{s^{n+1}},$ Hint: $\displaystyle(1+t)^{-1} = \sum_{n=0}^{\infty} (-1)^n\, t^n,$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}\Gamma\left(n + \frac{3}{2}\right)}{(2n+1)!\, s^{n+\frac{3}{2}}},$ $\displaystyle\sin(2\sqrt{t}) = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}\, t^{\left(n+\frac{1}{2}\right)}}{(2n+1)!}.$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n} \frac{\Gamma(n+1)}{s^{n+1}},$ Hint: $\displaystyle\log(1+t) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n},$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, a^{2n}\, \Gamma(2n+1)}{\{2^2 \cdot 4^2 \cdot \cdots \cdot (2n)^2\} s^{2n+1}},$ Hint: $\displaystyle J_0(at) = \sum_{n=0}^{\infty} \frac{(-1)^n (at)^{2n}}{2^2 \cdot 4^2 \cdot \cdots \cdot (2n)^2}.$

32. (a) $\mathscr{L}\{(t-a)^n\, H(t-a)\} = e^{-as}\, \mathscr{L}\{t^n\} = e^{-as}\, \dfrac{n!}{s^{n+1}}.$

(b) $\mathscr{L}\left\{t^2\, H(t-a)\right\} = e^{-as}\, \mathscr{L}\left\{(t+a)^2\right\}$
$= e^{-as}\, \mathscr{L}\left\{t^2 + 2at + a^2\right\} = e^{-as}\left(\frac{2}{s^3} + \frac{2a}{s^2} + \frac{a^2}{s}\right).$

(c) $f(t) = t - t\, H(t-a).$ Hence, $\bar{f}(s) = \dfrac{1}{s^2} - \mathscr{L}\{t\, H(t-a)\}$
$= \dfrac{1}{s^2} - e^{-as}\mathscr{L}\{t + a\} = \dfrac{1}{s^2} - e^{-as}\left(\dfrac{1}{s^2} + \dfrac{a}{s}\right).$

(d) $f(x) = w_0\left(1 - \dfrac{2x}{l}\right) - w_0\left(1 - \dfrac{2x}{l}\right) H\left(x - \dfrac{l}{2}\right)$
$= \dfrac{2w_0}{l}\left[\left(\dfrac{l}{2} - x\right) + \left(x - \dfrac{l}{2}\right) H\left(x - \dfrac{l}{2}\right)\right]$
$\bar{f}(s) = \dfrac{2w_0}{l}\left[\left(\dfrac{l}{2s} - \dfrac{1}{s^2}\right) + \dfrac{1}{s^2}\exp\left(-\dfrac{sl}{2}\right)\right].$

(e) $e^{-\pi s}\mathscr{L}\{\cos 2(t+\pi)\} = e^{-\pi s}\mathscr{L}\{\cos 2t\} = e^{-\pi s}\left(\dfrac{s}{s^2+4}\right)$.

(f) $f(t) = 2 - 4H(t-a)$, $\bar{f}(s) = \dfrac{2}{s} - \dfrac{4}{s}e^{-as}$.

33. $\begin{Bmatrix} a, & 0 \le t \le a \\ 0, & a < t < 2a \end{Bmatrix}$

34. $\begin{Bmatrix} a, & 0 \le t \le a \\ -a, & 1 < t < 2 \\ 0, & t < 2 \end{Bmatrix}$

36. (a) $(f_p * f_q)(t) = \displaystyle\int_0^t f_p(t-\tau)\, f_q(\tau)\, d\tau = t^{p+q-1}\, e^{-t}\, B(p, q), \quad x = 1 - \frac{\tau}{t}$
$= f_{p+q}(t)\, B(p,q)$.

(c) $(f_p * f_q)'(t) = f_p'(t) * f_q(t) = [(p-1) f_{p-1}(t) - f_p(t)] * f_q(t)$
$= (p-1)B(p-1,q) f_{p+q-1}(t) - B(p,q) f_{p+q}(t)$.

## 4.11 Exercises

1. (a) $\dfrac{1}{(a-b)}(e^{-bt} - e^{-at})$, (b) $2e^{-t} - t^2 - 2t - 2$,

(c) $\dfrac{1}{5}(2\cos t + \sin t + 3e^{-2t})$, (d) $2(e^{2t} - 1)$.

2. $x(t) = x_0 \exp(-kt)$.

3. (a) $x(t) = \dfrac{1}{2}(e^{3t} + e^{-t})$, $y(t) = \dfrac{1}{2}(e^{3t} - e^{-t})$.

(b) $x_1 = \dfrac{28}{9}e^{3t} - e^{-1} - \dfrac{t}{3} - \dfrac{1}{9}$, $x_2 = \dfrac{28}{9}e^{3t} + e^{-t} - \dfrac{t}{3} - \dfrac{1}{9}$.

(c) $x = 15\cos t + 20\sin t - 10e^{-t}$,
$y = 10\cos t + 5\sin t - 10e^{-t}$,
$z = -25\sin t$.

(d) $x = \dfrac{1}{5}(7e^{-t} + 3e^{4t})$, $y = \dfrac{1}{5}(7e^{-t} - 2e^{4t})$.

4. $x(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_0 \begin{pmatrix} 3e^{-t} - 2 \\ 3 - 3e^{-t} \end{pmatrix}$.

5. $x(t) = x_0 e^t, \quad y(t) = (x_0 + y_0)e^{2t} - x_0 e^t$.

6. (a) Write $s^4 + 2s^2(\ell + 2k^2) + \ell^2 = (s^2 + \alpha^2)(s^2 + \beta^2)$ so that $\alpha^2 + \beta^2 = 2(\ell + 2k^2)$, $(\alpha\beta)^2 = \ell^2$ and $\alpha = \sqrt{k^2 + \ell} + k$, $\beta = \sqrt{k^2 + \ell} - k$.

(b) $\overline{x}(s) = \overline{y}(s) = \dfrac{s(s^2+3)}{(s^2+2)^2-1}$.

7. $C(t) = \left(\dfrac{\alpha}{kV}\right)(1-e^{-kt})$.

8. $p(t) = p_0 \exp\left(-\dfrac{ct}{k}\right)$

$$+Ac\omega\left(\omega^2+\frac{c^2}{k^2}\right)^{-1}\left[\frac{c}{\omega k}\sin\omega t - \cos\omega t + \exp\left(-\frac{ct}{k}\right)\right].$$

10. $c(t) = c_0 \exp(-k_1 t)$.

11. Hint: $\dfrac{d}{dt}(c_1+c_2+c_3) = 0$ and so $c_1+c_2+c_3 = c_1(0)$.

$c_1(t) = c_1 e^{-k_1 t}$, $c_2(t) = \dfrac{k_1 c_1}{k_2-k_1}(e^{-k_1 t} - e^{-k_2 t})$ and

$c_3(t) = c_1(0) - c_1(t) - c_2(t)$.

12. (a) $x(t) = \left(1+\dfrac{1}{n^2-\omega^2}\right)\cos\,\omega t - \dfrac{\cos nt}{n^2-\omega^2}$.

(b) $x(t) = \dfrac{1}{3}(\sin t - \sin 2t)$.

(c) $2e^{-1} + \dfrac{1}{16}e^{-4t} + \dfrac{3t}{4} - \dfrac{31}{16}$.

(d) $\dfrac{1}{16}(3\sin 2t + 5\sinh 2t)$.

(e) Hint: $\bar{x}(s) = \dfrac{1}{(s-1)^2+1} - \dfrac{(s+2)}{(s+2)^2+1}$.

(f) $\bar{x}(s) = \dfrac{e^{-as}+\alpha(s+b)+\beta}{s(s+b)}$

$$= \tfrac{\alpha}{s} + \tfrac{\beta}{b}\left(\tfrac{1}{s} - \tfrac{1}{s+b}\right) + \tfrac{e^{-as}}{b}\left(\tfrac{1}{s} - \tfrac{1}{s+b}\right)$$

$x(t) = \alpha + \tfrac{\beta}{b}\left(1-e^{-bt}\right) + \tfrac{1}{b}H(t-a)\left(1-e^{-bt}\right)$.

(g) $v(t) = \dfrac{1}{C}\,\mathscr{L}^{-1}\left\{(e^{-as}-1)\dfrac{1}{(s+\frac{1}{2RC})^2+\omega^2}\right\}$,

where $\omega^2 = \frac{1}{LC} - \frac{1}{4R^2C^2}$.

(i) $\bar{x}(s) = \dfrac{1}{s^2(s-a)^2} - \dfrac{e^{-as}}{s^2(s-a)^2} - \dfrac{a\,e^{-as}}{s(s-a)^2}$.

Inversion yields the solution as

$x(t) = f(t) - f(t-a)\,H(t-a) - a\,g(t-a)\,H(t-a)$, where

$f(t) = \mathscr{L}^{-1}\left\{\dfrac{1}{s^2(s-a)^2}\right\} = \dfrac{1}{a^3}\left[2+at+(at-2)e^{at}\right]$,

$$g(t)=\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)^2}\right\}=\frac{1}{a^2}\left[1+(at-1)e^{at}\right].$$

14. $x(t)=a(\omega t-\sin\omega t),\ y(t)=a(1-\cos\omega t).$

16. $\dot{x}(t)=\dfrac{eE}{m\omega}\sin\omega t,\ \dot{y}(t)=\dfrac{eE}{m\omega}(\cos\omega t-1),\ \dot{z}=0.$

19. $\bar{y}(x,s)=\bar{f}(s)\dfrac{\sinh\left\{\frac{s}{c}(l-x)\right\}}{\sinh\left(\frac{sl}{c}\right)}.$

20. $V(x,t)=V_0\, erfc\left(\dfrac{x}{2\sqrt{\kappa t}}\right).$

21. $V(x,t)=V_0\left(t-\dfrac{x}{c}\right)H\left(t-\dfrac{x}{c}\right),$

(i) $V=V_0H\left(t-\dfrac{x}{c}\right),$ (ii) $V=V_0\cos\left\{\omega\left(t-\dfrac{x}{c}\right)\right\}H\left(t-\dfrac{x}{c}\right).$

22. $u(z,t)=Ut\left[(1+2\zeta^2)erfc(\zeta)-\dfrac{2\zeta}{\sqrt{\pi}}e^{-\zeta^2}\right]$ where $\zeta=\dfrac{z}{2\sqrt{\nu t}}.$

23. $$q(z,t)=\frac{a}{2}e^{i\omega t}\Big[e^{-\lambda_1 z}erfc\{\zeta-[it(2\Omega+\omega)]^{1/2}\}+e^{\lambda_1 z}erfc\{\zeta+[it(2\Omega+\omega)]^{1/2}\}\Big]$$
$$+\frac{b}{2}e^{-i\omega t}[e^{-\lambda_2 z}erfc\{\zeta-[it(2\Omega-\omega)]^{1/2}\}+e^{\lambda_2 z}erfc\{\zeta+[it(2\Omega-\omega)]^{1/2}\}],$$
where $\lambda_{1,2}=\left\{\dfrac{i(2\Omega\pm\omega)}{\nu}\right\}^{1/2}.$

$q(z,t)\sim a\exp(i\omega t-\lambda_1 z)+b\exp(-i\omega t-\lambda_2 z),\quad \delta_{1,2}=\left\{\dfrac{\nu}{|2\Omega\pm\omega|}\right\}^{1/2}.$

24. $\left(\dfrac{\nu}{2\Omega}\right)^{1/2}.$

25. (a) $\dfrac{1}{2}\left(t+\dfrac{3}{2}\sin 2t\right),$ (b) $(1-\cos t),$ (c) $aJ_0(at),$

(d) $3\sin t-\sqrt{2}\sin(\sqrt{2}t),$ (e) $\left(t^2+\dfrac{2t}{a}\right).$

(f) $\bar{x}(s)=\dfrac{s}{s^2-a^2},\quad x(t)=\cosh at.$

26. Hint: $\overline{f}(s)=\dfrac{1}{s(\sqrt{s}-a)}.$

27. $1-(1+t)e^{-t}.$

28. (a) $\dfrac{\pi}{2a^2}\,(1-e^{-at})$, (b) $\dfrac{\pi}{2}\,sgn\,t$, (c) $\dfrac{\pi}{a}\,e^{-at}$,
(d) $\pi e^{-at}$, (e) $\sqrt{\dfrac{\pi}{4t}}$, (f) $\sqrt{\dfrac{\pi}{8t}}$.

29. Hint: Use the Laplace transform of sine and cosine functions.

31. $EIs^4\bar{y}(s) = W\exp(-as) + As + B$,
where $A = EI\,y''(0)$ and $B = EI\,y'''(0)$.
$EI\;y(x) = \dfrac{W}{6}(x-a)^3\;H(x-a) + \dfrac{A}{2}x^2 + \dfrac{B}{6}x^3$,
$y(\ell) = 0 = y''(\ell)$ gives $A = Wa\;\ell^{-2}(\ell-a)^2$ and
$B = -W\ell^{-3}(\ell-a)^2(\ell+2a)$.

32. $EI\,s^4\bar{y}(s) = \dfrac{W}{s}\left[\exp\left(-\dfrac{\ell s}{2}\right) - \exp\left(-\dfrac{3\ell s}{2}\right)\right] + As + B$,
where $A = EI\;y''(0)$ and $B = EI\;y'''(0)$.
$$EI\,y(x) = \frac{W}{24}\left[\left(x-\frac{\ell}{2}\right)H\left(x-\frac{\ell}{2}\right) - \left(x-\frac{3\ell}{2}\right)^4 H\left(x-\frac{3\ell}{2}\right)\right] + \frac{Ax^2}{2} + \frac{Bx^3}{6}.$$
$y''(2\ell) = 0 = y'''(2\ell)$ gives $A = W\ell^2$, $B = -W\ell$.

33. $EI\,y^{(IV)}(x) = W[1 - H(x-\ell)], \quad 0 < x < 2\ell$.
$$EI\,y(x) = \frac{W}{8}\left[\frac{9}{8}(\ell x)^2 - \frac{19}{16}\ell x^3 + \frac{1}{3}\{x^4 - (x-\ell)^4 H(x-\ell)\}\right].$$

34. $EI\,s^4\bar{y}(s) = \dfrac{W}{s}[1 - \exp(-2\ell s)] + P\,\exp(-\ell s) + As + B$,
where $A = EI\,y''(0)$ and $B = EI\,y'''(0)$.
$$EI\,y(x) = \frac{W}{24}[x^4 - (x-2\ell)^4 H(x-2\ell)] + \frac{P}{6}(x-\ell)^3 H(x-\ell) + \frac{A}{2}x^2 + \frac{B}{6}x^3.$$

The second term inside the square bracket in $y(x)$ does not contribute because the beam extends over $0 \le x < 2\ell$. Thus

$$\begin{aligned} EI\,y(x) &= \frac{W}{24}x^4 + \frac{A}{2}x^2 + \frac{B}{6}x^3, && 0 \le x < \ell, \\ &= \frac{W}{24}x^4 + \frac{P}{6}(x-\ell)^3 + \frac{A}{2}x^2 + \frac{B}{6}x^3, && \ell < x \le 2\ell. \end{aligned}$$

$$y''(2\ell) = 0 = y'''(2\ell) \text{ gives} A = \ell(2W\ell + P) \text{ and } B = -(A/\ell).$$

$$M\left(\frac{\ell}{2}\right) = EI\,y''\left(\frac{\ell}{2}\right) = \frac{\ell}{8}(9W\ell + 4P) \text{ and}$$

$$S\left(\frac{\ell}{2}\right) = EI\,y'''\left(\frac{\ell}{2}\right) = -\left(\frac{3}{2}W\ell + P\right).$$

35. (a) $u_n = 3^n$, (b) $u_n = n2^{n-1}$,

(c) $u_n = (n+1)2^n$, (d) $u_n = 2(3^n - 2^{n-1})$,

(e) $u_n = n2^n$, (f) $u_n = A3^n + B2^n$,

where $A = (u_1 - 2u_0)$ and $B = (3u_0 - u_1)$,

(g) $u_n = 3^n$, (h) $u_n = ca^n$.

37. $u(t) = 1 + t + \dfrac{(t-1)^3}{3} + \cdots$.

38. $u(x,t) = \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} s^{-1}\exp\left[st - \frac{sx}{\sqrt{1+k^2s^2}}\right] ds.$

39. Hint:

$$\mathscr{L}^{-1}\left[\frac{1}{\sqrt{s^2-\alpha^2}}\exp\{-\beta(s^2-\alpha^2)^{1/2}\}\right] = I_0[\alpha(t^2-\beta^2)^{1/2}]H(t-\alpha).$$

42. $$u(x,t) = x + \exp\left[-\left(\frac{3\pi c}{a}\right)^2 t\right]\sin\left(\frac{3\pi x}{a}\right)$$

$$-a\left[\sum_{n=0}^{\infty} erfc\left\{\frac{(2n+1)a+x}{3c\sqrt{t}}\right\} - \sum_{n=0}^{\infty} erfc\left\{\frac{(2n+1)a-x}{2c\sqrt{t}}\right\}\right].$$

50. (a) $$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right] f(\xi)d\xi$$

$$+\int_0^t d\tau\int_{-\infty}^{\infty} q(\xi,\tau)G(x,t;\xi,\tau)d\xi,$$

where

$$G(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi\kappa(t-\tau)}}\exp\left[-\frac{(x-\xi)^2}{4\kappa(t-\tau)}\right].$$

(b) The Laplace transform solution is

$$\bar{x}(x,s) = \frac{1}{s(1+e^{-al})}\left[e^{-ax} + e^{-a(l-x)}\right],$$

where $a = \sqrt{\frac{s}{\kappa}}$.

Expanding the denominator, the solution is

$$u(x,t) = \sum_{n=0}^{\infty}(-1)^n \left[erfc\left(\frac{x+nl}{\sqrt{4\kappa t}}\right) - erfc\left(\frac{(n+1)l-x}{\sqrt{4\kappa t}}\right)\right].$$

52.

$$\begin{aligned}\bar{u}(z,s) &= U_0 \bar{f}(s)\exp\left(-z\sqrt{\frac{s}{\nu}}\right) = U_0 s f(s)\frac{1}{s}\exp\left(-z\sqrt{\frac{s}{\nu}}\right)\\ &= U_0\left[\mathscr{L}\{f'(t)+f(0)\}\right]\mathscr{L}\left[erfc\left(\frac{z}{\sqrt{4\nu t}}\right)\right]\end{aligned}$$

Using the convolution theorem, we obtain

$$u(z,t) = U_0\int_0^t [f'(t-\tau)+f(0)]\, erfc\left(\frac{z}{\sqrt{4\nu t}}\right) d\tau.$$

For the special case, we obtain

$$\begin{aligned}u(z,t) &= \frac{U_0}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{\omega}{s^2+\omega^2}\right)\exp\left(st - z\sqrt{\frac{s}{\nu}}\right) ds,\ c>0\\ &= \left(\frac{U_0\omega}{\pi}\right)\int_0^\infty \frac{e^{-\sigma t}\sin\left(\sqrt{\frac{\sigma}{\nu}}\,z\right)}{\sigma^2+\omega^2}\, d\sigma\\ &\quad + U_0\exp\left(-z\sqrt{\frac{\omega}{2\nu}}\right)\sin\left(\omega t - z\sqrt{\frac{\omega}{2\nu}}\right),\end{aligned}$$

where the first integral is due to the branch cut of the Bromwich integral and it tends to zero as $t\to\infty$, and represents the initial transient term that occurs because the disk starts from rest. The second term comes from the residues at the poles at $s = \pm i\omega$. It represents the oscillatory motion of the viscous fluid whose amplitude decays exponentially with $z$ and whose phase changes with $z$.

53. (a) $\bar{h}(s) = \left(s^2+2s+5\right)^{-1}$, $\quad h(t) = \frac{1}{2}e^{-t}\sin 2t$,
$x(t) = 2e^{-t}\cos 2t + (h*f)(t)$.

(b) $\bar{h}(s) = \left(s^2-2s+5\right)^{-1}$, $\quad h(t) = \frac{1}{2}e^{t}\sin 2t$,
$x(t) = e^{t}\sin 2t + (h*f)(t)$.

(c) $\bar{h}(s) = \left(s^2 + 3^2\right)^{-1}, \quad h(t) = \dfrac{1}{3}\sin 3t,$
$x(t) = (2\cos 3t - \sin 3t) + (h * f)(t).$

(d) $\bar{h}(s) = \left(s^2 - 2s + 5\right)^{-1}, \quad h(t) = \dfrac{1}{2}e^t \sin 2t,$

$$x(t) = \mathscr{L}^{-1}\left\{\frac{x_0 s + (x_1 - 2x_0)}{s^2 - 2s + 5}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{x_0(s-1) + (x_1 - x_0)}{(s-1)^2 + 2^2}\right\} + \frac{1}{2}\int_0^t e^{t-\tau}\sin 2(t-\tau) f(\tau)d\tau$$

$$x(t) = e^t\left\{x_0 \cos 2t + \frac{1}{2}(x_1 - x_0)\sin 2t\right\} + (h * f)(t).$$

54. (a) $\bar{h}(s) = \dfrac{\bar{x}(s)}{\bar{f}(s)} = \dfrac{3s+2}{s^2+2s+2}.$
The system is of order 2, and its characteristic equation is $s^2 + 2s + 2 = 0$ with roots $s = -1 \pm i$. Since the real parts are negative, the system is stable.

(b) $\bar{h}(s) = \dfrac{\bar{x}(s)}{\bar{f}(s)} = \dfrac{2s+3}{4s^2+16s+25}.$
Order 2, characteristic equation is $4s^2 + 16s + 25 = 0$,
roots $s = -2 \pm \frac{3}{2}i$. Stable.

(c) $\bar{h}(s) = \dfrac{2s^2 + s - 6}{36s^2 + 12s + 37}.$
Order 2, characteristic equation is $36s^2 + 12s + 37 = 0$,
roots $s = \frac{1}{6} \pm i$. Unstable.

(d) $\bar{h}(s) = \dfrac{2s-1}{s^2 - 6s + 10}.$
Order 2, characteristic equation is $s^2 - 6s + 10 = 0$,
roots $s = 3 \pm i$. Unstable.

55. $$\overline{h}(s) = \frac{1}{s^3 - as^2 + b^2 s - ab^2} = \frac{1}{(s-a)(s^2+b^2)}$$

$$= (a^2 + b^2)^{-1}\left[\frac{1}{(s-a)} - \frac{(s+a)}{(s^2+b^2)}\right].$$

It has simple poles at $s = a$ and $s = \pm ib (b \neq 0)$. The system is always unstable.

(a) The system has a pole in the right-half plane and hence, is unstable.

(b) The poles are at $\pm ib$ which lie on the imaginary axis. The system is unstable (or marginally unstable).

(c) The pole zero is of second-order and the system is unstable.

$$h(t) = \frac{1}{(a^2+b^2)}\left(e^{at} - \cos bt - \frac{a}{b}\sin bt\right).$$

## 6.8 Exercises

1. Hint: $E_{1,1}(x) = e^x$.

## 7.5 Exercises

2. (b) $u(r,z) = \int_0^\infty k e^{-kz} J_0(kr)\widetilde{f}(k)dk = a\int_0^\infty e^{-kz} J_0(kr)J_1(ak)dk.$

3. (b) Hint: $\tilde{f}(k) = \left(\frac{Q}{\pi a k}\right) J_1(ak).$

6. $u(r,t) = \int_0^\infty k\tilde{f}(k)\cos(btk^2)J_0(kr)dk.$

9. Hint: The solution of the dual integral equations

$$\int_0^\infty kJ_0(kr)A(k)dk = u_0, \qquad 0 \le r \le a,$$

$$\int_0^\infty k^2 J_0(kr)A(k)dk = 0, \qquad a < r < \infty,$$

is $A(k) = \left(\frac{2u_0}{\pi}\right)\frac{\sin(ak)}{k^2}.$

10. Hint: See Debnath, 1994, pp. 103–105.

11. $u(r,z) = \frac{1}{\pi a}\int_0^\infty k^{-1}J_1(ak)J_0(kr)\ \exp(-kz)dk.$

13. Hint: $\mathscr{L}^{-1}\left[\frac{\exp\left\{-k(s^2+a^2)^{\frac{1}{2}}\right\}}{(s^2+a^2)^{\frac{1}{2}}}\right] = H(t-k)J_0(a\sqrt{t^2-k^2}).$

14. $u(r,z) = b\int_0^\infty k^{-1}\left(\frac{\sinh kz}{\cosh ka}\right)J_1(bk)J_0(kr)dk.$

15. Hint: $\mathscr{H}_0\left[\dfrac{H(a-r)}{\sqrt{a^2-r^2}}\right]=\dfrac{\sin ak}{k}$ and

$$\mathscr{L}^{-1}\left\{\frac{\exp(-\sqrt{s}k)}{\sqrt{s}(\sqrt{s}-a)}\right\}=\exp(-ak-a^2t)\; erfc\left\{\frac{k}{2\sqrt{t}}-a\sqrt{t}\right\}.$$

16. $u(r,z)=\left(\dfrac{Q}{\pi aK}\right)\displaystyle\int_0^\infty k^{-1}e^{-|k|z}J_1(ak)J_0(kr)dr.$

17. Use the hint in exercise 9 with $a=1$ and $u_0=1$.

18. Hint: Use the joint Hankel and Laplace transform method.

20. Hint: Use the Hankel transform.

$$u(r,z,t)=\frac{1}{\rho}\int_0^\infty k\,\exp(kz)\;J_0(kz)\int_0^t\left(\int_0^{r_0(\tau)}\alpha p(\alpha,\tau)J_0(k\alpha)d\alpha\right)\times\cos[\omega(t-\tau)d\tau]dk,$$

where $\omega^2=gk$.

21.

$$\tilde{\phi}(k,z)=-\frac{q}{k}\;\frac{\cosh(kz)}{\cosh(ka)}\,e^{-ak},\quad \phi(r,z)=-q\int_0^\infty e^{-ak}J_0(kr)\;\frac{\cosh(kz)}{\cosh(ka)}\,dk.$$

## 8.8 Exercises

1. (e) $\tilde{f}(p)=-\frac{x_0^{p+z}}{p+z},\quad \mathrm{Re(p)}<-\mathrm{Re(z)}.$
   (f) $\tilde{f}(p)=-\frac{x_0^{p+z}}{p+z},\quad \mathrm{Re(p)}>-\mathrm{Re(z)}.$
   (g) $\tilde{f}(p)=p^{-1}\Gamma(p),$
   (h) $\tilde{f}(p)=(-p)!\,\{\Gamma(p)\}^2.$

2. Hint: Substitute $e^{-t}=x$ and $g(-\log x)=f(x)$.

3. Hint: Similar to Example 6.2.1(d).

4. Hint: Use (6.2.12) and the scaling property of the Mellin transform.

5. Hint: Use $\mathscr{F}_c\{x^{-n}J_n(ax)\}$ and $\mathscr{F}_c\{x^{p-1}\}$ and then the Parseval relation for the Fourier cosine transform.

16. (a) Hint: Use (8.4.8) and $\tilde{\phi}(p, \pm\alpha) = \frac{a^p}{p}, \quad \text{Re(p)} > 0.$
The solution of (8.4.8) is $\tilde{\phi}(p, \theta) = A(p)e^{ip\theta} + B(p)e^{-ip\theta}$.
Hence, $A = B = \frac{a^p}{2p\cos(p\alpha)}$, and $\tilde{\phi}(p, \theta) = \frac{a^p \cos p\theta}{p\cos(p\alpha)}$.
Inversion gives the solution.
(b) $\phi(r, \theta) = \mathscr{M}^{-1}\left\{\tilde{f}(p)\,\frac{\sin p\theta}{\sin p\alpha}\right\}$.

17. Hint: $\displaystyle\sum_{n=1}^{\infty} \frac{\cos kn}{n^2} = -\frac{k^2}{2}\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{2\pi}{k}\right)^p \frac{\zeta(1-p)}{(p-1)(p-2)}dp,$
and the integrand has three simple poles at $p = 0, 1, 2$ with residues
$-\dfrac{1}{2}, \dfrac{\pi}{k}, -\dfrac{\pi^2}{3k^2}$.

18. Hint: $\displaystyle\sum_{n=1}^{\infty} e^{-nx} = \frac{1}{(1-e^{-x})}.$

27. Hint: (a) Put $x = -e^t$, $dx = -e^{-t}\,dt$ in (8.2.5) to obtain
$$\widetilde{f}(p) = \mathscr{M}\,\{f(x); p\} = \int_{-\infty}^{\infty} e^{-pt} f(e^{-t})dt = \mathscr{L}\left\{f(e^{-t}); p\right\}.$$
(b) Put $p = a + i\omega$ to obtain
$$\widetilde{f}(p) = \mathscr{M}\,\{f(x);\, p\} = \int_{-\infty}^{\infty} f(e^{-t})e^{-at}e^{-i\omega t}dt = \mathscr{F}\left\{f(e^{-t})e^{-at}; \omega\right\}.$$

## 9.13 Exercises

1. (a) $(a^2+z^2)^{-1}\left[\dfrac{\pi z}{2a} - \log\left(\dfrac{z}{a}\right)\right]$.
(b) $(a-z)^{-1}(a^\alpha - z^\alpha)\pi \operatorname{cosec}\pi\alpha$.
(c) $-\exp(az)Ei(-az)$.
(d) $\Gamma(1-\alpha)z^{-\alpha}\,\exp(az)\Gamma(\alpha, az)$.
(e) $\left(\dfrac{\pi}{z}\right)[1-\exp(-a\sqrt{z})]$.
(f) $z^{-1}(\cos z - 1)$.

6. (a) $(a-z)^{-1}(z^{\alpha-1} - a^{\alpha-1})\pi\ \operatorname{cosec}\,(\alpha\pi)$.
(b) $(a^2+z^2)^{-1}\left[\left(\dfrac{\pi z}{2a}\right) - \log\left(\dfrac{z}{a}\right)\right]$.
(c) $(a^2+z^2)^{-1}\left[\left(\dfrac{\pi a}{2}\right) + z\log\left(\dfrac{z}{a}\right)\right]$.

9. Hint: Use $t = xu$ in the transform solution and then apply the convolution theorem for the Mellin transform.

14. Hint: $\dfrac{1}{\pi}\displaystyle\oint_{-\infty}^{\infty}\frac{f(t)}{t-x}dt = \lim_{\varepsilon\to 0}\left[\frac{1}{\pi}\left\{\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right\}\frac{f(t)}{t-x}dt\right]$ and then put $t - x = u$.

15. Hint: Use general Parseval's relation
$$\int_{-\infty}^{\infty} f_1(x) f_2(x) dx = \int_{-\infty}^{\infty} (\boldsymbol{H}\, f_1)(x)\,(\boldsymbol{H}\, f_2)(x) dx,$$
where $f_1 \in L^p(\mathbb{R})$ and $f_2 \in L^q(\mathbb{R})$ with $(p^{-1} + q^{-1}) = 1$.
Put $f_1(x) = f(x)$ and $f_2(x) = (\boldsymbol{H}\, g)(x)$ to obtain
$$\int_{-\infty}^{\infty} f(x)(\boldsymbol{H}\, g)(x) dx = \int_{-\infty}^{\infty} (\boldsymbol{H}\, f)(x)\boldsymbol{H}\,[\boldsymbol{H}\, g(x)]\,(x) dx$$
$$= -\int_{-\infty}^{\infty} (\boldsymbol{H}\, f)(x) g(x) dx.$$
Thus, (9.3.9) follows.

## 10.6 Exercises

1. $\dfrac{1}{3}a^3$ when $n = 0$, and $2\left(\dfrac{a}{n\pi}\right)^2 a(-1)^n, n = 1, 2, 3, \ldots.$

3. $u(x,t) = \left(\dfrac{2\pi\kappa}{a}\right)\displaystyle\sum_{n=1}^{\infty} n \sin\left(\frac{n\pi x}{a}\right)\int_0^t f(\tau)\exp\left[-\kappa(t-\tau)\left(\frac{n\pi}{a}\right)^2\right] d\tau.$

4. Hint: $\tilde{f}_s(n) = \displaystyle\int_0^a f(x)\sin(\xi_n x) dx$
$$f(x) = \mathscr{F}_s^{-1}\{\tilde{f}_s(n)\} = \frac{2}{a}\sum_{n=0}^{\infty}\frac{(h^2 + \xi_n^2)\tilde{f}_s(n)\sin(x\xi_n)}{h + (h^2 + \xi_n^2)}$$
where $\xi_n$ is the root of the equation $\xi\cot(a\xi) + h = 0$.
$$u(x,t) = \left(\frac{2}{a}\right)\sum_{n=1}^{\infty}\frac{\xi_n(h^2 + \xi_n^2)}{h + (h^2 + \xi_n^2)}\int_0^t f(\xi)\exp[-\kappa\xi_n(t-\xi)]\sin(x\xi_n) d\xi.$$

5. Hint: Use $\tilde{f}_c(n) = \int_0^a f(x)\cos(x\xi_n)dx$,

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(h^2+\xi_n^2)\tilde{f}_c(n)\cos(x\xi_n)}{h+a(h^2+\xi_n^2)},$$

where $\xi_n$ is the root of the equation $\xi\tan(a\xi) = h$.

6. Hint: Apply the finite Fourier cosine transform.

8. Hint: $\tilde{W}_s(n,t) = W_0\phi(t)H(Ut-\ell)\int_0^\ell \sin\left(\frac{\pi n x}{\ell}\right)\delta(x-Ut)dx$

$$= W_0\phi(t)H\left(t-\frac{\ell}{U}\right)\sin\left(\frac{n\pi U t}{\ell}\right).$$

12. Hint: $\frac{d\tilde{V}_s}{dt} + \kappa\left(\frac{n\pi}{a}\right)^2\tilde{V}_s = 0,\ \tilde{V}_s(n,t) = A\exp\left(-\frac{\kappa n^2\pi^2 t}{a^2}\right),$

$$A = \tilde{V}_s(n,0) = \frac{4aV_0}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = (-1)^r\frac{4aV_0}{(2r+1)^2\pi^2}$$

where $n = (2r+1),\ r = 0,1,2,\ldots$

$$V(x,t) = \left(\frac{8V_0}{\pi^2}\right)\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^2}\sin\left\{(2r+1)\frac{nx}{a}\right\}\exp\left\{-\frac{\kappa(2r+1)^2\pi^2 t}{a^2}\right\}.$$

15. Hint: Replace $P$ by $W_0\ d\xi\,d\eta$ and integrate with respect to $\xi$ and $\eta$ over the region $\alpha \le \xi \le \beta,\ \gamma \le \eta \le \delta$.

$$u(x,y) = \left(\frac{4W_0}{\mathscr{D}\pi^6}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left\{\cos\left(\frac{m\pi\alpha}{a}\right)-\cos\left(\frac{m\pi\beta}{a}\right)\right\}\right.$$
$$\left.\times\left\{\cos\left(\frac{n\pi\gamma}{b}\right)-\cos\left(\frac{n\pi\delta}{b}\right)\right\}\frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}{mn\ \omega_{mn}^4}\right].$$

16. Hint: $\frac{d^2\tilde{u}_s(m,n,t)}{dt^2} + \Omega_{mn}^2\tilde{u}_s(m,n,t) = 0, \quad \text{where} \quad \Omega_{mn}^2 = \frac{\mathscr{D}\pi^4\omega_{mn}^4}{\rho h}.$

$$u(x,y,t) = \left(\frac{4}{ab}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{A_{mn}\cos(\Omega_{mn}t)+B_{mn}\sin(\Omega_{mn}t)\}$$
$$\times\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

## 11.6 Exercises

1. (a) $\dfrac{s}{(s^2-a^2)} - \dfrac{\exp(-sT)}{(s^2-a^2)}\,(s\cosh aT + a\sinh aT)$.

   (d) $\dfrac{1}{s}(1-e^{-sT})H(T)$.

## 12.8 Exercises

1. (a) $\dfrac{z^3+4z^2+z}{(z-1)^4}$, use (12.4.13) and (12.3.14).

   (b) $\exp(a/z)$, (c) $\dfrac{z}{(z-e^a)^2}$, (d) $\left(1+\dfrac{1}{z}\right)$, (e) $\dfrac{z(z+a)}{(z-a)^2}$.

2. Hint: Put $b = ix$ in (12.4.14).

6. (a) $$Z^{-1}\left\{\frac{z}{z-2}\cdot\frac{z}{z-3}\right\} = \sum_{m=0}^{n} 2^m 3^{n-m} = 3^n \sum_{m=0}^{n}\left(\frac{2}{3}\right)^m, \\ = (3^{n+1}-2^{n+1}),$$

   (d) $na^{n-1}$,

   (e) $(n-1)a^{n-2}H(n-1)$,

   (f) $(2^{n-1}-n)$,

   (g) $2(-1)^n-(-2)^n$,

   (h) $\dfrac{1}{6}\left[(-1)^n + 2^{n+3} - 3(1)^n\right]$,

   (i) Hint:

   $$\frac{U(z)}{z} = \frac{2}{(z-1)} - \frac{1}{\left(z-\frac{1}{2}\right)},\ U(z) = \frac{2z}{(z-1)} - \frac{z}{\left(z-\frac{1}{2}\right)}$$

   $u(n) = (2-2^{-n})$.

   (j) $$f(n) = Z^{-1}\left\{\frac{z}{z-e^{-a}}\right\} = e^{-an},\quad g(n) = Z^{-1}\left\{\frac{z}{z-e^{-b}}\right\} = e^{-bn},$$

   $$h(n) = f(n) * g(n) = \sum_{m=0}^{\infty} e^{-am}e^{-b(n-m)} \\ = e^{-bn}\sum_{m=0}^{\infty} e^{-(a-b)m} = e^{-bn}\left[\frac{1-e^{(b-a)(n+1)}}{1-e^{(b-a)}}\right].$$

(k) Divide (12.3.6) by $z$ and differentiate both sides with respect to $z$

$$Z^{-1}\left\{(z-a)^{-k}\right\} = \frac{(n-k+1)_{k-1} a^{n-k} H(n-k)}{(k-1)!}, \qquad |z| > a > 0,$$

where $(a)_n = a(a+1)\ldots(a+n-1)$, $\quad (a)_0 = 1$, $\quad n = 1, 2, \ldots,$
and $Z^{-1}\left\{z^{-k}\right\} = \delta(n-k)$.

(l) $F(z) = (z+4) - \dfrac{1}{(z-1)^2} - \dfrac{5}{(z-1)} + \dfrac{16}{(z-2)}$.
Use 6(k) to obtain

$$Z^{-1}\{F(z)\} = \delta(n+1) + 4\delta(n) - (n-1)H(n-2) - 5H(n-1) \\ +16.\, 2^{n-1} H(n-1).$$

Since

$$f(-1) = 1,\ f(0) = 4,\ f(1) = 11,\ f(n) = -(n+4) + 16\, 2^{n-1},\ n \geq 2.$$

(m) $f(n) = \dfrac{1}{5} Z^{-1}\left(\dfrac{6}{z+2} - \dfrac{1}{z-\frac{1}{2}}\right) = \dfrac{6}{5}(-2)^{n-1} - \dfrac{1}{5}\left(\dfrac{1}{2}\right)^{n-1}$, $n \geq 1$.

7. (a) $\dfrac{1}{16}[17(-3)^n + 4n - 1]$, (c) $x_0 a^n + n a^{n-1}$,

(d) $x_0(1-a)^n + 1 - (1-a)^n$, (e) $\dfrac{3}{5}[3^n - (-2)^n]$, (i) $n a^n$.

(k) From the given equation and initial data, $f(-1) = \frac{1}{2}$ and $f(-2) = \frac{1}{4}$.
The $Z$ transform is

$$\begin{aligned} F(z) &= \sum_{k=-1}^{-1} f(k)\, z^{-(k+1)} + z^{-1} F(z) \\ &\qquad + 2\left[\sum_{k=-2}^{-1} f(k)\, z^{-(k+2)} + z^{-2} F(z)\right] \\ &= f(-1) + z^{-1}F(z) + 2\left[f(-2) + f(-1)z^{-1} + z^{-2}\, F(z)\right] \\ F(z) &= 1 + z^{-1} + z^{-1}F(z) + 2\, z^{-2}\, F(z) \\ F(z) &= \frac{z^2+z}{z^2-z-2} = \frac{z}{z-2}. \quad \text{Hence, } f(n) = 2^n. \end{aligned}$$

(l) $F(z) - a\, f(-1) - a\, z^{-1}\, F(z) = z(z-1)^{-1}$

$$\begin{aligned} F(z) &= \frac{2a}{(1-\frac{a}{z})} + \frac{1}{(1-a)} \frac{1}{(1-\frac{1}{z})} + \left(\frac{a}{a-1}\right) \frac{1}{(1-\frac{a}{z})} \\ f(n) &= 2\, a^{n+1} + (1-a)^{-1} + (a-1)^{-1}\, a^{n+1} \\ &= \frac{1}{1-a} + \left(\frac{2a-1}{a-1}\right) a^{n+1}. \end{aligned}$$

(m) $F(z) = \dfrac{z^2+5z}{(z+1)(z+2)} = \dfrac{4z}{z+1} - \dfrac{3z}{z+2}$. $\quad f(n) = 4(-1)^n - 3(-2)^n$.

(n) $F(z) = \dfrac{3z}{z-2}$, $\quad f(n) = 3.2^n$, $\quad n = 0, 1, 2, \ldots$.

9. (a) 1, (b) 0, (c) 1,

(d) $f(0) = 0,\ m > 0;\quad f(0) = 1,\ m = 0$.

11. (a) $(1 - ae^{ix})^{-1}$, (b) $e \log\left(1 + \dfrac{1}{e}\right)$, (c) $(2 \sinh x)^{-1}$.

12. $\dfrac{7}{4} - \dfrac{3}{4}\left(-\dfrac{1}{3}\right)^n$.

13. $U(z) = \dfrac{2z}{z^2-4}$, $u(n) = 2^{n-1}[1 + (-1)^{n+1}]$, $\quad \nu(n) = 2^{n-1}[1 + (-1)^n] - 1$,

where $n = 0, 1, 2, \ldots$.

14. Apply the $Z$ transform to obtain

$$z^3\left[U(z) - u(0) - u(1)z^{-1} - u(2)z^{-2}\right] - 3z^2\left[U(z) - u(0) - u(1)z^{-1}\right] + 3z\left[U(z) - u(0)\right] - U(z) = 0.$$

$$U(z) = \frac{1}{(z-1)^3}\,(z^3 - 3\,z^2 + 4z) = \frac{(1 - 3\,z^{-1} + 4\,z^{-2})}{(1 - z^{-1})^3}.$$

Use

$$Z\left\{n\,a^{n-1}\right\} = \frac{z^{-1}}{(1 - a\,z^{-1})^2}, \qquad Z\left\{n\,(n-1)a^{n-2}\right\} = \frac{2\,z^{-2}}{(1 - a\,z^{-2})},$$

$$\frac{z^{N-1}}{p_N(z)} = \frac{z^2}{(z-1)^3} = \frac{z^{-2}}{(1 - z^{-1})^3}.$$

Inversion gives

$$u(n) = Z^{-1}\left\{\frac{1}{(1 - z^{-1})^3}\right\} - Z^{-1}\left\{\frac{3z^{-1}}{(1 - z^{-1})^3}\right\} + Z^{-1}\left\{\frac{4\,z^{-2}}{(1 - z^{-1})^3}\right\}$$

$$= \frac{1}{2}\,(n+1)\,(n+2) - \frac{3}{2}\,n(n+1) + \frac{4}{2}\,n(n-1) = (n-1)^2.$$

16. (a) $z^2\left[U(z) - u(0) - u(1)z^{-1}\right] + 2\,\left[U(z) - u(1)\right] - 3\,U(z) = 0$

$$U(z) = \frac{z(z+2)}{(z^2 + 2\,z - 3)} = \frac{1}{4}\left(\frac{1}{1 + 3\,z^{-1}} + \frac{3}{1 - z^{-1}}\right)$$

$$u_n = \frac{1}{4}\left[(-3)^n - 3.1^n\right] = \frac{3}{4}\left[1 - (-3)^{n-1}\right],$$

(b) $u_n = 2\left[\left(\frac{2}{3}\right)^{n-1} - 1\right]$,

(c) $u_n = 5^{n/2}\left(2\sin nx + \frac{1}{2}\cos nx\right)$, where $x = \tan^{-1}\left(\frac{1}{2}\right)$.

## 13.5 Exercises

1. (a) $\frac{a^2}{k_i^2}\left(ak_i - \frac{4}{ak_i}\right) J_1(ak_i)$, (b) $\frac{ak_i}{(\alpha^2 - k_i^2)} J_0(a\alpha) J_1(ak_i)$.

10. $u(r,t) = \frac{2}{a^2}\sum_{i=1}^{\infty} \frac{J_0(rk_i)}{J_1^2(ak_i)} \int_0^t \tilde{Q}(k_i,\tau)\ \exp[-(t-\tau)k_i^2] d\tau.$

(a) $u(r,t) = \frac{Q_0}{4k}(a^2 - r^2) - \left(\frac{2Q_0}{ak}\right)\sum_{i=1}^{\infty} \frac{J_0(rk_i)}{k_i^3 J_1(ak_i)}\ \exp(-t\kappa k_i^2)$.

(b) $u(r,t) = \frac{2\kappa Q_0}{ka^2}\sum_{i=1}^{\infty} \frac{J_0(rk_i)}{J_1^2(ak_i)} \int_0^t f(\tau)\ \exp[-\kappa(t-\tau)k_i^2] d\tau$.

## 16.5 Exercises

1. (a) $f_0(n) = \exp(-a),\ n = 0,$

$f_0(n)\exp(-a)[L_n(a) - L_{n-1}(a)],\ n \geq 1.$

(b) $a^n(1+a)^{n+1}$, (c) $A\,\delta_{mn}$,

(d) $0,\ n > m;\ (-1)^n \binom{m}{n} m!, m \geq n$, (e) 1 for $n = 0, 1, 2, 3, \ldots$.

## 17.4 Exercises

1. (a) $2^{n-\frac{1}{2}}\Gamma\left(n + \frac{1}{2}\right)$, (b) $0;\ m = 0, 1, 2, \ldots, n-1$,

(c) $\left(n + \frac{1}{2}\right)\delta_n$.

3. Hint: Use Feldheim's result (1938).
$$H_n^2(x) = n!2^n \sum_{r=0}^{n} \binom{n}{r} \frac{H_{2r}(x)}{2^2\ r!}.$$

## 18.9 Exercises

1. Hint: Use the same method employed in Example 18.2.2 to solve (a) and (b). To solve (c) apply the linearity property.

2. Hint: $\widehat{f}(p, \mathbf{u}) = \sqrt{\pi}\exp(-p^2)$. Use $\mathbf{v} = s\mathbf{u}$, $s = \sqrt{v_1^2 + v_2^2}$ and then use (18.3.8b) to obtain $\widehat{f}(p, \mathbf{v}) = \widehat{f}(p, s\mathbf{u}) = \dfrac{\sqrt{\pi}}{s}\exp(-p^2/s^2)$.
Apply $\dfrac{\partial}{\partial v_k} = \dfrac{\partial s}{\partial v_k}\dfrac{\partial}{\partial s}, \qquad (k = 1, 2) \qquad$ to find
$$\frac{\partial \widehat{f}}{\partial v_k} = \sqrt{\pi}\left(\frac{v_k}{s}\right)\frac{\partial}{\partial s}\left[\frac{1}{s}\exp(-p^2/s^2)\right]$$
$$= \sqrt{\pi}\left(\frac{v_k}{s}\right)(2p^2 - s^2)\exp(-p^2/s^2).$$
Finally, replace $\mathbf{v}$ and $\mathbf{u}$ with $s = 1$ to get
$$\frac{\partial \widehat{f}}{\partial u_k} = \sqrt{\pi} u_k (2p^2 - 1)\exp(-p^2).$$

## 19.5 Exercises

1. (a) $$\mathscr{F}\{\psi(x)\} = \int_{-\infty}^{\infty} e^{-i\omega x}\psi(x)\,dx = \int_0^{\frac{1}{2}} e^{-i\omega x}\,dx - \int_{\frac{1}{2}}^{1} e^{-i\omega x}dx$$
$$= \frac{4i}{\omega}\exp(-\frac{i\omega}{2})\,\sin^2(\frac{i\omega}{4}).$$
Using scaling and shifting properties of the Fourier transform gives $\mathscr{F}\{\psi(2^m x - n)\}$.

(b) The amplitude spectrum decays like $\omega^{-1}$ and hence, tends to zero slowly as $|\omega| \to \infty$. This shows that the Haar wavelet has poor frequency localization. But it has a very good time localization.

2. (a) $$\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(\frac{\omega}{2})}{(\frac{\omega}{2})}\right]^2,$$

(b) $$\hat{\psi}(\omega) = \sqrt{2\pi}\,\omega^2\exp\left(-\frac{\omega^2}{2}\right),$$

(c) $\hat{\psi}(\omega) = \sqrt{2\pi}\, \exp\left[-\frac{1}{2}(\omega - \omega_0)^2\right].$

8. (a)–(c) See pages 434-437 of a book by Debnath (2002).
9. See Debnath (2002) pages 434-438.
10. See Debnath (2002) pages 440-441.
11. A *window function* is a $f \in L^2(\mathbb{R})$ with norm unity.

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2\, dt = \left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-2at^2}\, dt = \left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \cdot \sqrt{\frac{\pi}{2a}} = 1.$$

$$\hat{f}(\omega) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t}\, \exp(-at^2) dt = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \mathscr{F}\left\{\exp(-at^2)\right\}.$$

$$= \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\omega^2}{4a}\right) = \frac{1}{\sqrt{2a}}\, f\left(\frac{\omega}{2a}\right).$$

The Gabor windows have optimal time and frequency localization properties.

12.

$$\text{(a)} \quad \|f\|^2 = \left(\frac{3}{2a^3}\right) \int_{-\infty}^{\infty} |(a - |t|)^2|\, \chi_{[-a,a]}(t)\, dt$$

$$= \left(\frac{3}{2a^3}\right) \int_{-a}^{a} |(a - |t|)|^2\, dt = \left(\frac{3}{2a^3}\right) 2 \int_0^a (a - t)^2\, dt = 1.$$

$$\text{(b)} \quad \hat{f}(\omega) = \sqrt{\frac{3}{2a^3}}\, \mathscr{F}\left\{\chi_{[-\frac{a}{2},\frac{a}{2}]}(t) * \chi_{[-\frac{a}{2},\frac{a}{2}]}(t)\right\},$$

$$= \sqrt{\frac{3}{2a^3}} \left[\mathscr{F}\left\{\chi_{[-\frac{a}{2},\frac{a}{2}]}(t)\right\}\right]^2, \text{by convolution theorem}$$

$$= \sqrt{\frac{3}{2a^4}} \left(\frac{2}{\pi}\right) \left(\frac{\sin \frac{a\omega}{2}}{\omega}\right)^2 = \sqrt{\frac{3a}{2}} \left(\frac{2}{\pi}\right) \left(\frac{\sin \frac{a\omega}{2}}{a\omega}\right)^2.$$

It is better localized in the frequency domain, as $\hat{f}(\omega)$ decays like $|\omega|^{-2}$ as $|\omega| \to \infty$.

$$\text{(c)} \quad \nabla_f^2 = \int_{-\infty}^{\infty} t^2\, |f(t)|^2\, dt = \int_{-a}^{a} t^2 \left(\sqrt{\frac{3}{2a^3}}(a - |t|)\right)^2 dt$$

$$= \frac{3}{a^3} \int_0^a t^2 (a - t)^2\, dt = \frac{1}{10}\, a^2$$

$$\nabla_{\hat{f}}^2 = \frac{1}{4\pi^2}\, \|f'\|^2 = \frac{3}{8\pi^2 a^3}\left[\int_{-a}^{0} 1^2\, dt + \int_0^a (-1)^2\, dt\right] = \frac{3}{4\pi^2 a^3}$$

$$(4\, \nabla_f\, \nabla_{\hat{f}}) = \frac{1}{\pi}\sqrt{\frac{6}{5}} > \frac{1}{\pi}.$$

# Bibliography

The following bibliography is not by any means a complete one for the subject. For the most part, it consists of books and papers to which reference is made in the text. Many other selected books and papers related to material of the subject have been included so that they may serve to stimulate new interest in future study and research.


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